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Representation Theory of the Symmetric Groups

The Okounkov–Vershik Approach, Character Formulas, and Partition Algebras

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REPRESENTATION THEORY OF THE SYMMETRIC GROUPS

The representation theory of the symmetric groups is a classical topic that, since the pioneering work of Frobenius, Schur and Young, has grown into a huge body of theory, with many important connections to other areas of mathematics and physics.

This self-contained book provides a detailed introduction to the subject, covering classical topics such as the Littlewood–Richardson rule and the Schur–Weyl duality. Importantly, the authors also present many recent advances in the area, including M. Lassalle's character formulas, the theory of partition algebras, and an exhaustive exposition of the approach developed by A. M. Vershik and A. Okounkov.

A wealth of examples and exercises makes this an ideal textbook for graduate students. It will also serve as a useful reference for more experienced researchers across a range of areas, including algebra, computer science, statistical mechanics and theoretical physics.

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To Katiuscia, Giacomo, and Tommaso

To my parents, Cristina, and Nadiya

To my Mom, Rossella, and Stefania

Contents

Preface

Since the pioneering works of Frobenius, Schur and Young more than a hundred years ago, the representation theory of the finite symmetric group has grown into a huge body of theory, with many important and deep connections to the representation theory of other groups and algebras as well as with fruitful relations to other areas of mathematics and physics. In this monograph, we present the representation theory of the symmetric group along the new lines developed by several authors, in particular by A. M. Vershik, G. I. Olshanskii and A. Okounkov. The tools/ingredients of this new approach are either completely new, or were not fully understood in their whole importance by previous authors. Such tools/ingredients, that in our book are presented in a fully detailed and exhaustive exposition, are:

– the algebras of conjugacy-invariant functions, the algebras of bi-K-invariant functions, the Gelfand pairs and their spherical functions;
– the Gelfand–Tsetlin algebras and their corresponding bases;
– the branching diagrams, the associated posets and the content of a tableau;
– the Young–Jucys–Murphy elements and their spectral analysis;
– the characters of the symmetric group viewed as spherical functions.

The first chapter is an introduction to the representation theory of finite groups. The second chapter contains a detailed discussion of the algebras of conjugacy-invariant functions and their relations with Gelfand pairs and Gelfand–Tsetlin bases. In the third chapter, which constitutes the core of the whole book, we present an exposition of the Okounkov–Vershik approach to the representation theory of the symmetric group. We closely follow the original sources. However, we enlighten the presentation by establishing a connection between the algebras of conjugacy-invariant functions and Gelfand pairs, and by deducing the Young rule from the analysis of a suitable poset.

We also derive, in an original way, the Pieri rule. In the fourth chapter we present the theory of symmetric functions focusing on their relations with the representation theory of the symmetric group. We have added some nonstandard material, closely related to the subject. In particular, we present two proofs of the Jucys–Murphy theorem which characterizes the center of the group algebra of the symmetric group as the algebra of symmetric polynomials in the Jucys–Murphy elements. The first proof is the original one given by Murphy, while the second one, due to A. Garsia, also provides an explicit expression for the characters of $\mathfrak{S}_n$ as symmetric polynomials in the Jucys–Murphy elements. In the fifth chapter we give some recent formulas by Lassalle and Corteel–Goupil–Schaeffer. In these formulas, the characters of the symmetric group, viewed as spherical functions, are expressed as symmetric functions on the content of the tableaux, or, alternatively, as shifted symmetric functions (a concept introduced by Olshanskii and Okounkov) on the partitions. Chapter 6 is entirely dedicated to the Littlewood–Richardson rule and is based on G. D. James' approach. The combinatorial theory developed by James is extremely powerful and, besides giving a proof of the Littlewood–Richardson rule, provides explicit orthogonal decompositions of the Young modules. We show that the decompositions obtained in Chapter 3 (via the Gelfand–Tsetlin bases) are particular cases of those obtained with James' method and, following Sternberg, we interpret such decompositions in terms of Radon transforms (P. Diaconis also alluded to this idea in his book [26]). Moreover, we introduce the Specht modules and the generalized Specht modules. It is important to point out that this part is closely related to the theory developed in Chapter 3 starting from the branching rule and the elementary notions on Young modules (in fact these notions and the related results suffice). The seventh chapter is an introduction to finite dimensional algebras and their representation theory. In order to avoid technicalities and to get as fast as possible to the fundamental results, we limit ourselves to the operator $*$-algebras on a finite dimensional Hilbert space. We have included a detailed account on reciprocity laws based on recent ideas of R. Howe and their exposition in the book by Goodman–Wallach, and a related abstract construction that naturally leads to the notion of partition algebra. In Chapter 8 we present an exposition of the Schur–Weyl duality emphasizing the connections with the results from Chapters 3 and 4. We do not go deeply into the representation theory of the general linear group $GL(n, \mathbb{R})$, because it requires tools like Lie algebras, but we include an elementary account on partition algebras, mainly based on a recent expository paper of T. Halverson and A. Ram.

The style of our book is the following. We explicitly want to remain at an elementary level, without introducing the notions in their wider generality and avoiding too many technicalities. On the other hand, the book is absolutely self-contained (apart from the elementary notions of linear algebra and group theory, including group actions) and the proofs are presented in full details. Our goal is to introduce the (possibly inexperienced) reader to an active area of research, with a text that is, therefore, far from being a simple compilation of papers and other books. Indeed, in several places, our treatment is original, even for a few elementary facts. Just to draw a comparison against two other books, the theory of Okounkov and Vershik is treated in a complete way in the first chapter of Kleshchev's book, but this monograph is at an extremely more advanced level than ours. Also, the theory of symmetric functions is masterly and remarkably treated in the classical book by Macdonald; in comparison with this book, by which we were inspired at several stages, our treatment is slightly more elementary and less algebraic. However, we present many recent results not included in Macdonald's book.

We express our deep gratitude to Alexei Borodin, Adriano Garsia, Andrei Okounkov, Grigori Olshanski, and especially to Arun Ram and Anatoly Vershik, for their interest in our work, useful comments and continuous encouragement.

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Roma, 21 May 2009 TCS, FS and FT

1
Representation theory of finite groups

This chapter is a basic course in representation theory of finite groups. It is inspired by the books by Serre [109], Simon [111], Sternberg [115], Fulton and Harris [43] and by our recent [20]. With respect to the latter, we do not separate the elementary and the advanced topics (Chapter 3 and Chapter 9 therein). Here, the advanced topics are introduced as soon as possible.

The presentation of the character theory is based on the book by Fulton and Harris [43], while the section on induced representations is inspired by the books by Serre [109], Bump [15], Sternberg [115] and by our expository paper [18].

1.1 Basic facts

1.1.1 Representations

Let G be a finite group and V a finite dimensional vector space over the complex field $\mathbb{C}$. We denote by $GL(V)$ the group of all bijective linear maps $T : V \to V$. A (linear) *representation* of G on V is a homomorphism

$$\sigma : G \to GL(V).$$

This means that for every $g \in G$, $\sigma(g)$ is a linear bijection of V into itself and that

- $\sigma(g_1 g_2) = \sigma(g_1)\sigma(g_2)$ for all $g_1, g_2 \in G$;
- $\sigma(1_G) = I_V$, where 1_G is the identity of G and I_V the identity map on V;
- $\sigma(g^{-1}) = \sigma(g)^{-1}$ for all $g \in G$.

To emphasize the role of V, a representation will be also denoted by the pair (σ, V) or simply by V. Note that a representation may be also seen as an action of G on V such that $\sigma(g)$ is a linear map for all $g \in G$.

A subspace $W \leq V$ is *σ-invariant* (or *G-invariant*) if $\sigma(g)w \in W$ for all $g \in G$ and $w \in W$. Clearly, setting $\rho(g) = \sigma(g)|_W$, then (ρ, W) is also a representation of G. We say that ρ is a *sub-representation* of σ.

The trivial subspaces V and $\{0\}$ are always invariant. We say that (σ, V) is *irreducible* if V has no non-trivial invariant subspaces; otherwise we say that it is *reducible*.

Suppose now that V is a *unitary* space, that is, it is endowed with a Hermitian scalar product $\langle \cdot, \cdot \rangle_V$. A representation (σ, V) is *unitary* provided that $\sigma(g)$ is a unitary operator for all $g \in G$. This means that $\langle \sigma(g)v_1, \sigma(g)v_2 \rangle_V = \langle v_1, v_2 \rangle_V$ for all $g \in G$ and $v_1, v_2 \in V$. In particular, $\sigma(g^{-1})$ equals $\sigma(g)^*$, the *adjoint* of $\sigma(g)$.

Let (σ, V) be a representation of G and let $K \leq G$ be a subgroup. The *restriction* of σ from G to K, denoted by $\mathrm{Res}^G_K \sigma$ (or $\mathrm{Res}^G_K V$) is the representation of K on V defined by $[\mathrm{Res}^G_K \sigma](k) = \sigma(k)$ for all $k \in K$.

1.1.2 Examples

Example 1.1.1 (The trivial representation) For every group G, we define the *trivial representation* as the one-dimensional representation $(\iota_G, \mathbb{C})$ defined by setting $\iota_G(g) = 1$, for all $g \in G$.

Example 1.1.2 (Permutation representation (homogeneous space)) Suppose that G acts on a finite set X; for $g \in G$ and $x \in X$ denote by gx the g-image of x. Denote by $L(X)$ the vector space of all complex-valued functions defined on X. Then we can define a representation λ of G on $L(X)$ by setting

$$[\lambda(g)f](x) = f(g^{-1}x)$$

for all $g \in G$, $f \in L(X)$ and $x \in X$. This is indeed a representation:

$$[\lambda(g_1g_2)f](x) = f(g_2^{-1}g_1^{-1}x) = [\lambda(g_2)f](g_1^{-1}x) = \{\lambda(g_1)[\lambda(g_2)f]\}(x),$$

that is, $\lambda(g_1g_2) = \lambda(g_1)\lambda(g_2)$ (and clearly $\lambda(1_G) = I_{L(X)}$). λ is called the *permutation representation* of G on $L(X)$.

If we introduce a scalar product $\langle \cdot, \cdot \rangle_{L(X)}$ on $L(X)$ by setting

$$\langle f_1, f_2 \rangle_{L(X)} = \sum_{x \in X} f_1(x)\overline{f_2(x)}$$

for all $f_1, f_2 \in L(X)$, then λ is unitary.

Another useful notation is the following. For $x \in X$, we denote by δ_x the *Dirac function* centered at x, which is defined by

$$\delta_x(y) = \begin{cases} 1 & \text{if } y = x \\ 0 & \text{if } y \neq x. \end{cases}$$

Note that $\{\delta_x : x \in X\}$ constitutes an orthonormal basis for $L(X)$ and, in particular, $f = \sum_{x \in X} f(x)\delta_x$ for all $f \in L(X)$. Moreover, $\lambda(g)\delta_x = \delta_{gx}$ for all $g \in G$ and $x \in X$.

Example 1.1.3 (Left and right regular representations) This is a particular case of the previous example. Consider the left Cayley action of G on itself: $g_0 \overset{g}{\mapsto} gg_0$, $g, g_0 \in G$. The associated permutation representation is called the *left regular representation* and it is always denoted by λ. In other words, $[\lambda(g)f](g_0) = f(g^{-1}g_0)$ for all $g, g_0 \in G$ and $f \in L(G)$.

Analogously, the permutation representation associated with the right Cayley action of G on itself: $g_0 \overset{g}{\mapsto} g_0 g^{-1}$, $g, g_0 \in G$, is called the *right regular representation* and it is always denoted by ρ. In other words, $[\rho(g)f](g_0) = f(g_0 g)$ for all $g, g_0 \in G$ and $f \in L(G)$.

Example 1.1.4 (The alternating representation) Let $\mathfrak{S}_n$ be the *symmetric group* of degree n (the group of all bijections, called *permutations*, $\pi : \{1, 2, \dots, n\} \to \{1, 2, \dots, n\}$). The *alternating representation* of $\mathfrak{S}_n$ is the one-dimensional representation $(\varepsilon, \mathbb{C})$ defined by setting

$$\varepsilon(\pi) = \begin{cases} 1 & \text{if } \pi \text{ is even} \\ -1 & \text{if } \pi \text{ is odd.} \end{cases}$$

Remark 1.1.5 Every finite dimensional representation (σ, V) of a finite group G is *unitarizable*, that is, it is possible to define a scalar product $\langle \cdot, \cdot \rangle$ on V which makes σ unitary.

Indeed, given an arbitrary scalar product $(\cdot, \cdot)$ on V, setting

$$\langle v_1, v_2 \rangle = \sum_{g \in G} (\sigma(g)v_1, \sigma(g)v_2)$$

we have that

$$\begin{aligned} \langle \sigma(g)v_1, \sigma(g)v_2 \rangle &= \sum_{h \in G} (\sigma(h)\sigma(g)v_1, \sigma(h)\sigma(g)v_2) \\ (s := gh) \quad &= \sum_{s \in G} (\sigma(s)v_1, \sigma(s)v_2) \\ &= \langle v_1, v_2 \rangle. \end{aligned}$$

By virtue of this remark, from now on we consider only unitary representations.

1.1.3 Intertwining operators

Let V and W be two vector spaces. We denote by $\mathrm{Hom}(V, W)$ the space of all linear maps $T : V \to W$. If (σ, V) and (ρ, W) are two representations of a group G and $T \in \mathrm{Hom}(V, W)$ satisfies

$$T\sigma(g) = \rho(g)T \tag{1.1}$$

for all $g \in G$, we say that T *intertwines* σ and ρ (or V and W) or that T is an *intertwining operator*. We denote by $\mathrm{Hom}_G(V, W)$ (or by $\mathrm{Hom}_G(\sigma, \rho)$) the vector space of all operators that intertwine σ and ρ.

If σ and ρ are unitary, then

$$\begin{aligned} \mathrm{Hom}_G(\sigma, \rho) &\to \mathrm{Hom}_G(\rho, \sigma) \\ T &\mapsto T^* \end{aligned} \tag{1.2}$$

is an antilinear (that is, $(\alpha T_1 + \beta T_2)^* = \overline{\alpha} T_1^* + \overline{\beta} T_2^*$, for all $\alpha, \beta \in \mathbb{C}$ and $T_1, T_2 \in \mathrm{Hom}_G(\sigma, \rho)$) isomorphism. Indeed, taking the adjoint of both sides, (1.1) is equivalent to

$$\sigma(g^{-1})T^* = T^*\rho(g^{-1})$$

and therefore $T \in \mathrm{Hom}_G(\sigma, \rho)$ if and only if $T^* \in \mathrm{Hom}_G(\rho, \sigma)$.

Two representations (σ, V) and (ρ, W) are said to be *equivalent*, if there exists $T \in \mathrm{Hom}_G(\sigma, \rho)$ which is bijective. If this is the case, we call T an *isomorphism* and we write $\sigma \sim \rho$ and $V \cong W$; if not, we write $\sigma \not\sim \rho$. If in addition σ and ρ are unitary representations and T is a unitary operator, then we say that σ and ρ are *unitarily equivalent*.

The following lemma shows that for unitary representations the notions of equivalence and of unitary equivalence coincide. We first recall that a bijective operator $T \in \mathrm{Hom}(V, W)$ has the following, necessarily unique, *polar decomposition*: $T = U|T|$, where $|T| \in GL(V)$ is the square root of the positive operator T^*T and $U \in \mathrm{Hom}(V, W)$ is unitary, see [75].

Lemma 1.1.6 *Suppose that (ρ, V) and (σ, W) are unitary representations of a finite group G. If they are equivalent then they are also unitarily equivalent.*

Proof Let $T \in \mathrm{Hom}_G(V, W)$ be a linear bijection. Composing with $T^* \in \mathrm{Hom}_G(W, V)$, one obtains an operator $T^*T \in \mathrm{Hom}_G(V, V) \cap GL(V)$. Denote by $T = U|T|$ the polar decomposition of T. Then, for all $g \in G$,

$$\sigma(g^{-1})|T|^2\sigma(g) = \sigma(g^{-1})T^*T\sigma(g) = T^*\rho(g^{-1})\rho(g)T = T^*T = |T|^2$$

and, since $[\sigma(g^{-1})|T|\sigma(g)]^2 = \sigma(g^{-1})|T|^2\sigma(g)$, the uniqueness of the polar decomposition implies that

$$\sigma(g^{-1})|T|\sigma(g) = |T|,$$

in other words, $|T| \in Hom_G(V, V)$. It then follows that

$$U\sigma(g) = T|T|^{-1}\sigma(g) = T\sigma(g)|T|^{-1} = \rho(g)U$$

for all $g \in G$, and therefore U implements the required unitary equivalence. □

Definition 1.1.7 We denote by Irr(G) the set of all (unitary) irreducible representations of G and by $\widehat{G} = \mathrm{Irr}(G)/\sim$ the set of its equivalence classes. More concretely, we shall often think of $\widehat{G}$ as a fixed set of irreducible (unitary) pairwise inequivalent representations of G.

1.1.4 Direct sums and complete reducibility

Suppose that (σ_j, V_j), $j = 1, 2, \ldots, m$, $m \in \mathbb{N}$, are representations of a group G. Their *direct sum* is the representation (σ, V) defined by setting $V = \bigoplus_{j=1}^m V_j$ and, for $v = \sum_{j=1}^m v_j \in V$ (with $v_j \in V_j$) and $g \in G$,

$$\sigma(g)v = \sum_{j=1}^{m} \sigma_j(g)v_j.$$

Usually, we shall write $(\sigma, V) = (\bigoplus_{j=1}^m \sigma_j, \bigoplus_{j=1}^m V_j)$.

Conversely, let (σ, V) be a representation of G and suppose that $V = \bigoplus_{j=1}^m V_j$ with all V_j's σ-invariant subspaces. Set, for all $g \in G$, $\sigma_j(g) = \sigma(g)|_{V_j}$. Then σ is the direct sum of the representations σ_j's: $\sigma = \bigoplus_{j=1}^m \sigma_j$.

Lemma 1.1.8 *Let (σ, V) be a finite dimensional representation of a finite group G. Then it can be decomposed into a direct sum of irreducible representations, namely,*

$$V = \bigoplus_{j=1}^{m} V_j \tag{1.3}$$

where the V_j's are irreducible. Moreover, if σ is unitary, the decomposition (1.3) can be chosen to be orthogonal.

Proof By virtue of Remark 1.1.5, we can reduce to the case that σ is unitary.

If (σ, V) is not irreducible, let $W \leq V$ be a non-trivial σ-invariant subspace. We show that its orthogonal complement $W^\perp = \{v \in V : \langle v, w\rangle_V = 0$ for all $w \in W\}$ is also invariant. Indeed, if $g \in G$ and $v \in W^\perp$, one has

$$\langle \sigma(g)v, w\rangle_V = \langle v, \sigma(g^{-1})w\rangle_V = 0$$

for all $w \in W$, because $\sigma(g^{-1})w \in W$ and $v \in W^\perp$. Thus $\sigma(g)v \in W^\perp$. In other words, we have the orthogonal decomposition

$$V = W \bigoplus W^\perp.$$

If both W and $W^\perp$ are irreducible we are done. Otherwise we can iterate this process decomposing W and/or $W^\perp$. As $\dim W, \dim W^\perp < \dim V$, this procedure necessarily stops after a finite number of steps (the one-dimensional representations are clearly irreducible). □

1.1.5 The adjoint representation

We recall that if V is a finite dimensional vector space over $\mathbb{C}$, its *dual* V' is the space of all linear functions $f : V \to \mathbb{C}$. If in addition V is endowed with a scalar product $\langle \cdot, \cdot\rangle_V$, then we define the *Riesz map*

$$V \ni v \mapsto \theta_v \in V' \tag{1.4}$$

where $\theta_v(w) = \langle w, v\rangle_V$ for all $w \in V$. This map is antilinear (that is, $\theta_{\alpha v+\beta w} = \overline{\alpha}\theta_v + \overline{\beta}\theta_w$) and bijective (Riesz representation theorem). The dual scalar product on V' is defined by setting

$$\langle \theta_v, \theta_w\rangle_{V'} = \langle w, v\rangle_V \tag{1.5}$$

for all $v, w \in V$. If $\{v_1, v_2, \ldots, v_n\}$ is an orthonormal basis of V, the associated *dual basis* of V' is given by $\{\theta_{v_1}, \theta_{v_2}, \ldots, \theta_{v_n}\}$. Note that it is characterized by the following property: $\theta_{v_i}(v_j) = \delta_{i,j}$.

Suppose now that (σ, V) is a representation of G. The *adjoint* (or *conjugate*, or *contragredient*) representation is the representation (σ', V') of G defined by setting

$$[\sigma'(g)f](v) = f(\sigma(g^{-1})v)) \tag{1.6}$$

for all $g \in G$, $f \in V'$ and $v \in V$. Note that we have

$$\begin{aligned}[\sigma'(g)\theta_w](v) &= \theta_w(\sigma(g^{-1})v)\\ &= \langle \sigma(g^{-1})v, w\rangle_V\\ &= \langle v, \sigma(g)w\rangle_V\\ &= \theta_{\sigma(g)w}(v),\end{aligned}$$

that is,

$$\sigma'(g)\theta_w = \theta_{\sigma(g)w} \tag{1.7}$$

for all $g \in G$ and $w \in V$. In particular, σ is irreducible if and only if σ' is irreducible.

1.1.6 Matrix coefficients

Let (σ, V) be a unitary representation of G. Let $\{v_1, v_2, \dots, v_n\}$ be an orthonormal basis of V ($n = \dim V$). The *matrix coefficients* associated with this basis are given by

$$\varphi_{i,j}(g) = \langle \sigma(g)v_j, v_i\rangle_V$$

$g \in G, i, j = 1, 2, \dots, n$. We now present some basic properties of the matrix coefficients.

Lemma 1.1.9 *Let $U(g) = (\varphi_{i,j}(g))_{i,j=1}^n \in M_{n,n}(\mathbb{C})$ be the matrix with entries the matrix coefficients of (σ, V). Then, for all $g, g_1, g_2 \in G$ we have*

(i) *$U(g_1g_2) = U(g_1)U(g_2)$;*
(ii) *$U(g^{-1}) = U(g)^*$;*
(iii) *$U(g)$ is unitary;*
(iv) *the matrix coefficients of the adjoint representation σ' with respect to the dual basis $\theta_{v_1}, \theta_{v_2}, \dots, \theta_{v_n}$ are*

$$\langle \sigma'(g)\theta_{v_j}, \theta_{v_i}\rangle_{V'} = \overline{\varphi_{i,j}(g)}.$$

Proof All these properties are easy consequences of the fact that $U(g)$ is the representation matrix of the linear operator $\sigma(g)$ with respect to the basis $v_1, v_2, \dots, v_n$.

(i) We have

$$\sigma(g_2)v_j = \sum_{k=1}^n \langle \sigma(g_2)v_j, v_k\rangle_V v_k$$

and therefore

$$\begin{aligned}\varphi_{i,j}(g_1g_2) &= \langle \sigma(g_1)\sigma(g_2)v_j, v_i\rangle_V \\ &= \sum_{k=1}^n \langle \sigma(g_2)v_j, v_k\rangle_V \cdot \langle \sigma(g_1)v_k, v_i\rangle_V \\ &= \sum_{k=1}^n \varphi_{i,k}(g_1)\varphi_{k,j}(g_2).\end{aligned}$$

(ii) We have

$$\begin{aligned}\varphi_{i,j}(g^{-1}) &= \langle \sigma(g^{-1})v_j, v_i\rangle_V \\ &= \langle v_j, \sigma(g)v_i\rangle_V \\ (\sigma(g^{-1}) = \sigma(g)^*) \quad &= \overline{\langle \sigma(g)v_i, v_j\rangle_V} \\ &= \overline{\varphi_{j,i}(g)}.\end{aligned}$$

(iii) This is an immediate consequence of (i) and (ii).

(iv) Recalling (1.5) and (1.7) we have

$$\begin{aligned}\langle \sigma'(g)\theta_{v_j}, \theta_{v_i}\rangle_{V'} &= \langle \theta_{\sigma(g)v_j}, \theta_{v_i}\rangle_{V'} \\ &= \langle v_i, \sigma(g)v_j\rangle_V \\ &= \overline{\langle \sigma(g)v_j, v_i\rangle_V}.\end{aligned}$$

□

We shall say that the $\varphi_{i,j}(g)$'s are *unitary* matrix coefficients and that $g \mapsto U(g)$ is a *unitary matrix realization* of $\sigma(g)$.

1.1.7 Tensor products

We first recall the notion of tensor product of finite dimensional unitary spaces (we follow the monograph by Simon [111] and our book [20]).

Suppose that V and W are finite dimensional unitary spaces over $\mathbb{C}$. The *tensor product* $V \bigotimes W$ is the vector spaces consisting of all maps

$$B : V \times W \to \mathbb{C}$$

which are bi-antilinear:

$$\begin{aligned}B(\alpha_1 v_1 + \alpha_2 v_2, w) &= \overline{\alpha_1} B(v_1, w) + \overline{\alpha_2} B(v_2, w) \\ B(v, \alpha_1 w_1 + \alpha_2 w_2) &= \overline{\alpha_1} B(v, w_1) + \overline{\alpha_2} B(v, w_2)\end{aligned}$$

for all $\alpha_1, \alpha_2 \in \mathbb{C}$, $v_1, v_2, v \in V$ and $w_1, w_2, w \in W$.

For $v \in V$ and $w \in W$ we define the *simple tensor* $v \otimes w$ in $V \bigotimes W$ by setting

$$[v \otimes w](v_1, v_2) = \langle v, v_1\rangle_V \langle w, w_1\rangle_W.$$

The map

$$\begin{aligned}V \times W &\to V \bigotimes W \\ (v, w) &\mapsto v \otimes w\end{aligned}$$

is bilinear:

$$(\alpha_1 v_1 + \alpha_2 v_2) \otimes (\beta_1 w_1 + \beta_2 w_2) = \sum_{i,j=1}^{2} \alpha_i \beta_j v_i \otimes w_j$$

for all $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{C}$, $v_1, v_2 \in V$ and $w_1, w_2 \in W$.

The simple tensors span the whole $V \bigotimes W$: more precisely, if $\mathcal{B}_V = \{v_1, v_2, \ldots, v_n\}$ and $\mathcal{B}_W = \{w_1, w_2, \ldots, w_m\}$ are orthonormal bases for V and W, respectively, then

$$\{v_i \otimes w_j\}_{\substack{i=1,2,\ldots,n \\ j=1,2,\ldots,m}} \tag{1.8}$$

is a basis for $V \bigotimes W$. In particular, $\dim(V \bigotimes W) = \dim V \cdot \dim W$. Indeed, if $B \in V \bigotimes W$, $v = \sum_{i=1}^{n} \alpha_i v_i \in V$ and $w = \sum_{j=1}^{m} \beta_j w_j \in W$, then $(v_i \otimes w_j)(v, w) = \overline{\alpha_i}\overline{\beta_j}$ and therefore

$$\begin{aligned} B(v, w) &= \sum_{i=1}^{n} \sum_{j=1}^{m} \overline{\alpha_i}\overline{\beta_j} B(v_i, w_j) \\ &= \left[\sum_{i=1}^{n} \sum_{j=1}^{m} B(v_i, w_j) v_i \otimes w_j \right] (v, w) \end{aligned}$$

so that $B = \sum_{i=1}^{n} \sum_{j=1}^{m} B(v_i, w_j) v_i \otimes w_j$.

Moreover, one introduces a scalar product on $V \bigotimes W$ by setting

$$\langle v_i \otimes w_k, v_j \otimes w_\ell \rangle_{V \bigotimes W} = \langle v_i, v_j \rangle_V \cdot \langle w_k, w_\ell \rangle_W$$

$v_i, v_j \in \mathcal{B}_V$, $w_k, w_\ell \in \mathcal{B}_W$, and then extending it by linearity. It then follows that (1.8) is also orthonormal.

Finally, if $T \in \mathrm{Hom}(V, V)$ and $S \in \mathrm{Hom}(W, W)$ one defines a linear action of $T \otimes S$ on $V \otimes W$ by setting $(T \otimes S)(v \otimes w) = (Tv) \otimes (Sw)$ for all $v \in \mathcal{B}_V$ and $w \in \mathcal{B}_W$, and extending it by linearity. This determines an isomorphism $\mathrm{Hom}(V \bigotimes W) \cong \mathrm{Hom}(V, V) \bigotimes \mathrm{Hom}(W, W)$. Indeed, we have $T \otimes S = 0$ if and only if $T = 0$ or $S = 0$. Moreover, $\dim\mathrm{Hom}(V \otimes W) = \dim[\mathrm{Hom}(V, V) \otimes \mathrm{Hom}(W, W)] = (\dim V)^2 (\dim W)^2$.

Suppose now that G_1 and G_2 are two finite groups. Let (σ_i, V_i) be a representation of G_i, $i = 1, 2$. The *outer tensor product* of σ_1 and σ_2 is the representation $\sigma_1 \boxtimes \sigma_2$ of $G_1 \times G_2$ on $V_1 \otimes V_2$ defined by setting

$$[\sigma_1 \boxtimes \sigma_2](g_1, g_2) = \sigma_1(g_1) \otimes \sigma_2(g_2)$$

for all $g_1 \in G_1$ and $g_2 \in G_2$.

With the notation above, if $G_1 = G_2 = G$, we define a representation $\sigma_1 \otimes \sigma_2$ of G on $V_1 \bigotimes V_2$ by setting

$$[\sigma_1 \otimes \sigma_2](g) = \sigma_1(g) \otimes \sigma_2(g)$$

for all $g \in G$. This is called the *internal* tensor product of σ_1 and σ_2. Note that

$$\sigma_1 \otimes \sigma_2 = \mathrm{Res}^{G\times G}_{\widetilde{G}}(\sigma_1 \boxtimes \sigma_2)$$

where $\widetilde{G} = \{(g_1, g_2) \in G \times G : g_1 = g_2\}$ is the diagonal subgroup of $G \times G$ and Res denotes the restriction as in the end of Section 1.1.1.

1.1.8 Cyclic and invariant vectors

Let (σ, V) be a unitary representation of G. A vector $v \in V$ is *cyclic* if the translates $\sigma(g)v$, with $g \in G$, span the whole V. In formulæ: $< \sigma(g)v : g \in G >= V$. For instance, if G acts transitively on X, then every δ_x, with $x \in X$, is cyclic in the permutation representation λ of G on $L(X)$ (indeed, $\lambda(g)\delta_x = \delta_{gx}$). Note that, if W is a proper σ-invariant subspace of V, then no vector $w \in W$ can be cyclic.

On the other hand, we say that a vector $u \in V$ is σ-*invariant* (or *fixed*) if $\sigma(g)u = u$ for all $g \in G$. We denote by $V^G = \{u \in V : \sigma(g)u = u$ for all $g \in G\}$ the subspace of all σ-invariant vectors.

Lemma 1.1.10 *Suppose that $u \in V^G$ is a non-trivial invariant vector. If $v \in V$ is orthogonal to u, namely $\langle u, v\rangle_V = 0$, then v cannot be cyclic.*

Proof Indeed, v is contained in the orthogonal complement $< u >^{\perp}$, which is a proper σ-invariant subspace of V (σ is unitary). □

The following result will be useful in the study of the representation theory of the symmetric group.

Corollary 1.1.11 *Suppose that there exist a cyclic vector $v \in V$, $g \in G$ and $\lambda \in \mathbb{C}$, $\lambda \neq 1$, such that $\sigma(g)v = \lambda v$. Then $V^G = \{0\}$.*

Proof Let $u \in V^G$, so that $\sigma(g)u = u$. As u and v admit distinct eigenvalues w.r. to the unitary operator $\sigma(g)$, they are orthogonal. By the previous lemma, $u = 0$. □

Exercise 1.1.12 Show that if $\dim V^G \geq 2$ then V has no cyclic vectors.

[1.1.12 *Hint.* If $u, w \in V^G$ are non-trivial and orthogonal, then for $v \in V \setminus V^G$ one has $\dim < u, w, v >= 3$ and therefore there exists $u_0 \in < u, w > \subseteq V^G$, with $u_0 \neq 0$, such that $< u_0, v >= 0$.]

For instance, if G acts transitively on X, then $\dim L(X)^G = 1$ because for $f \in L(G)$ one has $\lambda(g)f = f$ for all $g \in G$ if and only if f is constant on X. Moreover, as we remarked before, the vectors δ_x, $x \in X$, are cyclic.

1.2 Schur's lemma and the commutant

Let G be a finite group.

1.2.1 Schur's lemma

Lemma 1.2.1 (Schur's lemma) *Let (σ, V) and (ρ, W) be two irreducible representations of G.*

(i) *If $T \in \mathrm{Hom}_G(\sigma, \rho)$ then either $T = 0$ or T is an isomorphism.*
(ii) $\mathrm{Hom}_G(\sigma, \sigma) = \mathbb{C}I_V$.

Proof (i) If $T \in \mathrm{Hom}_G(\sigma, \rho)$, then $\mathrm{Ker}T \leq V$ and $\mathrm{Im}T \leq W$ are G-invariant. Therefore, by irreducibility, either $\mathrm{Ker}T = V$, so that $T \equiv 0$, or $\mathrm{Ker}T = \{0\}$, and necessarily $\mathrm{Im}T = W$, so that T is an isomorphism.

(ii) Let $T \in \mathrm{Hom}_G(\sigma, \sigma)$. Since $\mathbb{C}$ is algebraically closed, T has at least one eigenvalue: there exists $\lambda \in \mathbb{C}$ such that $\mathrm{Ker}(T - \lambda I_V)$ is nontrivial. But $T - \lambda I_V \in \mathrm{Hom}_G(\sigma, \sigma)$ so that by part (i), necessarily $T - \lambda I_V \equiv 0$, in other words $T = \lambda I_V$. □

Corollary 1.2.2 *If σ and ρ are irreducible representations of G then*

$$\dim\mathrm{Hom}_G(\sigma, \rho) = \begin{cases} 1 & \text{if } \sigma \sim \rho \\ 0 & \text{if } \sigma \not\sim \rho. \end{cases}$$

Corollary 1.2.3 *If G is an Abelian group then every irreducible G-representation is one-dimensional.*

Proof Let (σ, V) be an irreducible representation of G. For $g_0 \in G$ we have

$$\sigma(g_0)\sigma(g) = \sigma(g)\sigma(g_0)$$

for all $g \in G$, that is, $\sigma(g_0) \in \mathrm{Hom}_G(\sigma, \sigma)$. It follows from Lemma 1.2.1.(ii), that $\sigma(g_0) = \chi(g_0)I_V$, where $\chi : G \to \mathbb{C}$. This implies that $\dim V = 1$ as σ is irreducible. □

Example 1.2.4 Let $C_n =< a >$ be the cyclic group of order n generated by the element a. A one-dimensional representation of C_n is just a map $\chi : C_n \to \mathcal{T} := \{z \in \mathbb{C} : |z| = 1\}$ such that $\chi(a^k) = \chi(a)^k$ for all $k \in \mathbb{Z}$. In particular,

$\chi(a)^n = 1$, that is, $\chi(a)$ is an nth root of 1. Thus there exists $j \in \{0, 1, \ldots, n-1\}$ such that $\chi = \chi_j$, where

$$\chi_j(a^k) = \exp\left(\frac{2\pi i}{n} kj\right)$$

for all $k \in \mathbb{Z}$. It follows that $\chi_0, \chi_1, \ldots, \chi_{n-1}$ are the irreducible representations of C_n. Indeed they are pairwise inequivalent as two one-dimensional representations are equivalent if and only if they coincide.

In particular, the irreducible representations of $C_2 \equiv \mathfrak{S}_2$ are the trivial representation ι (see Example 1.1.1), and the alternating representation ε (see Example 1.1.4).

1.2.2 Multiplicities and isotypic components

First of all, we recall some basic facts on projections.

Let V be a vector space. A linear transformation $E \in \mathrm{Hom}(V, V)$ is a *projection* if it is *idempotent*: $E^2 = E$. If the range $W = \mathrm{Im}E$ is orthogonal to the null space $\mathrm{Ker}E$, we say that E is an *orthogonal projection* of V onto W. It is easy to see that a projection E is orthogonal if and only if it is self-adjoint, that is, $E = E^*$.

Let now (V, σ) be a representation of G and suppose that

$$V = \bigoplus_{\rho \in J} m_\rho W_\rho$$

is an orthogonal decomposition of V into irreducible sub-representations (see Lemma 1.1.8). This means that J is a set of inequivalent irreducible representations, and that, for $\rho \in J$, $m_\rho W_\rho$ is the orthogonal direct sum of m_ρ copies of W_ρ.

More precisely, there exist $I_{\rho,1}, I_{\rho,2}, \ldots, I_{\rho,m_\rho} \in \mathrm{Hom}_G(W_\rho, V)$, necessarily linearly independent, such that

$$m_\rho W_\rho = \bigoplus_{j=1}^{m_\rho} I_{\rho,j} W_\rho. \tag{1.9}$$

Any vector $v \in V$ may be uniquely written in the form $v = \sum_{\rho \in J} \sum_{j=1}^{m_\rho} v_{\rho,j}$, with $v_{\rho,j} \in I_{\rho,j} W_\rho$. The operator $E_{\rho,j} \in \mathrm{Hom}(V, V)$ defined by setting $E_{\rho,j}(v) = v_{\rho,j}$ is the orthogonal projection from V onto $I_{\rho,j} W_\rho$. In particular, $I_V = \sum_{\rho \in J} \sum_{j=1}^{m_\rho} E_{\rho,j}$.

Observe that if $v = \sum_{\rho \in J} \sum_{j=1}^{m_\rho} v_{\rho,j}$ then $\sigma(g)v = \sum_{\rho \in J} \sum_{j=1}^{m_\rho} \sigma(g) v_{\rho,j}$. As $\sigma(g) v_{\rho,j} \in I_{\rho,j} W_\rho$, by the uniqueness of such a decomposition we have that $E_{\rho,j}\sigma(g)v = \sigma(g)v_{\rho,j} = \sigma(g)E_{\rho,j}v$. Therefore, $E_{\rho,j} \in \mathrm{Hom}_G(\sigma, \sigma)$.

Lemma 1.2.5 *The space* $\mathrm{Hom}_G(W_\rho, V)$ *is spanned by* $I_{\rho,1}, I_{\rho,2}, \ldots, I_{\rho,m_\rho}$. *In particular,* $m_\rho = \dim\mathrm{Hom}_G(W_\rho, V)$.

Proof If $T \in \mathrm{Hom}_G(W_\rho, V)$, then

$$T = I_V T = \sum_{\theta \in J} \sum_{k=1}^{m_\theta} E_{\theta,k} T.$$

Since $\mathrm{Im}E_{\theta,k} = I_{\theta,k} W_\theta$, if follows from Corollary 1.2.2 that, if $\theta \neq \rho$, then $E_{\theta,k}T = 0$. Moreover, again from Corollary 1.2.2, one deduces that $E_{\rho,k}T = \alpha_k I_{\rho,k}$ for some $\alpha_k \in \mathbb{C}$. Thus,

$$T = \sum_{k=1}^{m_\rho} E_{\rho,k} T = \sum_{k=1}^{m_\rho} \alpha_k I_{\rho,k}.$$ □

Corollary 1.2.6 *If* $V = \bigoplus_{\rho \in J'} m'_\rho W_\rho$ *is another decomposition of* V *into irreducible inequivalent sub-representations, then necessarily* $J' = J$, $m'_\rho = m_\rho$ *and* $m'_\rho W_\rho = m_\rho W_\rho$ *for all* $\rho \in J$.

Proof Just note that if $\theta \notin J$ and $T \in \mathrm{Hom}_G(\theta, \sigma)$, then, arguing as in the proof of the previous lemma, one deduces that $T = 0$. □

By applying (1.2) we also get the following.

Corollary 1.2.7 *With the above notation we have* $m_\rho = \dim\mathrm{Hom}_G(W_\rho, V)$.

Exercise 1.2.8 Prove the above corollary directly using the same arguments of Lemma 1.2.5.

Clearly, the decomposition of $m_\rho W_\rho$ into irreducible sub-representations is not unique: it corresponds to the choice of a basis in $\mathrm{Hom}_G(W_\rho, V)$.

Definition 1.2.9 The positive integer m_ρ is called the *multiplicity* of ρ in σ (or of W_ρ in V). Moreover, the invariant subspace $m_\rho W_\rho$ is the ρ-*isotypic component* of ρ in σ.

Suppose again that $V = \bigoplus_{\rho \in J} m_\rho W_\rho$ is the decomposition of V into irreducible sub-representations. Also let $I_{\rho,k}$ and $E_{\rho,k}$ be as before.

For all $\rho \in J$ and $1 \leq j, k \leq m_\rho$ there exist non-trivial operators $T^\rho_{k,j} \in \mathrm{Hom}_G(V, V)$ such that

$$\mathrm{Im}T^\rho_{k,j} = I_{\rho,k} W_\rho$$
$$\mathrm{Ker}T^\rho_{k,j} = V \ominus I_{\rho,j} W_\rho$$

and

$$T^{\rho}_{k,j} T^{\theta}_{s,t} = \delta_{\rho,\theta}\delta_{j,s} T^{\rho}_{k,t}. \tag{1.10}$$

We may construct the operators $T^{\rho}_{k,j}$ in the following way. Denote by $I^{-1}_{\rho,j}$ the inverse of $I_{\rho,j} : W_\rho \to I_{\rho,j} W_\rho \leq V$. Then we may set

$$T^{\rho}_{k,j} v = \begin{cases} I_{\rho,k} I^{-1}_{\rho,j} v & \text{if } v \in I_{\rho,j} W_\rho \\ 0 & \text{if } v \in V \ominus I_{\rho,j} W_\rho. \end{cases}$$

In particular,

$$T^{\rho}_{j,j} \equiv E_{\rho,j}.$$

Moreover, by Corollary 1.2.2 we have that

$$\mathrm{Hom}_G(I_{\rho,j} W_\rho, I_{\rho,k} W_\rho) = \mathbb{C} T^{\rho}_{k,j}.$$

We shall use these operators to study the structure of $\mathrm{Hom}_G(\sigma, \sigma)$. But first we need to recall some basic notions on associative algebras.

1.2.3 Finite dimensional algebras

An *(associative) algebra* over $\mathbb{C}$ is a vector space $\mathcal{A}$ over $\mathbb{C}$ endowed with a product such that: $\mathcal{A}$ is a ring with respect to the sum and the product and the following associative laws hold for the product and multiplication by a scalar:

$$\alpha(AB) = (\alpha A)B = A(\alpha B)$$

for all $\alpha \in \mathbb{C}$ and $A, B \in \mathcal{A}$.

The basic example is $\mathrm{End}(V) := \mathrm{Hom}(V, V)$ with the usual operations of sum and product of operators, and of multiplication by scalars.

A *subalgebra* of an algebra $\mathcal{A}$ is a subspace $\mathcal{B} \leq \mathcal{A}$ which is closed under multiplication.

An *involution* is a bijective map $A \mapsto A^*$ such that

- $(A^*)^* = A$
- $(\alpha A + \beta B)^* = \overline{\alpha} A^* + \overline{\beta} B^*$
- $(AB)^* = B^* A^*$

for all $\alpha, \beta \in \mathbb{C}$ and $A, B \in \mathcal{A}$.

For instance, if $\mathcal{A} = \mathrm{Hom}(V, V)$, then $T \to T^*$ (where T^* is the adjoint of T) is an involution on $\mathcal{A}$.

An algebra $\mathcal{A}$ is *commutative* (or *Abelian*) if it is commutative as a ring, namely if $AB = BA$ for all $A, B \in \mathcal{A}$.

An algebra $\mathcal{A}$ is *unital* if it has a *unit*, that is there exists an element $I \in \mathcal{A}$ such that $AI = IA = A$ for all $A \in \mathcal{A}$. Note that a unit is necessarily unique and self-adjoint. Indeed, if I' is another unit in $\mathcal{A}$, then $I = II' = I'$. Moreover,

$$I^*A = ((I^*A)^*)^* = (A^*(I^*)^*)^* = (A^*I)^* = (A^*)^* = A$$

and therefore $I = I^*$.

An algebra with involution is called an *involutive algebra* or $*$-algebra.

Given an algebra $\mathcal{A}$, its *center* $Z(\mathcal{A})$ is the commutative subalgebra

$$Z(\mathcal{A}) = \{B \in \mathcal{A} : AB = BA \text{ for all } A \in \mathcal{A}\}. \tag{1.11}$$

Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be two involutive algebras and let $\phi\colon \mathcal{A}_1 \to \mathcal{A}_2$ be a map. One says that ϕ is a $*$-*homomorphism* provided that

- $\phi(\alpha A + \beta B) = \alpha\phi(A) + \beta\phi(B)$ (linearity)
- $\phi(AB) = \phi(A)\phi(B)$ (multiplicative property)
- $\phi(A^*) = [\phi(A)]^*$ (preservation of involution)

for all $\alpha, \beta \in \mathbb{C}$ and $A, B \in \mathcal{A}_1$. If in addition ϕ is a bijection, then it is called a $*$-*isomorphism* between $\mathcal{A}_1$ and $\mathcal{A}_2$. One then says that $\mathcal{A}_1$ and $\mathcal{A}_2$ are $*$-isomorphic.

On the other hand, ϕ is a $*$-*anti-homomorphism* if the multiplicativity property is replaced by

$$\phi(AB) = \phi(B)\phi(A) \quad \text{(anti-multiplicative property).}$$

for all $A, B \in \mathcal{A}_1$. Finally, ϕ is a $*$-*anti-isomorphism* if it is a bijective $*$-anti-homomorphism. One then says that $\mathcal{A}_1$ and $\mathcal{A}_2$ are $*$-anti-isomorphic.

Let $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_k$, $k \geq 2$, be associative algebras. Their *direct sum*, denoted by $\mathcal{A}_1 \oplus \mathcal{A}_2 \oplus \cdots \oplus \mathcal{A}_k$, is the algebra whose vector space structure is the direct sum of the corresponding vector spaces endowed with the product

$$(A_1, A_2, \dots, A_k)(B_1, B_2, \dots, B_k) = (A_1B_1, A_2B_2, \dots, A_kB_k)$$

for all $A_i, B_i \in \mathcal{A}_i$.

Let $\mathcal{A}$ be an algebra and let $\mathcal{B} \subset \mathcal{A}$ be a subset. The *algebra generated* by $\mathcal{B}$, denoted by $< \mathcal{B} >$, is the smallest subalgebra of $\mathcal{A}$ containing $\mathcal{B}$. Equivalently, $< \mathcal{B} >$ is the set of all linear combinations of products of elements of $\mathcal{B}$.

An algebra $\mathcal{A}$ is *finite dimensional* if the corresponding vector space is finite dimensional. If this is the case, denoting by $e_1, e_2, \dots, e_d$ a linear basis of $\mathcal{A}$, the d^3 (structure) coefficients $c_{i,j,k} \in \mathbb{C}$ such that

$$e_ie_j = \sum_{k=1}^{d} c_{i,j,k}e_k$$

uniquely determine the product in $\mathcal{A}$.

Example 1.2.10 Let V be a unitary space. Then the space $\mathrm{End}(V) = \mathrm{Hom}(V, V) = \{T : V \to V, T \text{ linear}\}$ is a unital $*$-algebra with respect to the composition of operators. Moreover, if (σ, V) is a unitary representation of a finite group G, then the subalgebra $\mathrm{Hom}_G(V, V)$ of $\mathrm{Hom}(V, V)$ is also a unital $*$-algebra.

Example 1.2.11 Let $m \geq 1$ and denote by $M_{m,m}(\mathbb{C}) = \{\left(a_{i,j}\right)_{1,j=1}^{m}\}$ the algebra of $m \times m$ complex matrices (product is just the ordinary rows by columns multiplication). A linear basis is given by the *matrix units* namely, by the matrices $e_{i,j}$ where 1 is the only nonzero entry in the position (i, j), $1 \leq i, j \leq m$. The product is given by

$$e_{i,j}e_{k,\ell} = \delta_{j,k}e_{i,\ell}$$

and the involution is given by

$$e_{i,j}^* = e_{j,i}.$$

This algebra is unital with unit $I = e_{1,1} + e_{2,2} + \cdots + e_{m,m}$. Note that setting $V = \mathbb{C}^m$ we clearly have $M_{m,m}(\mathbb{C}) \cong \mathrm{Hom}(V, V)$ (cf. the previous example).

Example 1.2.12 A finite dimensional commutative algebra of dimension m is isomorphic to $\mathbb{C}^m = \bigoplus_{k=1}^{m} \mathbb{C}$. The center of the matrix algebra $M_{m,m}(\mathbb{C})$ is isomorphic to $\mathbb{C}$: it consists of all scalar matrices λI with $\lambda \in \mathbb{C}$. A maximal Abelian subalgebra $\mathcal{A}$ of $M_{m,m}(\mathbb{C})$ is isomorphic to the subalgebra consisting of all diagonal matrices, so that $\mathcal{A} \cong \mathbb{C}^m$.

1.2.4 The structure of the commutant

Definition 1.2.13 Let (σ, V) be a representation of a finite group G. The algebra $\mathrm{Hom}_G(V, V)$ is called the *commutant* of (σ, V).

In the following theorem we shall use the operators $T_{k,j}^{\rho}$ constructed at the end of Section 1.2.2.

Theorem 1.2.14 *The set*

$$\{T_{k,j}^{\rho} : \rho \in J, 1 \leq k, j \leq m_\rho\} \tag{1.12}$$

is a basis for $\mathrm{Hom}_G(V, V)$. *Moreover, the map*

$$\begin{array}{rcl} \mathrm{Hom}_G(V, V) & \to & \bigoplus_{\rho \in J} M_{m_\rho, m_\rho}(\mathbb{C}) \\ T & \mapsto & \bigoplus_{\rho \in J} \left(\alpha_{k,j}^{\rho}\right)_{k,j=1}^{m_\rho} \end{array}$$

where the $\alpha^{\rho}_{k,j}$'s are the coefficients of T with respect to the basis (1.12), that is,

$$T = \sum_{\rho \in J} \sum_{k,j=1}^{m_\rho} \alpha^{\rho}_{k,j} T^{\rho}_{k,j},$$

is an isomorphism of algebras.

Proof Let $T \in \mathrm{Hom}_G(V, V)$. We have

$$T = I_V T I_V = \left(\sum_{\rho \in J} \sum_{k=1}^{m_\rho} E_{\rho,k} \right) T \left(\sum_{\theta \in J} \sum_{j=1}^{m_\theta} E_{\theta,j} \right)$$
$$= \sum_{\rho,\theta \in J} \sum_{k=1}^{m_\rho} \sum_{j=1}^{m_\theta} E_{\rho,k} T E_{\theta,j}.$$

Observe that $\mathrm{Im} E_{\rho,k} T E_{\theta,j} \leq I_{\rho,k} W_\rho$, and the restriction to $I_{\theta,j} W_\theta$ of $E_{\rho,k} T E_{\theta,j}$ is an intertwining operator from $I_{\theta,j} W_\theta$ to $I_{\rho,k} W_\rho$. From Corollary 1.2.2, it follows that $E_{\rho,k} T E_{\theta,j} = 0$ if $\theta \not\sim \rho$, while, if $\rho \sim \theta$, $E_{\rho,k} T E_{\theta,j}$ is a multiple of $T^{\rho}_{k,j}$, that is, there exist $\alpha^{\rho}_{k,j} \in \mathbb{C}$ such that

$$E_{\rho,k} T E_{\theta,j} = \alpha^{\rho}_{k,j} T^{\rho}_{k,j}.$$

This proves that the $T^{\rho}_{k,j}$'s generate $\mathrm{Hom}_G(V, V)$. To prove independence, suppose that we can express the 0-operator as

$$0 = \sum_{\rho \in J} \sum_{k,j=1}^{m_\rho} \alpha^{\rho}_{k,j} T^{\rho}_{k,j}.$$

If $0 \neq v \in I_{\rho,j} W_\rho$ we obtain that $0 = \sum_{k=1}^{m_\rho} \alpha^{\rho}_{k,j} T^{\rho}_{k,j} v$ and this in turn implies that $\alpha^{\rho}_{k,j} = 0$ for all $k = 1, 2, \ldots, m_\rho$, as $T^{\rho}_{k',j} v$ and $T^{\rho}_{k,j} v$ belong to independent subspaces in V if $k \neq k'$. The isomorphism of the algebras follows from (1.10). □

Corollary 1.2.15 $\dim \mathrm{Hom}_G(V, V) = \sum_{\rho \in J} m_\rho^2$.

Definition 1.2.16 The representation (σ, V) is *multiplicity-free* if $m_\rho = 1$ for all $\rho \in J$.

Corollary 1.2.17 *The representation (σ, V) is multiplicity-free if and only if $\mathrm{Hom}_G(V, V)$ is commutative.*

Observe that

$$E_\rho = \sum_{j=1}^{m_\rho} E_{\rho,j} \equiv \sum_{j=1}^{m_\rho} T^\rho_{j,j}$$

is the projection from V onto the ρ-isotypic component $m_\rho W_\rho$. It is called the *minimal central projection* associated with ρ.

Corollary 1.2.18 *The center $\mathcal{Z} = Z(\mathrm{Hom}_G(V, V))$ is isomorphic to $\mathbb{C}^J$. Moreover, the minimal central projections $\{E_\rho : \rho \in J\}$ constitute a basis for $\mathcal{Z}$.*

Exercise 1.2.19 Let (σ, V) and (η, U) be two G-representations. Suppose that $V = \bigoplus_{\rho\in J} m_\rho W_\rho$ and $U = \bigoplus_{\rho\in K} n_\rho W_\rho$, $J, K \subseteq \widehat{G}$, are the decompositions of V and U into irreducible sub-representations. Show that

$$\mathrm{Hom}_G(U, V) \cong \bigoplus_{\rho\in K\cap J} M_{n_\rho,m_\rho}(\mathbb{C})$$

as vector spaces.

1.2.5 Another description of $\mathrm{Hom}_G(W, V)$

In this section, we present an alternative and useful description of $\mathrm{Hom}_G(W, V)$.

First of all, we recall the canonical isomorphism

$$\begin{array}{ccc} W' \otimes V & \cong & \mathrm{Hom}(W, V) \\ \varphi \otimes v & \mapsto & T_{\varphi,v} \end{array} \tag{1.13}$$

where $T_{\varphi,v} \in \mathrm{Hom}(W, V)$ is defined by setting

$$T_{\varphi,v} w = \varphi(w)v \tag{1.14}$$

for all $w \in W$.

Suppose that (σ, V) and (ρ, W) are two G-representations. We define a G-representation η on $\mathrm{Hom}(W, V)$ by setting

$$\eta(g)T = \rho(g)T\sigma(g^{-1}) \tag{1.15}$$

for all $g \in G$ and $T \in \mathrm{Hom}(W, V)$.

Lemma 1.2.20 *The isomorphism (1.13) is also an isomorphism between $\sigma' \otimes \rho$ and η.*

Proof For $g \in G$, $\varphi \in W'$, $v \in V$ and $w \in W$ we have

$$[\sigma'(g) \otimes \rho(g)](\varphi \otimes v) = [\sigma'(g)\varphi] \otimes [\rho(g)v] \mapsto T_{\sigma'(g)\varphi,\rho(g)v} \tag{1.16}$$

and

$$\begin{aligned} T_{\sigma'(g)\varphi,\rho(g)v} w &= [\sigma'(g)\varphi](w) \cdot \rho(g)v \\ &= \varphi(\sigma(g^{-1})w) \cdot \rho(g)v \\ &= \rho(g)[\varphi(\sigma(g^{-1})w) \cdot v] \\ &= \rho(g) T_{\varphi,v} \sigma(g^{-1}) w, \end{aligned}$$

that is,

$$T_{\sigma'(g)\varphi,\rho(g)v} = \rho(g) T_{\varphi,v} \sigma(g^{-1}) = \eta(g) T_{\varphi,v}. \tag{1.17}$$

Then (1.16) and (1.17) ensure that the map $\varphi \otimes v \mapsto T_{\varphi,v}$ is an isomorphism between $\sigma' \otimes \rho$ and η. □

Definition 1.2.21 Let (ρ, V) be a G-representation and $K \leq G$ be a subgroup of G. We denote by

$$V^K = \{v \in V : \rho(k)v = v, \text{ for all } k \in K\}$$

the subspace of all K-*invariant vectors* in V.

In other words, V^K is the ι_K-isotypic component in $\mathrm{Res}^G_K V$, where ι_K is the trivial representation of K.

As a consequence of Lemma 1.2.20 we have the following.

Corollary 1.2.22 $\mathrm{Hom}_G(W, V) = [\mathrm{Hom}(W, V)]^G \cong \mathrm{Hom}_G(\iota_G, \sigma' \otimes \rho)$.

Proof $\mathrm{Hom}_G(W, V)$ and $[\mathrm{Hom}(W, V)]^G$ coincide as subspaces of $\mathrm{Hom}(W, V)$. Indeed, $T \in \mathrm{Hom}_G(W, V)$ if and only if $\eta(g)T \equiv \rho(g)T\sigma(g^{-1}) = T$. Since $\mathrm{Hom}[(W, V)]^G$ is the ι_G-isotypic component of η, the isomorphism $[\mathrm{Hom}(W, V)]^G \cong \mathrm{Hom}_G(\iota_G, \sigma' \otimes \rho)$ follows from Lemma 1.2.20 and Lemma 1.2.5. □

1.3 Characters and the projection formula

1.3.1 The trace

Let V be a finite dimensional vector space over $\mathbb{C}$. We recall that the *trace* is a linear operator $tr : \mathrm{Hom}(V, V) \to \mathbb{C}$ such that

(i) $tr(TS) = tr(ST)$, for all $S, T \in \mathrm{Hom}(V, V)$;
(ii) $tr(I_V) = \dim V$.

Moreover, tr is uniquely determined by the above properties. In particular, if V is endowed with a Hermitian scalar product $\langle \cdot, \cdot \rangle$ and $v_1, v_2, \ldots, v_n$ is an orthonormal basis, then

$$tr(T) = \sum_{j=1}^{n} \langle T v_j, v_j \rangle$$

for all $T \in \mathrm{Hom}(V, V)$.

Let W be another vector space. It is easy to check that

$$tr(T \otimes S) = tr(T)tr(S) \tag{1.18}$$

for all $T \in \mathrm{Hom}(W, W)$ and $S \in \mathrm{Hom}(V, V)$. Indeed, if $w_1, w_2, \ldots, w_m$ and $v_1, v_2, \ldots, v_n$ are orthonormal bases for W and V, respectively, then $w_i \otimes v_j$, $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$ is an orthonormal basis for $W \otimes V$ so that

$$\begin{aligned} tr(T \otimes S) &= \sum_{i=1}^{m} \sum_{j=1}^{n} \langle (T \otimes S)(w_i \otimes v_j), w_i \otimes v_j \rangle_{W \otimes V} \\ &= \sum_{i=1}^{m} \sum_{j=1}^{n} \langle T w_i, w_i \rangle_W \langle S v_j, v_j \rangle_V \\ &= tr(T)tr(S). \end{aligned}$$

1.3.2 Central functions and characters

Definition 1.3.1 A function $f \in L(G)$ is *central* (or is a *class function*) if $f(g_1 g_2) = f(g_2 g_1)$ for all $g_1, g_2 \in G$.

Lemma 1.3.2 *A function $f \in L(G)$ is central if and only if it is conjugacy invariant, that is,*

$$f(g^{-1} s g) = f(s)$$

for all $g, s \in G$. In other words, a function is central if and only if it is constant on each conjugacy class.

Proof If f is central, then, for all $g, s \in G$,

$$f(g^{-1} s g) = f(s g g^{-1}) = f(s),$$

while, if f is conjugacy invariant, then, for all $g_1, g_2 \in G$,

$$f(g_1 g_2) = f(g_1 g_2 g_1 g_1^{-1}) = f(g_2 g_1). \qquad \square$$

Definition 1.3.3 Let G be a finite group and (σ, V) a G-representation. The *character* of (σ, V) is the function $\chi^\sigma : G \to \mathbb{C}$ given by

$$\chi^\sigma(g) = tr(\sigma(g))$$

for all $g \in G$.

The elementary properties of the characters are summarized in the following proposition.

Proposition 1.3.4 *Let (σ, V) and (ρ, W) be two G-representations and let χ^σ and χ^ρ be their characters.*

(i) $\chi^\sigma(1_G) = \dim V$.
(ii) $\chi^\sigma(g^{-1}) = \overline{\chi^\sigma(g)}$, *for all* $g \in G$.
(iii) $\chi^\sigma \in L(G)$ *is a central function.*
(iv) *If* $\sigma = \sigma_1 \oplus \sigma_2$ *then* $\chi^\sigma = \chi^{\sigma_1} + \chi^{\sigma_2}$.
(v) *If* $G = G_1 \times G_2$ *and* $\sigma = \sigma_1 \boxtimes \sigma_2$, *with* σ_i *a* G_i*-representation,* $i = 1, 2$, *then* $\chi^{\sigma_1 \boxtimes \sigma_2} = \chi^{\sigma_1}\chi^{\sigma_2}$.
(vi) *If* σ' *is the adjoint of* σ, *then* $\chi^{\sigma'}(g) = \overline{\chi^\sigma(g)}$ *for all* $g \in G$.
(vii) $\chi^{\sigma \otimes \rho} = \chi^\sigma \chi^\rho$.
(viii) *With* η *as in (1.15), we have* $\chi^\eta = \overline{\chi^\rho}\chi^\sigma$.

Proof (i) and (iv) are trivial.

(ii) Let $\{v_1, v_2, \dots, v_d\}$ be an orthonormal basis in V; then

$$\chi^\sigma(g^{-1}) = \sum_{i=1}^{d} \langle \sigma(g^{-1})v_i, v_i\rangle = \sum_{i=1}^{d} \langle v_i, \sigma(g)v_i\rangle = \sum_{i=1}^{d} \overline{\langle \sigma(g)v_i, v_i\rangle} = \overline{\chi^\sigma(g)}.$$

(iii) is a consequence of the defining property of the trace and (v) follows from (1.18).

(vi) Let $\{v_1, v_2, \dots, v_d\}$ be an orthonormal basis in V and let $\{\theta_{v_1}, \theta_{v_2}, \dots, \theta_{v_d}\}$ be the corresponding dual basis. Then, from (1.5) and (1.7) it follows that

$$\begin{aligned} tr[\sigma'(g)] &= \sum_{i=1}^{d} \langle \sigma'(g)\theta_{v_i}, \theta_{v_i}\rangle_{V'} \\ &= \sum_{i=1}^{d} \langle \theta_{\sigma(g)v_i}, \theta_{v_i}\rangle_{V'} \\ &= \sum_{i=1}^{d} \langle v_i, \sigma(g)v_i\rangle_V \\ &= \sum_{i=1}^{d} \overline{\langle \sigma(g)v_i, v_i\rangle_V} \\ &= \overline{\chi^\sigma(g)}. \end{aligned}$$

(vii) is a consequence of (v).

(viii) follows from Lemma 1.2.20, (vi) and (vii):

$$\chi^\eta = \chi^{\rho' \otimes \sigma} = \chi^{\rho'}\chi^\sigma = \overline{\chi^\rho}\chi^\sigma.$$

□

1.3.3 Central projection formulas

Following the monograph of Fulton and Harris [43], we deduce several important properties of the characters from a simple projection formula. We recall that if $E : V \to V$ is a projection, then

$$\dim \mathrm{Im} E = tr(E). \tag{1.19}$$

Lemma 1.3.5 (Basic projection formula) *Let (σ, V) be a G-representation and K a subgroup of G. Then the operator*

$$E = E_\sigma^K = \frac{1}{|K|}\sum_{k\in K}\sigma(k)$$

is the orthogonal projection of V onto V^K.

Proof For $v \in V$ we have $Ev \in V^K$. Indeed,

$$\sigma(k_1)Ev = \frac{1}{|K|}\sum_{k\in K}\sigma(k_1 k)v = Ev,$$

for all $k_1 \in K$. Moreover if $v \in V^K$ then

$$Ev = \frac{1}{|K|}\sum_{k\in K}\sigma(k)v = \frac{1}{|K|}\sum_{k\in K}v = v.$$

Therefore E is a projection from V onto V^K. It is orthogonal because σ is unitary:

$$E^* = \frac{1}{|K|}\sum_{k\in K}\sigma(k)^* = \frac{1}{|K|}\sum_{k\in K}\sigma(k^{-1}) = E.$$ □

Corollary 1.3.6 $\dim V^K = \frac{1}{|K|}\langle \chi^{\mathrm{Res}_K^G \sigma}, \iota_K\rangle_{L(K)}$.

Proof From (1.19) and the previous lemma we get

$$\dim V^K = tr\left[\frac{1}{|K|}\sum_{k\in K}\sigma(k)\right] = \frac{1}{|K|}\sum_{k\in K}\chi^\sigma(k) = \frac{1}{|K|}\langle \chi^{\mathrm{Res}_K^G \sigma}, \iota_K\rangle_{L(K)}.$$ □

Corollary 1.3.7 (Orthogonality relations for the characters of irreducible representations) *Let (σ, V) and (ρ, W) be two irreducible representations of G. Then*

$$\frac{1}{|G|}\langle \chi^\sigma, \chi^\rho\rangle_{L(G)} = \begin{cases} 1 & \text{if } \sigma \sim \rho \\ 0 & \text{if } \sigma \not\sim \rho. \end{cases}$$

Proof We have

$$\begin{aligned}
\frac{1}{|G|}\langle\chi^\sigma,\chi^\rho\rangle_{L(G)} &= \frac{1}{|G|}\langle\overline{\chi^\rho}\chi^\sigma,\iota_G\rangle_{L(G)}\\
\text{(by Proposition 1.3.4.(viii))}\quad &= \frac{1}{|G|}\langle\chi^\eta,\iota_G\rangle_{L(G)}\\
\text{(by Corollary 1.3.6)}\quad &= \dim\mathrm{Hom}(\rho,\sigma)^G\\
\text{(by Corollary 1.2.22)}\quad &= \dim\mathrm{Hom}_G(\rho,\sigma)\\
\text{(by Schur's lemma)}\quad &= \begin{cases}1 & \text{if } \sigma\sim\rho\\ 0 & \text{if } \sigma\not\sim\rho.\end{cases}
\end{aligned}$$

□

Corollary 1.3.8 *Let (σ, V) be a representation of G and let $\sigma = \oplus_{\rho\in J} m_\rho\rho$ be its decomposition into irreducible sub-representations (cf. Corollary 1.2.6). Then*

(i) $m_\rho = \frac{1}{|G|}\langle\chi^\rho,\chi^\sigma\rangle_{L(G)}$, *for all $\rho \in J$.*
(ii) $\frac{1}{|G|}\langle\chi^\sigma,\chi^\sigma\rangle_{L(G)} = \sum_{\rho\in J} m_\rho^2$.
(iii) $\frac{1}{|G|}\langle\chi^\sigma,\chi^\sigma\rangle_{L(G)} = 1$ *if and only if σ is irreducible.*

Note that this gives another proof of the uniqueness of the multiplicity m_ρ. Moreover, it also shows that σ is uniquely determined (up to equivalence) by its character.

Lemma 1.3.9 (Fixed points character formula) *Suppose that G acts on a finite set X and denote by $(\lambda, L(X))$ the corresponding permutation representation. Then*

$$\chi^\lambda(g) = |\{x\in X : gx = x\}|.$$

Proof Take the Dirac functions $\{\delta_x : x \in X\}$ as an orthonormal basis for $L(X)$. Clearly,

$$[\lambda(g)\delta_x](y) = \delta_x(g^{-1}y) = \begin{cases}1 & \text{if } g^{-1}y = x\\ 1 & \text{if } g^{-1}y \neq x\end{cases} = \delta_{gx}(y),$$

that is $\lambda(g)\delta_x = \delta_{gx}$ for all $g \in G$ and $x \in X$. Thus,

$$\begin{aligned}
\chi^\lambda(g) &= \sum_{x\in X}\langle\lambda(g)\delta_x,\delta_x\rangle_{L(X)}\\
&= \sum_{x\in X}\langle\delta_{gx},\delta_x\rangle_{L(X)}\\
&= |\{x\in X : gx = x\}|.
\end{aligned}$$

□

Again, a very simple formula has deep consequences.

Corollary 1.3.10 *The multiplicity of an irreducible representation* (ρ, V_ρ) *in the left regular representation* $(\lambda, L(G))$ *is equal to* $d_\rho = \dim V_\rho$*, that is,*

$$L(G) = \bigoplus_{\rho \in \widehat{G}} d_\rho V_\rho.$$

In particular, $|G| = \sum_{\rho \in \widehat{G}} d_\rho^2$.

Proof The character of λ is given by:

$$\chi^\lambda(g) = |\{h \in G : gh = h\}| = \begin{cases} |G| & \text{if } g = 1_G \\ 0 & \text{otherwise} \end{cases} = |G|\delta_{1_G}(g)$$

and therefore

$$\frac{1}{|G|}\langle \chi^\rho, \chi^\lambda \rangle_{L(G)} = \chi^\rho(1_G) = \dim V_\rho. \qquad \square$$

We shall also use the following version of the fixed points character formula, specialized to the case of a transitive action.

Corollary 1.3.11 *Suppose that* G *acts transitively on a finite set* X *and denote by* K *the stabilizer of a point* $x_0 \in X$ *(so that* $X \cong G/K$ *as* G*-spaces). If* $\mathcal{C}$ *is a conjugacy class of* G *and* $g_0 \in \mathcal{C}$ *then the character of the permutation representation* λ *of* G *on* X *is given by*

$$\chi^\lambda(g_0) = \frac{|X|}{|\mathcal{C}|}|\mathcal{C} \cap K|.$$

Proof First note that for all $x \in X$ one has

$$|\{g \in \mathcal{C} : gx = x\}| = |\{h \in \mathcal{C} : hx_0 = x_0\}| = |\mathcal{C} \cap K|.$$

Indeed, if $s \in G$ is such that $sx_0 = x$, then the map $g \mapsto h = s^{-1}gs$ yields a bijection implementing the first equality above. Also, applying the fixed points character formula we get, for $g_0 \in \mathcal{C}$,

$$\begin{aligned} \chi^\lambda(g_0) &= |\{x \in X : g_0 x = x\}| \\ &= \frac{|\{(x, g) \in X \times \mathcal{C} : gx = x\}|}{|\mathcal{C}|} \\ &= \sum_{x \in X} \frac{|\{g \in \mathcal{C} : gx = x\}|}{|\mathcal{C}|} \\ &= \frac{|X|}{|\mathcal{C}|}|\mathcal{C} \cap K|. \end{aligned} \qquad \square$$

Note that $\mathcal{C} \cap K$ is a union of conjugacy classes of K.

Definition 1.3.12 (Fourier transform) Let (σ, V) be a representation of G and $f \in L(G)$. The operator

$$\sigma(f) = \sum_{g \in G} f(g)\sigma(g) \in \mathrm{Hom}(V, V) \tag{1.20}$$

is called the *Fourier transform* of f at σ.

Lemma 1.3.13 (Fourier transform of central functions) *Suppose that $f \in L(G)$ is central and let (ρ, W) be an irreducible representation. Then*

$$\rho(f) = \frac{1}{d_\rho}\langle \chi^\rho, \overline{f}\rangle_{L(G)} I_W.$$

Proof We first show that $\rho(f) \in \mathrm{Hom}_G(W, W)$. For $g_0 \in G$ we have

$$\begin{aligned}
\rho(f)\rho(g_0) &= \sum_{g \in G} f(g)\rho(gg_0)\\
&= \sum_{g \in G} f(gg_0^{-1})\rho(g)\\
(f \text{ is central}) \quad &= \sum_{g \in G} f(g_0^{-1}g)\rho(g)\\
&= \sum_{g \in G} f(g)\rho(g_0g)\\
&= \rho(g_0)\rho(f).
\end{aligned}$$

Therefore, by Schur's lemma there exists $c \in \mathbb{C}$ such that

$$\rho(f) = cI_W.$$

As

$$tr[\rho(f)] = tr[\sum_{g \in G} f(g)\rho(g)] = \sum_{g \in G} f(g)\chi^\rho(g) = \langle \chi^\rho, \overline{f}\rangle_{L(G)}$$

and

$$tr[cI_W] = cd_\rho$$

we immediately deduce that $c = \frac{1}{d_\rho}\langle \chi^\rho, \overline{f}\rangle_{L(G)}$ and the statement follows. □

From the orthogonality relations for the characters (see Corollary 1.3.7) we deduce the following.

Corollary 1.3.14 *Let (σ, V) and (ρ, W) be two irreducible representations. Then*

$$\sigma(\overline{\chi^\rho}) = \begin{cases} \frac{|G|}{d_\sigma} I_V & \text{if } \sigma \sim \rho \\ 0 & \text{otherwise.} \end{cases}$$

We now give an explicit expression formula for the minimal central projections $E_\rho \in \mathrm{Hom}_G(V, V)$ (see Corollary 1.2.18).

Corollary 1.3.15 (Projection onto an isotypic component) *Let (σ, V) and (ρ, W) be two representations, with ρ irreducible. Then*

$$E_\rho := \frac{d_\rho}{|G|}\sigma(\overline{\chi^\rho})$$

is the orthogonal projection onto the ρ-isotypic component of V.

Proof If $\sigma = \bigoplus_{\theta \in J} m_\theta \theta$ is the decomposition of σ into irreducible subrepresentations, then

$$\frac{d_\rho}{|G|}\sigma(\overline{\chi^\rho}) = \frac{d_\rho}{|G|}\sum_{\theta \in J} m_\theta \theta(\overline{\chi^\rho}) = \frac{d_\rho}{|G|} m_\rho \rho(\overline{\chi^\rho}) = m_\rho I_{W_\rho} \equiv E_\rho. \qquad \square$$

We now determine the number of irreducible pairwise nonequivalent representations of G, that is, the cardinality of $\widehat{G}$ (see Definition 1.1.7). For this purpose, we use the following elementary formula:

$$f = \lambda(f)\delta_{1_G}. \tag{1.21}$$

Indeed

$$f = \sum_{g \in G} f(g)\delta_g = \sum_{g \in G} f(g)\lambda(g)\delta_{1_G} = \lambda(f)\delta_{1_G}.$$

Corollary 1.3.16 *The characters χ^ρ, $\rho \in \widehat{G}$ form an orthogonal basis for the space of central functions of G. In particular, $|\widehat{G}|$ equals the number of conjugacy classes of G.*

Proof From the orthogonality relations for the characters (Corollary 1.3.7) we have that $\{\chi^\rho : \rho \in \widehat{G}\}$ is an orthogonal system for the space of central functions in $L(G)$. We now show that it is complete. Suppose that f is central and that $\langle f, \chi^\rho \rangle_{L(G)} = 0$ for all $\rho \in \widehat{G}$. Then, from Corollary 1.3.10 and Lemma 1.3.13 we have

$$\lambda(f) = \bigoplus_{\rho \in \widehat{G}} d_\rho \rho(f) = 0$$

(observe that $\overline{\chi^\rho} = \chi^{\rho'}$ by Proposition 1.3.4.(vi)). From (1.21) we deduce that $f \equiv 0$.

Denoting by $\mathcal{C}$ the set of all conjugacy classes of G, we have that the characteristic functions $\mathbf{1}_C$, $C \in \mathcal{C}$, form another basis for the space of central functions. Therefore the dimension of this space is equal to $|\mathcal{C}|$ and thus $|\widehat{G}| = |\mathcal{C}|$. $\square$

Theorem 1.3.17 *Let G_1 and G_2 be two finite groups. Then*

$$\widehat{G_1} \times \widehat{G_2} \ni (\rho_1, \rho_2) \mapsto \rho_1 \boxtimes \rho_2 \in \widehat{G_1 \times G_2}$$

is a bijection.

Proof From (v) in Proposition 1.3.4 we know that $\chi^{\rho_1 \boxtimes \rho_2} = \chi^{\rho_1}\chi^{\rho_2}$. Therefore, if $\rho_1, \sigma_1 \in \widehat{G_1}$ and $\rho_2, \sigma_2 \in \widehat{G_2}$, we have, by Corollary 1.3.7,

$$\begin{aligned}\frac{1}{|G_1 \times G_2|}\langle \chi^{\rho_1 \boxtimes \rho_2}, \chi^{\sigma_1 \boxtimes \sigma_2}\rangle_{L(G_1 \times G_2)} &= \frac{1}{|G_1|}\langle \chi^{\rho_1}, \chi^{\sigma_1}\rangle_{L(G_1)} \frac{1}{|G_2|}\langle \chi^{\rho_2}, \chi^{\sigma_2}\rangle_{L(G_2)} \\ &= \delta_{\rho_1,\sigma_1}\delta_{\rho_2,\sigma_2}.\end{aligned}$$

Another application of Corollary 1.3.7 and (iii) in Corollary 1.3.8 ensure that $\{\rho_1 \boxtimes \rho_2 : \rho_1 \in \widehat{G_1}, \rho_2 \in \widehat{G_2}\}$ is a system of pairwise inequivalent irreducible representations of $G_1 \times G_2$.

Finally, $|\widehat{G_1 \times G_2}|$ equals the number of conjugacy classes of $G_1 \times G_2$ which clearly equals the product of the number of conjugacy classes of G_1 and the number of conjugacy classes of G_2. By Corollary 1.3.16, the latter in turn equals the product $|\widehat{G_1}| \cdot |\widehat{G_2}|$, and this ends the proof. □

1.4 Permutation representations

1.4.1 Wielandt's lemma

Let G be a finite group. Suppose that G acts on a finite set X and denote by λ the permutation representation of G on $L(X)$ (see Example 1.1.2).

We may identify $\mathrm{Hom}(L(X), L(X))$ with $L(X \times X)$ via the isomorphism

$$\begin{array}{ccc} L(X \times X) & \to & \mathrm{Hom}(L(X), L(X)) \\ F & \mapsto & T_F \end{array} \tag{1.22}$$

where

$$(T_F f)(x) = \sum_{y \in X} F(x, y) f(y)$$

for all $f \in L(X)$ and $x \in X$. In other words, we may think of $L(X \times X)$ as the space of $X \times X$-matrices with coefficients in $\mathbb{C}$. Indeed, if $F_1, F_2 \in L(X \times X)$, then

$$T_{F_1} T_{F_2} = T_{F_1 F_2}$$

where the product $F_1F_2 \in L(X \times X)$ is defined by setting

$$(F_1F_2)(x, y) = \sum_{z\in X} F_1(x, z)F_2(z, y)$$

for all $x, y \in X$. In particular, this shows that (1.22) is in fact an isomorphism of algebras.

The group G acts on $X \times X$ diagonally, that is,

$$g(x, y) = (gx, gy),$$

for all $g \in G$ and $x, y \in X$.

In particular, $L(X \times X)^G$ is the set of all $F \in L(X \times X)$ which are G-*invariant*: $F(gx, gy) = F(x, y)$ for all $g \in G$ and $x, y \in X$. Thus, $F \in L(X \times X)$ is G-invariant if and only if it is constant on the G-orbits on $X \times X$.

In the present situation, Corollary 1.2.22 has the following more particular form.

Proposition 1.4.1 $\mathrm{Hom}_G(L(X), L(X)) \cong L(X \times X)^G$.

Proof Taking into account (1.22), we have

$$[\lambda(g)T_F f](x) = \sum_{y\in X} F(g^{-1}x, y)f(y) = \sum_{y\in X} F(g^{-1}x, g^{-1}y)f(g^{-1}y)$$

and

$$[T_F\lambda(g)f](x) = \sum_{y\in X} F(x, y)f(g^{-1}y)$$

for all $g \in G$, $F \in L(X \times X)$, $f \in L(X)$ and $x \in X$. Therefore $T_F\lambda(g) = \lambda(g)T_F$ if and only if F is G-invariant. □

Exercise 1.4.2 Show that the permutation representation of G on $L(X \times X)$ is equivalent to the tensor product of the permutation representation $L(X)$ with itself.

Exercise 1.4.3 Suppose that G acts on another finite set Y. Show that $\mathrm{Hom}_G(L(X), L(Y)) \cong L(X \times Y)^G$.

In the same notation as Exercise 1.4.3, an *incidence relation* between X and Y is just a subset $\mathcal{I} \subseteq X \times Y$. If $\mathcal{I}$ is G-invariant (that is, $(gx, gy) \in \mathcal{I}$ whenever $(x, y) \in \mathcal{I}$ and $g \in G$), the *Radon Transform* associated with $\mathcal{I}$ is the operator $R_{\mathcal{I}} \in \mathrm{Hom}_G(L(X), L(Y))$ given by:

$$(R_{\mathcal{I}} f)(y) = \sum_{\substack{x\in X:\\ (x,y)\in\mathcal{I}}} f(x)$$

for all $y \in Y$ and $f \in L(X)$. Clearly, $\mathcal{I}$ is G-invariant if and only if it is the union of orbits of G on $X \times Y$, and Exercise 1.4.3 may be reformulated in the following way: the Radon transforms associated with the orbits of G on $X \times Y$ form a basis for $\mathrm{Hom}_G(L(X), L(Y))$.

Corollary 1.4.4 (Wielandt's lemma) *Let G act on a finite set X and let $\lambda = \oplus_{\rho \in J} m_\rho \rho$ be the decomposition into irreducibles of the associated permutation representation. Then*

$$\sum_{\rho \in J} m_\rho^2 = |\text{orbits of } G \text{ on } X \times X|.$$

Proof On the one hand,

$$\dim \mathrm{Hom}_G(L(X), L(X)) = \dim L(X \times X)^G = |\text{orbits of } G \text{ on } X \times X|$$

because the characteristic functions of the orbits of G on $X \times X$ form a basis for $L(X \times X)^G$.

On the other hand, $\dim \mathrm{Hom}_G(L(X), L(X)) = \sum_{\rho \in J} m_\rho^2$, by Corollary 1.2.15. □

Example 1.4.5 Let $G = \mathfrak{S}_n$ act in the natural way on $X = \{1, 2, \ldots, n\}$. The stabilizer of n is $\mathfrak{S}_{n-1}$, the symmetric group on $\{1, 2, \ldots, n-1\}$, and $X = \mathfrak{S}_n/\mathfrak{S}_{n-1}$. In $L(X)$ we find the invariant subspaces V_0, the space of constant functions, and $V_1 = \{f \in L(X) : \sum_{j=1}^n f(j) = 0\}$, the space of mean-zero functions. Moreover, the orthogonal direct sum

$$L(X) = V_0 \oplus V_1 \tag{1.23}$$

holds. On the other hand, $\mathfrak{S}_n$ has exactly two orbits on $X \times X$, namely $\Omega_0 = \{(i, i) : i = 1, 2, \ldots, n\}$ and $\Omega_1 = \{(i, j) : i \neq j, i, j = 1, 2, \ldots, n\}$. Therefore, Wielandt's lemma ensures that (1.23) is the decomposition of $L(X)$ into irreducible $\mathfrak{S}_n$-sub-representations.

Exercise 1.4.6 Suppose that a group G acts transitively on a space X. Denote by V_0 the space of constant functions on X and by $V_1 = \{f \in L(X) : \sum_{x \in X} f(x) = 0\}$ the space of mean-zero functions. Show that $L(X) = V_0 \oplus V_1$ is the decomposition into irreducible G-sub-representations of $L(X)$ if and only if the action of G on X is *doubly transitive*, that is, for all $x, y, z, u \in X$ such that $x \neq y$ and $z \neq u$ there exists $g \in G$ such that $g(x, y) = (z, u)$.

Exercise 1.4.7 In the same notation as Exercise 1.4.3, suppose that $\bigoplus_{\sigma \in I} m'_\sigma \sigma$ is the decomposition of the permutation representation of G on $L(Y)$. Show that $\Sigma_{\rho \in J \cap I} m_\rho m_{\rho'} = |\text{orbits of } G \text{ on } X \times Y|$.

1.4.2 Symmetric actions and Gelfand's lemma

We say that an orbit Ω of G on $X \times X$ is *symmetric* if $(x, y) \in \Omega$ implies $(y, x) \in \Omega$. Also, the action of G on X is *symmetric* if all orbits of G on $X \times X$ are symmetric. Finally a function $F \in L(X \times X)$ is *symmetric* if $F(x, y) = F(y, x)$ for all $x, y \in X$.

Proposition 1.4.8 (Gelfand's lemma; symmetric case) *Suppose that the action of G on X is symmetric. Then* $\mathrm{Hom}_G(L(X), L(X))$ *is commutative and the permutation representation of G on $L(X)$ is multiplicity-free.*

Proof First observe that if the action of G is symmetric then any $F \in L(X \times X)^G$, being constant on the G-orbits on $X \times X$, is clearly symmetric. Thus,

$$\begin{aligned}(F_1 F_2)(x, y) &= \sum_{z \in X} F_1(x, z) F_2(z, y) \\ &= \sum_{z \in X} F_1(z, x) F_2(y, z) \\ &= (F_2 F_1)(y, x) = (F_2 F_1)(x, y)\end{aligned}$$

($F_2 F_1$ is in $L(X \times X)^G$), for all $F_2, F_1 \in L(X \times X)^G$ and $x, y \in X$. This shows that $L(X \times X)^G$ is commutative. To end the proof, apply Proposition 1.4.1 and Corollary 1.2.17. $\square$

Observe that if $\mathcal{A}$ is a subalgebra of $M_{n,n}(\mathbb{C})$ such that all $A \in \mathcal{A}$ are symmetric, then $\mathcal{A}$ is commutative, indeed

$$AB = A^T B^T = (BA)^T = BA$$

for all $A, B \in \mathcal{A}$.

1.4.3 Frobenius reciprocity for a permutation representation

In what follows, we assume that the action of G on X is transitive. We fix $x_0 \in X$ and denote by $K = \{g \in G : gx_0 = x_0\} \leq G$ its *stabilizer*. Thus, $X = G/K$, that is, we identify X with the space of right cosets of K in G. Note also, that

$$\sum_{x \in X} f(x) = \frac{1}{|K|} \sum_{g \in G} f(gx_0)$$

for all $f \in L(X)$.

Definition 1.4.9 Suppose that G acts transitively on X. If the corresponding permutation representation is multiplicity-free, we say that (G, K) is a *Gelfand* *pair*. If in addition, the action of G on X is symmetric, we say that (G, K) is a *symmetric Gelfand pair*.

Example 1.4.10 Let $\mathfrak{S}_n$ be the symmetric group of all permutations of the set $\{1, 2, \ldots, n\}$. Fix $0 \le k \le n/2$ and define $\Omega_{n-k,k}$ as the set of all subsets of $\{1, 2, \ldots, n\}$ of cardinality k, that is,

$$\Omega_{n-k,k} = \{A \subseteq \{1, 2, \ldots, n\} : |A| = k\}$$

(we also say that $A \in \Omega_{n-k,k}$ is a k-subset of $\{1, 2, \ldots, n\}$). The group $\mathfrak{S}_n$ acts on $\Omega_{n-k,k}$: for $\pi \in \mathfrak{S}_n$ and $A \in \Omega_{n-k,k}$, then $\pi A = \{\pi(j) : j \in A\}$.

Fix $\overline{A} \in \Omega_{n-k,k}$ and denote by K its stabilizer. Clearly, K is isomorphic to $\mathfrak{S}_{n-k} \times \mathfrak{S}_k$, where the first factor is the symmetric group on $\overline{A}^C = \{1, 2, \ldots, n\} \setminus \overline{A}$ and the second is the symmetric group on $\overline{A}$. As the action is transitive, we may write $\Omega_{n-k,k} = \mathfrak{S}_n/(\mathfrak{S}_{n-k} \times \mathfrak{S}_k)$.

Note that (A, B) and (A', B') in $\Omega_{n-k,k} \times \Omega_{n-k,k}$ are in the same $\mathfrak{S}_n$-orbit if and only if $|A \cap B| = |A' \cap B'|$. The "only if" part is obvious; conversely, if $|A \cap B| = |A' \cap B'|$, then using the decomposition

$$\begin{aligned}\{1, 2, \ldots, n\} &= (A \cup B)^C \coprod [A \setminus (A \cap B)] \coprod [B \setminus (A \cap B)] \coprod (A \cap B) \\ &= (A' \cup B')^C \coprod [A' \setminus (A' \cap B')] \coprod [B' \setminus (A' \cap B')] \coprod (A' \cap B')\end{aligned}$$

we can construct $\pi \in \mathfrak{S}_n$ such that

- $\pi(A \cap B) = A' \cap B'$
- $\pi[A \setminus (A \cap B)] = A' \setminus (A' \cap B')$
- $\pi[B \setminus (A \cap B)] = B' \setminus (A' \cap B')$

so that $\pi(A, B) = (A', B')$.

Setting

$$\Theta_j = \{(A, B) \in \Omega_{n-k,k} \times \Omega_{n-k,k} : |A \cap B| = j\}$$

we have that

$$\Omega_{n-k,k} \times \Omega_{n-k,k} = \coprod_{j=0}^{k} \Theta_j$$

is the decomposition of $\Omega_{n-k,k} \times \Omega_{n-k,k}$ into $\mathfrak{S}_n$-orbits.

Observe that every orbit Θ_j is symmetric: $|A \cap B| = |B \cap A|$, so that $(\mathfrak{S}_n, \mathfrak{S}_{n-k} \times \mathfrak{S}_k)$ is a symmetric Gelfand pair. Moreover, since there are exactly $k + 1$ orbits of $\mathfrak{S}_n$ on $\Omega_{n-k,k} \times \Omega_{n-k,k}$, by Wielandt's lemma (Lemma 1.4.4) we deduce that $L(\Omega_{n-k,k})$ decomposes into $k + 1$ irreducible inequivalent $\mathfrak{S}_n$-representations.

Exercise 1.4.11 (1) Show that, if $0 \le h \le k \le n/2$, then $\mathfrak{S}_n$ has precisely $h+1$ orbits on $\Omega_{n-k,k} \times \Omega_{n-h,h}$.

(2) Suppose that $L(\Omega_k) = \bigoplus_{j=0}^{k} V_{k,j}$ is the decomposition of $L(\Omega_k)$ into irreducible representations (see Example 1.4.10). Use (1) and Exercise 1.4.7 to show that it is possible to number the representations $V_{k,0}, V_{k,1}, \dots, V_{k,k}$ in such a way that $V_{h,j} \sim V_{k,j}$ for all $j = 0, 1, \dots, h$ and $0 \le h \le k \le n/2$. [*Hint.* Every representation in $L(\Omega_{n-h,h})$ is also in $L(\Omega_{n-k,k})$.]

Suppose that (ρ, W) is an irreducible representation of G. Set $d_\rho = \dim W$ and suppose that W^K (the subspace of K-invariant vectors in W) is non-trivial. For every $v \in W^K$, we define a linear map $\mathcal{T}_v : W \to L(X)$ by setting:

$$(\mathcal{T}_v u)(gx_0) = \sqrt{\frac{d_\rho}{|X|}} \langle u, \rho(g)v \rangle_W \tag{1.24}$$

for all $g \in G$ and $u \in W$. Since G is transitive on X, this is defined for all $x \in X$. Moreover, if $g_1, g_2 \in G$ and $g_1 x_0 = g_2 x_0$, then $g_1^{-1} g_2 \in K$ and therefore (v is K-invariant)

$$(\mathcal{T}_v u)(g_2 x_0) = \sqrt{\frac{d_\rho}{|X|}} \langle u, \rho(g_1)\rho(g_1^{-1} g_2) v \rangle_W = (\mathcal{T}_v u)(g_1 x_0).$$

This shows that $\mathcal{T}_v u$ is well defined.

Theorem 1.4.12 (Frobenius reciprocity for a permutation representation) *With the above notation we have the following.*

(i) $\mathcal{T}_v \in \mathrm{Hom}_G(W, L(X))$ *for all* $v \in W^K$.
(ii) (**Orthogonality relations**) *For all* $v, u \in W^K$ *and* $w, z \in V$ *we have*

$$\langle \mathcal{T}_u w, \mathcal{T}_v z \rangle_{L(X)} = \langle w, z \rangle_W \langle v, u \rangle_W.$$

(iii) *The map*

$$\begin{array}{rcl} W^K & \to & \mathrm{Hom}_G(W, L(X)) \\ v & \mapsto & \mathcal{T}_v \end{array} \tag{1.25}$$

is an antilinear isomorphism. In particular, the multiplicity of ρ *in the permutation representation on* $L(X)$ *equals* $\dim W^K$.

Proof (i) This is trivial: if $g, g_0 \in G$ and $u \in W$, then

$$\begin{aligned}[\lambda(g)\mathcal{T}_v u](g_0x_0) &= [\mathcal{T}_v u](g^{-1}g_0x_0) \\ &= \sqrt{\frac{d_\rho}{|X|}}\langle u, \rho(g^{-1})\rho(g_0)v\rangle_W \\ &= [\mathcal{T}_v \rho(g)u](g_0x_0).\end{aligned}$$

(ii) This is another application of Lemma 1.3.5. For $u, v \in W^K$, define a linear map $R_{u,v} : W \to W$ by setting $R_{u,v}w = \langle w, u\rangle_W v$, for all $w \in W$. Note that

$$tr(R_{u,v}) = \langle v, u\rangle_W. \tag{1.26}$$

Indeed, taking an orthonormal basis $w_1, w_2, \dots, w_{d_\rho}$ for W we have

$$tr(R_{u,v}) = \sum_{j=1}^{d_\rho}\langle R_{u,v}w_j, w_j\rangle_W = \sum_{j=1}^{d_\rho}\langle w_j, u\rangle_W\langle v, w_j\rangle_W = \langle v, u\rangle_W.$$

Also, with the notation in (1.14), we have $R_{u,v} = T_{\theta_u,v}$. In general, $R_{u,v}$ does not belong to $\mathrm{Hom}_G(W, W)$, but its projection (see the first equality in Corollary 1.2.22 and Lemma 1.3.5)

$$R = \frac{1}{|G|}\sum_{g\in G}\rho(g)R_{u,v}\rho(g^{-1}) \tag{1.27}$$

does. Since W is irreducible, $R = cI_W$ for some $c \in \mathbb{C}$. We now compute c by taking traces on both sides of (1.27). Clearly $tr(cI_W) = cd_\rho$, while, by (1.26) and (1.27) we have

$$\begin{aligned}tr(R) &= \frac{1}{|G|}\sum_{g\in G}tr[\rho(g)R_{u,v}\rho(g^{-1})] \\ &= \frac{1}{|G|}\sum_{g\in G}tr[R_{u,v}] \\ &= \langle v, u\rangle_W.\end{aligned}$$

We deduce that $c = \frac{1}{d_\rho}\langle v, u\rangle_W$ and therefore

$$R = \frac{1}{d_\rho}\langle v, u\rangle_W I_W. \tag{1.28}$$

Then, for $w, z \in W$ we have:

$$\begin{aligned}
\langle \mathcal{T}_u w, \mathcal{T}_v z\rangle_{L(X)} &= \frac{1}{|K|}\sum_{g\in G}(\mathcal{T}_u w)(gx_0)\overline{(\mathcal{T}_v z)(gx_0)} \\
&= \frac{d_\rho}{|G|}\sum_{g\in G}\langle w, \rho(g)u\rangle_W \overline{\langle z, \rho(g)v\rangle_W} \\
&= \frac{d_\rho}{|G|}\sum_{g\in G}\langle \rho(g)R_{u,v}\rho(g^{-1})w, z\rangle_W \\
&= d_\rho\langle Rw, z\rangle_W \\
&= \langle w, z\rangle_W \langle v, u\rangle_W,
\end{aligned}$$

where the last equality follows from (1.28).

(iii) It is obvious that the map $v \mapsto \mathcal{T}_v$ is antilinear. We now show that it is also bijective. Suppose that $T \in \mathrm{Hom}_G(W, L(X))$. Then $W \ni u \mapsto (Tu)(x_0) \in \mathbb{C}$ is a linear map, and therefore there exists $v \in W$ such that $(Tu)(x_0) = \langle u, v\rangle_W$, for all $u \in W$. Then we have

$$\begin{aligned}
[Tu](gx_0) &= [\lambda(g^{-1})Tu](x_0) \\
(T \in \mathrm{Hom}_G(W, L(X)) \quad &= [T\rho(g^{-1})u](x_0) \\
&= \langle \rho(g^{-1})u, v\rangle_W \\
&= \langle u, \rho(g)v\rangle_W,
\end{aligned} \tag{1.29}$$

that is,

$$T = \sqrt{\frac{|X|}{d_\rho}}\mathcal{T}_v.$$

Clearly, $v \in W^K$: if $k \in K$ then

$$\langle u, \rho(k)v\rangle_W = (Tu)(kx_0) = (Tu)(x_0) = \langle u, v\rangle_W$$

for all $u \in W$, and therefore $\rho(k)v = v$.

Moreover, it is obvious that the vector v in (1.29) is uniquely determined, so that (1.25) is a bijection.

Finally, the multiplicity of ρ in the permutation representation on $L(X)$ is equal to $\dim\mathrm{Hom}_G(W, L(X))$ (see Lemma 1.2.5) and therefore to $\dim W^K$. $\square$

Corollary 1.4.13 *(G, K) is a Gelfand pair if and only if* $\dim W^K \leq 1$ *for every irreducible G-representation W. In particular,* $\dim W^K = 1$ *if and only if W is a sub-representation of $L(X)$.*

1.4.4 The structure of the commutant of a permutation representation

In this section, we give, for a permutation representation, an explicit form for the operators $T^{\rho}_{k,j}$ constructed at the end of Section 1.2.2 (see also Theorem 1.2.14).

Suppose that $L(X) = \bigoplus_{\rho \in J} m_\rho W_\rho$ is the decomposition of $L(X)$ into irreducible sub-representations. For every $\rho \in J$ let $\{v_1^\rho, v_2^\rho, \ldots, v_{m_\rho}^\rho\}$ be an orthonormal basis for W_ρ^K and denote by $\mathcal{T}_j^\rho \in \mathrm{Hom}_G(W_\rho, L(X))$ the intertwining operator associated with v_j^ρ (cf. (1.24)).

Corollary 1.4.14

$$L(X) = \bigoplus_{\rho \in J} \bigoplus_{j=1}^{m_\rho} \mathcal{T}_j^\rho W_\rho$$

is an explicit orthogonal decomposition of $L(X)$ into irreducible sub-representations and every $\mathcal{T}_j^\rho$ is an isometric immersion of W_ρ into $L(X)$.

Let $v_1^\rho, v_2^\rho, \ldots, v_{m_\rho}^\rho$ be as before. For $\rho \in J$ and $i, j = 1, 2, \ldots, m_\rho$ set, for $g, h \in G$,

$$\phi^\rho_{i,j}(gx_0, hx_0) = \frac{d_\rho}{|X|} \langle \rho(h) v_j^\rho, \rho(g) v_i^\rho \rangle_{W_\rho}. \tag{1.30}$$

Then $\phi^\rho_{i,j} \in L(X \times X)$ is well defined as v_i^ρ and v_j^ρ are both K-invariant. Moreover it is G-invariant ($\phi^\rho_{i,j} \in L(X \times X)^G$):

$$\phi^\rho_{i,j}(sgx_0, shx_0) = \frac{d_\rho}{|X|} \langle \rho(s)\rho(h) v_j^\rho, \rho(s)\rho(g) v_i^\rho \rangle_{W_\rho} = \phi^\rho_{i,j}(gx_0, hx_0).$$

Thus, we can define $\Phi^\rho_{i,j} \in \mathrm{Hom}_G(L(X), L(X))$ as the operator associated with the G-invariant matrix $(\phi^\rho_{i,j}(x, y))_{x,y \in X}$, namely,

$$(\Phi^\rho_{i,j} f)(x) = \sum_{y \in X} \phi^\rho_{i,j}(x, y) f(y) \tag{1.31}$$

for all $f \in L(X)$ and $x \in X$.

We now present two elementary identities relating the operators $\Phi^\rho_{i,j} \in \mathrm{Hom}_G(L(X), L(X))$ and $\mathcal{T}_j^\rho \in \mathrm{Hom}_G(W_\rho, L(X))$.

Lemma 1.4.15 *For $g \in G$ and $f \in L(X)$ we have:*

(i) $[\Phi^\rho_{i,j} f](gx_0) = \frac{\sqrt{d_\rho}}{\sqrt{|X||K|}} \mathcal{T}_i^\rho [\sum_{h \in G} \rho(h) f(hx_0) v_j^\rho](gx_0)$.

(ii) $[\Phi^\rho_{i,j} f](gx_0) = \frac{\sqrt{d_\rho}}{\sqrt{|X||K|}} \langle f, \mathcal{T}_j^\rho \rho(g) v_i^\rho \rangle_{L(X)}$.

Proof (i) We have

$$[\Phi^\rho_{i,j} f](gx_0) = \frac{d_\rho}{|G|}\sum_{h\in G}\langle\rho(h)v^\rho_j, \rho(g)v^\rho_i\rangle_{W_\rho} f(hx_0)$$
$$= \frac{\sqrt{d_\rho}}{\sqrt{|X||K|}}\sum_{h\in G}[\mathcal{T}^\rho_i\rho(h)v^\rho_j](gx_0)\cdot f(hx_0)$$
$$= \frac{\sqrt{d_\rho}}{\sqrt{|X||K|}}\mathcal{T}^\rho_i\left[\sum_{h\in G}\rho(h)f(hx_0)v^\rho_j\right](gx_0).$$

The proof of (ii) is similar and it is left to the reader. □

We now show that, in the present situation, the operators $\Phi^\rho_{i,j}$ coincide with the operators $T^\rho_{i,j}$ in Theorem 1.2.14.

Theorem 1.4.16 *For every* $\sigma, \rho \in J$, $i, j = 1, 2, \ldots, m_\rho$ *and* $s, r = 1, 2, \ldots, m_\sigma$ *we have:*

(i) $\mathrm{Im}\Phi^\rho_{i,j} = \mathcal{T}^\rho_i W_\rho$
(ii) $\mathrm{Ker}\Phi^\rho_{i,j} = L(X) \ominus \mathcal{T}^\rho_j W_\rho$
(iii) $\Phi^\rho_{i,j}\Phi^\sigma_{s,r} = \delta_{j,s}\delta_{\rho,\sigma}\Phi^\rho_{i,r}$.

Proof (i) This is an immediate consequence of (i) in Lemma 1.4.15.

(ii) If in (ii) of Lemma 1.4.15 we take $f = \mathcal{T}^\sigma_r w$, with $w \in W_\sigma$, then we have

$$[\Phi^\rho_{i,j}\mathcal{T}^\sigma_r w](gx_0) = \frac{\sqrt{d_\rho}}{|G|}\langle\mathcal{T}^\sigma_r w, \mathcal{T}^\rho_j\rho(g)v^\rho_i\rangle_{L(X)}$$

and this is equal to zero if $\sigma \neq \rho$ or if $\sigma = \rho$ but $r \neq j$. Note that (ii) may also be deduced from (i) and (iii).

(iii) Arguing as in the proof of Lemma 1.4.15, we get, for all $g, h \in G$,

$$[\Phi^\rho_{i,j}\Phi^\sigma_{s,r}](gx_0, hx_0) = \frac{d_\rho d_\sigma}{|X|^2|K|}\sum_{t\in G}\langle\rho(t)v^\rho_j, \rho(g)v^\rho_i\rangle_{W_\rho}\langle\sigma(h)v^\sigma_r, \sigma(t)v^\sigma_s\rangle_{W_\sigma}$$
$$= \frac{\sqrt{d_\rho}\sqrt{d_\sigma}}{|X||K|}\sum_{t\in G}[\mathcal{T}^\sigma_s\rho(h)v^\sigma_r](tx_0)\overline{[\mathcal{T}^\rho_j\rho(g)v^\rho_i](tx_0)}$$
$$= \frac{\sqrt{d_\rho d_\sigma}}{|X|}\langle\mathcal{T}^\sigma_s\rho(h)v^\sigma_r, \mathcal{T}^\rho_j\rho(g)v^\rho_i\rangle_{L(X)}$$
$$=_{(*)} \delta_{\sigma,\rho}\frac{d_\rho}{|X|}\langle v^\rho_j, v^\rho_s\rangle_{W_\rho}\langle\rho(h)v^\rho_r, \rho(g)v^\rho_i\rangle_{W_\rho}$$
$$= \delta_{\sigma,\rho}\delta_{j,s}\phi^\rho_{i,r}(gx_0, hx_0),$$

where $(*)$ follows from Theorem 1.4.12 and Corollary 1.4.14. □

Corollary 1.4.17

(i) $\Phi^{\rho}_{i,i}$ *is the orthogonal projection onto* $\mathcal{T}^{\rho}_{i} W_{\rho}$.

(ii) $\sum_{i=1}^{m_\rho} \Phi^{\rho}_{i,i}$ *is the orthogonal projection onto the isotypic component* $m_\rho W_\rho$.

1.5 The group algebra and the Fourier transform

1.5.1 $L(G)$ and the convolution

In this section we study the left regular representation $(\lambda, L(G))$ of a finite group G (see Example 1.1.3). This is a particular case of the theory developed in the previous section.

For $f_1, f_2 \in L(G)$ we define the *convolution* $f_1 * f_2 \in L(G)$ by setting

$$[f_1 * f_2](g) = \sum_{h\in G} f_1(gh^{-1}) f_2(h).$$

Equivalently,

$$[f_1 * f_2](g) = \sum_{\substack{s,t\in G:\\ st=g}} f_1(s) f_2(t) = \sum_{h\in G} f_1(h) f_2(h^{-1}g).$$

A third way to write the convolution is the following

$$f_1 * f_2 = \sum_{h\in G} f_1(h)\lambda(h) f_2.$$

In particular, $\delta_{g_1} * \delta_{g_2} = \delta_{g_1 g_2} = \lambda(g_1)\delta_{g_2}$ and

$$\delta_g * f = \lambda(g) f \tag{1.32}$$

for all $g_1, g_2, g \in G$ and $f \in L(G)$. Similarly, one has

$$f * \delta_g = \rho(g^{-1}) f \tag{1.33}$$

for all $g \in G$ and $f \in L(G)$. It follows that

$$\rho(g_1)\lambda(g_2) = \lambda(g_2)\rho(g_1) \tag{1.34}$$

for all $g_1, g_2 \in G$, that is, the left and the right regular representations commute.

The vector space $L(G)$, endowed with the multiplication $*$ becomes an associative algebra, called the *group algebra* (or the *convolution algebra*) of G. It has a unit, namely the function δ_{1_G} and it is commutative if and only if G is commutative. Moreover, the map $L(G) \ni \psi \mapsto \check{\psi} \in L(G)$, where $\check{\psi}(g) = \overline{\psi(g^{-1})}$ for all $g \in G$, is an involution. We only check the anti-multiplicative property:

$$\begin{aligned}[\check{\psi}_1 * \check{\psi}_2](g) &= \sum_{s\in G} \check{\psi}_1(gs)\check{\psi}_2(s^{-1}) \\ &= \sum_{s\in G} \overline{\psi_1(s^{-1}g^{-1})\psi_2(s)} \\ &= \overline{[\psi_2 * \psi_1](g^{-1})} \\ &= [\psi_2 * \psi_1]\check{}(g)\end{aligned}$$

for all $\psi_1, \psi_2 \in L(G)$ and $g \in G$.

Remark 1.5.1 A function f is in the center of $L(G)$ (cf. (1.11)) if and only if $f * \delta_g = \delta_g * f$ for all $g \in G$, that is, $f(g^{-1}h) = f(hg^{-1})$ for all $g, h \in G$, i.e., if and only if f is central (cf. Definition 1.3.1).

The group algebra is a fundamental notion in the representation theory of finite groups, as the following proposition shows.

Proposition 1.5.2 *For every $\psi \in L(G)$ define a linear map $T_\psi : L(G) \to L(G)$ by setting*

$$T_\psi f = f * \psi,$$

for all $f \in L(G)$. Then the map

$$L(G) \ni \psi \mapsto T_\psi \in \mathrm{Hom}_G(L(G), L(G))$$

is a $*$-anti-isomorphism of algebras (here, $\mathrm{Hom}_G(L(G), L(G))$ *is the commutant (cf. Definition 1.2.13) of the left regular representation of G).*

Proof This is just a specialization of Proposition 1.4.1 for the left regular representation.

Suppose that $F \in L(G \times G)^G$, that is $F(gg_1, gg_2) = F(g_1, g_2)$ for all $g, g_1, g_2 \in G$. Then, $F(g_1, g_2) = F(g_2 g_2^{-1} g_1, g_2) = F(g_2^{-1} g_1, 1_G)$, that is

$$F(g_1, g_2) = \Psi(g_2^{-1} g_1) \tag{1.35}$$

where $\Psi(g) = F(g, 1_G)$ for all $g \in G$.

Conversely, if F is of the form (1.35) then clearly $F \in L(G \times G)^G$.

Moreover, assuming (1.35), for every $f \in L(G)$ and $g \in G$ we have

$$(T_\Psi f)(g) = \sum_{h\in G} f(h)\Psi(h^{-1}g) = \sum_{h\in G} F(g, h) f(h) = (T_F f)(g)$$

where T_F is as in (1.22).

We now show that the map $T \mapsto T_\Psi$ is a $*$-anti-isomorphism. Indeed, for Ψ_1, Ψ_2 and $f \in L(G)$ we have

$$T_{\Psi_1}(T_{\Psi_2} f) = (f * \Psi_2) * \Psi_1 = f * (\Psi_2 * \Psi_1) = T_{\Psi_2 * \Psi_1} f,$$

that is, $T_{\Psi_1} T_{\Psi_2} = T_{\Psi_2 * \Psi_1}$ (anti-multiplicative property). Moreover, for f_1, f_2 and $\Psi \in L(G)$, we have

$$\begin{aligned} \langle T_\Psi f_1, f_2 \rangle_{L(G)} &= \sum_{g \in G} \sum_{s \in G} f_1(gs)\Psi(s^{-1})\overline{f_2(g)} \\ \text{(setting } t = gs) \quad &= \sum_{t \in G} \sum_{s \in G} f_1(t)\Psi(s^{-1})\overline{f_2(ts^{-1})} \\ &= \langle f_1, T_{\check{\Psi}} f_2 \rangle_{L(G)}, \end{aligned}$$

that is $(T_\Psi)^* = T_{\check{\Psi}}$. □

Exercise 1.5.3 Give a direct proof of the above proposition by showing that if $T \in \mathrm{Hom}_G(L(X), L(X))$, then $Tf = f * \Psi$, where $\Psi = T(\delta_{1_G})$.

Let now (ρ, W) be an irreducible representation of G. Note that the stabilizer K of 1_G (that plays the same role of x_0), for the left action of G on itself reduces to the trivial subgroup $\{1_G\}$. This means that the space of K-invariant vectors is the whole space W. Therefore, as in (1.24) we can define, for every $v \in W$, an operator $\mathcal{T}_v : W \to L(G)$ by setting

$$[\mathcal{T}_v u](g) = \sqrt{\frac{d_\rho}{|G|}} \langle u, \rho(g)v \rangle_W,$$

for all $u \in W$ and $g \in G$. Then, the Frobenius reciprocity theorem (Theorem 1.4.12) now becomes:

Theorem 1.5.4

(i) *The map* $W \ni v \mapsto \mathcal{T}_v \in \mathrm{Hom}_G(W, L(G))$ *is an antilinear isomorphism.*
(ii) (**Orthogonality relations**) *For all* $v, u, w, z \in W$ *we have:*

$$\langle \mathcal{T}_u w, \mathcal{T}_v z \rangle_{L(G)} = \langle w, z \rangle_W \langle v, u \rangle_W.$$

From now on, we suppose that $\widehat{G}$ is a fixed complete set of irreducible, pairwise inequivalent representations of G. We know (cf. Corollary 1.3.16) that $|\widehat{G}|$ equals the number of conjugacy classes in G.

For every $\rho \in \widehat{G}$, fix an orthonormal basis $\{v_1^\rho, v_2^\rho, \ldots, v_{d_\rho}^\rho\}$ in the representation space W_ρ. For simplicity of notation, denote by $\mathcal{T}_j^\rho$ the operator $\mathcal{T}_{v_j^\rho}^\rho$. Then we have the following more explicit version of Corollary 1.3.10.

Corollary 1.5.5 *$L(G) = \bigoplus_{\rho\in\widehat{G}} \bigoplus_{j=1}^{d_\rho} \mathcal{T}_j^\rho W_\rho$ is an explicit orthogonal decomposition of $L(G)$ into irreducible sub-representations of the left regular representation. Moreover, every $\mathcal{T}_j^\rho$ is an isometric immersion of W_ρ into $L(G)$.*

In correspondence with the orthonormal basis $\{v_1^\rho, v_2^\rho, \ldots, v_{d_\rho}^\rho\}$, we define the unitary matrix coefficients

$$\varphi_{i,j}^\rho(g) = \langle \rho(g) v_j^\rho, v_i^\rho \rangle_{W_\rho}.$$

The coefficients of the G-invariant matrix (cf. (1.30)) now become

$$\phi_{i,j}^\rho(g,h) = \frac{d_\rho}{|G|} \langle \rho(h) v_j^\rho, \rho(g) v_i^\rho \rangle_{W_\rho}$$

and therefore

$$\varphi_{i,j}^\rho(h) = \frac{|G|}{d_\rho} \phi_{i,j}^\rho(1_G, h).$$

Moreover, the associated intertwining operators (cf. (1.31)) now have the form

$$\Phi_{i,j}^\rho f = \frac{d_\rho}{|G|} f * \overline{\varphi_{j,i}^\rho}, \tag{1.36}$$

as one easily verifies from the definitions (see also (1.35)) and the trivial identity $\varphi_{i,j}^\rho(t^{-1}) = \overline{\varphi_{j,i}^\rho(t)}$.

From Theorem 1.4.16 we immediately deduce the following.

Theorem 1.5.6 *For every $\sigma, \rho \in \widehat{G}$, $i, j = 1, 2, \ldots, d_\rho$ and $s, r = 1, 2, \ldots, d_\sigma$ we have:*

(i) $\mathrm{Im}\Phi_{i,j}^\rho = \mathcal{T}_i^\rho W_\rho$
(ii) $\mathrm{Ker}\Phi_{i,j}^\rho = L(G) \ominus \mathcal{T}_j^\rho W_\rho$
(iii) $\Phi_{i,j}^\rho \Phi_{s,r}^\sigma = \delta_{j,s}\delta_{\rho,\sigma}\Phi_{i,r}^\rho$.

In Lemma 1.1.9 we have given the basic properties of the matrix coefficients. We now add some specific properties that hold in the case the representations are irreducible.

Lemma 1.5.7 *Let ρ and σ be two irreducible representations of G.*

(i) (**Orthogonality relations**) $\langle \varphi_{i,j}^\rho, \varphi_{s,t}^\sigma \rangle_{L(G)} = \frac{|G|}{d_\rho} \delta_{\rho,\sigma}\delta_{i,s}\delta_{j,t}$.
(ii) $\varphi_{i,j}^\rho * \varphi_{s,t}^\sigma = \frac{|G|}{d_\rho} \delta_{\rho,\sigma}\delta_{j,s}\varphi_{i,t}^\rho$.

Proof (i) With the same notation of Theorem 1.5.4 we have

$$\begin{aligned}\langle \varphi^\rho_{i,j}, \varphi^\sigma_{s,t}\rangle_{L(G)} &= \sqrt{\frac{|G|}{d_\rho}}\sqrt{\frac{|G|}{d_\sigma}}\langle \mathcal{T}^\sigma_t v^\sigma_s, \mathcal{T}^\rho_j v^\rho_i\rangle_{L(G)}\\ \text{(by Theorem 1.5.4.(ii))} \quad &= \frac{|G|}{d_\rho}\delta_{\rho,\sigma}\delta_{i,s}\delta_{j,t}.\end{aligned}$$

(ii)

$$\begin{aligned}[\varphi^\rho_{i,j} * \varphi^\sigma_{s,t}](g) &= \sum_{h\in G}\varphi^\rho_{i,j}(gh)\varphi^\sigma_{s,t}(h^{-1})\\ \text{(by Lemma 1.1.9)} \quad &= \sum_{h\in G}\sum_{k=1}^{d_\rho}\varphi^\rho_{i,k}(g)\varphi^\rho_{k,j}(h)\overline{\varphi^\sigma_{t,s}(h)}\\ \text{(by (i))} \quad &= \frac{|G|}{d_\rho}\delta_{\rho,\sigma}\delta_{j,s}\varphi^\rho_{i,t}(g).\end{aligned}$$

Alternatively, (ii) is just a reformulation of (iii) in Theorem 1.5.6. □

Corollary 1.5.8 *The set*

$$\{\varphi^\rho_{i,j} : \rho \in \widehat{G}, i, j = 1, 2, \ldots, d_\rho\} \tag{1.37}$$

is an orthogonal system in $L(G)$ *which is also complete.*

Proof From Corollary 1.3.10 we have that $\dim L(G) = \sum_{\rho\in\widehat{G}} d^2_\rho$ and therefore it is equal to the cardinality of (1.37). □

We observe that Corollary 1.4.17, in the setting of the left regular representation, can be completed with the following expression for the projection onto the isotypic component:

$$\sum_{i=1}^{d_\rho}\Phi^\rho_{i,i} f = \frac{d_\rho}{|G|} f * \overline{\chi^\rho}$$

where χ^ρ is the character of the representation ρ (see (1.36)).

1.5.2 The Fourier transform

We begin this section by observing that, in the notation of Definition 1.3.12, for a representation (σ, V) of G and $f_1, f_2 \in L(G)$, we have

$$\begin{aligned}\sigma(f_1 * f_2) &= \sum_{g\in G}[f_1 * f_2](g)\sigma(g)\\ &= \sum_{g\in G}\sum_{h\in G} f_1(gh)f_2(h^{-1})\sigma(gh)\sigma(h^{-1})\\ &= \sigma(f_1)\sigma(f_2).\end{aligned} \tag{1.38}$$

In what follows, we set $\mathcal{A}(G) = \bigoplus_{\rho\in\widehat{G}} \mathrm{Hom}(W_\rho, W_\rho)$.

The *Fourier transform* is the linear map

$$\begin{aligned}\mathcal{F} : L(G) &\to \quad \mathcal{A}(G)\\ f \quad &\mapsto \bigoplus_{\rho\in\widehat{G}} \rho(f).\end{aligned} \tag{1.39}$$

In other words, $\mathcal{F}f$ is the direct sum of the linear operators $\rho(f) : W_\rho \to W_\rho$, $\rho \in \widehat{G}$. The *Hilbert–Schmidt scalar product* on $\mathcal{A}(G)$ is defined by setting

$$\left\langle \bigoplus_{\rho\in\widehat{G}} T_\rho, \bigoplus_{\sigma\in\widehat{G}} S_\sigma \right\rangle_{HS} = \frac{1}{|G|}\sum_{\rho\in\widehat{G}} d_\rho tr[T_\rho S_\rho^*], \tag{1.40}$$

where $T_\rho, S_\rho \in \mathrm{Hom}(W_\rho, W_\rho)$ for all $\rho \in \widehat{G}$.

Exercise 1.5.9 Prove that (1.40) defines a scalar product.

Recall that if $\{v_1^\rho, v_2^\rho, \ldots, v_{d_\rho}^\rho\}$ is an orthonormal basis for W_ρ, then

$$tr[T_\rho S_\rho^*] = \sum_{j=1}^{d_\rho} \langle T_\rho v_j^\rho, S_\rho v_j^\rho\rangle_{W_\rho}.$$

We now define the map

$$\mathcal{F}^{-1} : \mathcal{A}(G) \to L(G)$$

by setting

$$\left[\mathcal{F}^{-1}\left(\bigoplus_{\rho\in\widehat{G}} T_\rho\right)\right](g) = \frac{1}{|G|}\sum_{\rho\in\widehat{G}} d_\rho tr\left[\rho(g)^* T_\rho\right],$$

for all $\bigoplus_{\rho\in\widehat{G}} T_\rho \in \mathcal{A}(G)$ and $g \in G$.

For all $\rho \in \widehat{G}$ and $i, j = 1, 2, \ldots, d_\rho$ we define the element $T^\rho_{i,j} \in \mathcal{A}(G)$ by setting, if $w \in W_\sigma$, $\sigma \in \widehat{G}$,

$$T^\rho_{i,j} w = \delta_{\rho,\sigma} \langle w, v^\rho_j \rangle_{W_\rho} v^\rho_i.$$

In other words, the operator $T^\rho_{i,j}$ is trivial on the complement of $\mathrm{Hom}(W_\rho, W_\rho)$ and on this space it coincides with the linear operator associated with the tensor product $\theta_{v^\rho_j} \otimes v^\rho_i \in W'_\rho \otimes W_\rho (\cong \mathrm{Hom}(W_\rho, W_\rho))$; see (1.13).

Similarly to Corollary 1.5.8, we have:

Lemma 1.5.10 *The set $\{T^\rho_{i,j} : \rho \in \widehat{G}, i, j = 1, 2, \ldots, d_\rho\}$ is an orthogonal basis for $\mathcal{A}(G)$ and*

$$\langle T^\rho_{i,j}, T^\sigma_{s,t} \rangle_{HS} = \frac{d_\rho}{|G|} \delta_{\rho,\sigma} \delta_{i,s} \delta_{j,t}. \tag{1.41}$$

Proof Indeed,

$$\begin{aligned}
\langle T^\rho_{i,j}, T^\sigma_{s,t} \rangle_{HS} &= \frac{d_\rho}{|G|} \delta_{\rho,\sigma} \sum_{k=1}^{d_\rho} \langle T^\rho_{i,j} v^\rho_k, T^\sigma_{s,t} v^\rho_k \rangle_{W_\rho} \\
&= \frac{d_\rho}{|G|} \delta_{\rho,\sigma} \sum_{k=1}^{d_\rho} \delta_{j,k} \delta_{t,k} \langle v^\rho_i, v^\rho_s \rangle_{W_\rho} \\
&= \frac{d_\rho}{|G|} \delta_{\rho,\sigma} \delta_{i,s} \delta_{j,t}
\end{aligned}$$

and clearly $\dim \mathcal{A}(G) = \sum_{\rho \in \widehat{G}} d^2_\rho$. □

Theorem 1.5.11 *The Fourier transform $\mathcal{F}$ is an isometric $*$-isomorphism between the algebras $L(G)$ and $\mathcal{A}(G)$. Moreover, $\mathcal{F}^{-1}$ is the inverse of $\mathcal{F}$ and we have:*

$$\mathcal{F} \overline{\varphi^\rho_{i,j}} = \frac{|G|}{d_\rho} T^\rho_{i,j} \tag{1.42}$$

and

$$\mathcal{F}^{-1} T^\rho_{i,j} = \frac{d_\rho}{|G|} \overline{\varphi^\rho_{i,j}}. \tag{1.43}$$

Proof From the multiplicative property of $\mathcal{F}$ (see (1.38)) one immediately gets that $\mathcal{F}$ is a homomorphism of algebras. Also

$$\begin{aligned}
\langle \rho(\check{f})w_1, w_2\rangle_W &= \langle \sum_{g\in G} \check{f}(g)\rho(g)w_1, w_2\rangle_W \\
&= \langle \sum_{g\in G} \overline{f(g^{-1})}\rho(g)w_1, w_2\rangle_W \\
\text{(setting } s = g^{-1}\text{)} \quad &= \langle \sum_{s\in G} \overline{f(s)}\rho(s^{-1})w_1, w_2\rangle_W \\
&= \langle w_1, \sum_{s\in G} f(s)\rho(s)w_2\rangle_W \\
&= \langle w_1, \rho(f)w_2\rangle_W
\end{aligned}$$

for all $f \in L(G)$ and $w_1, w_2 \in W$. We deduce that $\rho(\check{f}) = \rho(f)^*$ for all $f \in L(G)$, that is, $\mathcal{F}$ preserves the involutions.

Moreover, since $\|\varphi^{\rho}_{i,j}\|^2_{L(G)} = |G|/d_\rho$ (the orthogonal relations, Lemma 1.5.7) and $\|T^{\rho}_{i,j}\|^2_{HS} = d_\rho/|G|$ by (1.41), to end the proof of the whole statement we just need to show (1.42) and (1.43).

Recall that $\overline{\varphi^{\rho}_{i,j}} = \varphi^{\rho'}_{i,j}$, where ρ' is the adjoint of ρ (see Lemma 1.1.9). Now, for $\sigma, \rho \in \widehat{G}$ we have

$$\sigma(\overline{\varphi^{\rho}_{i,j}}) = \sum_{g\in G} \sigma(g)\overline{\varphi^{\rho}_{i,j}(g)}$$

and therefore

$$\begin{aligned}
\langle \sigma(\overline{\varphi^{\rho}_{i,j}})v^{\sigma}_t, v^{\sigma}_s\rangle_{W_\sigma} &= \sum_{g\in G} \overline{\varphi^{\rho}_{i,j}(g)}\langle \sigma(g)v^{\sigma}_t, v^{\sigma}_s\rangle_{W_\sigma} \\
\text{(by the orthogonality relations)} \quad &= \frac{|G|}{d_\rho}\delta_{\sigma,\rho}\delta_{i,s}\delta_{j,t} \\
&= \frac{|G|}{d_\rho}\delta_{\sigma,\rho}\langle T^{\rho}_{i,j}v^{\sigma}_t, v^{\sigma}_s\rangle_{W_\sigma}
\end{aligned}$$

and this shows (1.42). We now prove (1.43).

$$\begin{aligned}
[\mathcal{F}^{-1}T^{\rho}_{i,j}](g) &= \frac{d_\rho}{|G|} tr[\rho(g)^* T^{\rho}_{i,j}] \\
&= \frac{d_\rho}{|G|}\sum_{t=1}^{d_\rho} \langle T^{\rho}_{i,j}v^{\rho}_t, \rho(g)v^{\rho}_t\rangle_{W_\rho} \\
&= \frac{d_\rho}{|G|}\langle v^{\rho}_i, \rho(g)v^{\rho}_j\rangle_{W_\rho} \\
&= \frac{d_\rho}{|G|}\overline{\varphi^{\rho}_{i,j}(g)}. \qquad \square
\end{aligned}$$

Corollary 1.5.12 *Let $\mathcal{C}$ be a conjugacy class of G, $\mathbf{1}_{\mathcal{C}}$ its characteristic function and $T_{\mathbf{1}_{\mathcal{C}}}$ the associated convolution operator (that is, $T_{\mathbf{1}_{\mathcal{C}}} f = f * \mathbf{1}_{\mathcal{C}}$ for all $f \in L(G)$). Then for each $\rho \in \widehat{G}$ the isotypic component $d_\rho W_\rho$ in $L(G)$ is an eigenspace of $T_{\mathcal{C}}$, and the associated eigenvalue is*

$$|\mathcal{C}| \frac{\chi^\rho(g)}{d_\rho},$$

where $\chi^\rho(g)$ does not depend on the particular element $g \in \mathcal{C}$.

Proof It is an immediate consequence of Theorem 1.5.11 and Lemma 1.3.13. □

Corollary 1.5.13 *For $f \in L(G)$, the operator $\oplus_{\rho\in\widehat{G}} \rho(f)$ is self-adjoint (resp. normal) if and only if $f = \check{f}$ (resp. $f * \check{f} = \check{f} * f$).*

Corollary 1.5.14 (Fourier inversion formula) *For every $f \in L(G)$ we have*

$$f(g) = \frac{1}{|G|} \sum_{\rho\in\widehat{G}} d_\rho tr[\rho(g)^* \rho(f)].$$

Corollary 1.5.15 (Plancherel formula) *For $f_1, f_2 \in L(G)$ we have*

$$\langle f_1, f_2 \rangle_{L(G)} = \frac{1}{|G|} \sum_{\rho\in\widehat{G}} d_\rho tr[\rho(f_1)\rho(f_2)^*].$$

From Lemma 1.3.13, the fact that the characters span the space of central functions, and the Fourier inversion formula we get:

Corollary 1.5.16 (Characterization of central functions) *A function $f \in L(G)$ is central if and only if the Fourier transform $\rho(f)$ is a multiple of the identity I_W for all irreducible representations $(\rho, W) \in \widehat{G}$.*

With the choice of an orthonormal basis $\{v_1^\rho, v_2^\rho, \ldots, v_{d_\rho}^\rho\}$ in each irreducible representation W_ρ, $\rho \in \widehat{G}$, we associate the maximal commutative subalgebra $\mathcal{B}$ of $L(G) \cong \mathcal{A}(G)$ (by the Fourier transform, see (1.39)) consisting of all $f \in L(G)$ whose Fourier transform $\oplus_{\rho\in\widehat{G}} \rho(f)$ is a diagonal operator in the given basis. In other words, $f \in L(G)$ belongs to $\mathcal{B}$ if and only if each vector v_j^ρ, $j = 1, 2, \ldots, d_\rho$, is an eigenvector of $\rho(f)$, for all $\rho \in \widehat{G}$.

The *primitive idempotent* associated with the vector v_j^ρ is the group algebra element e_j^ρ given by

$$e_j^\rho(g) = \frac{d_\rho}{|G|} \overline{\varphi_{j,j}^\rho(g)} \equiv \frac{d_\rho}{|G|} \overline{\langle \rho(g) w_j^\rho, w_j^\rho \rangle_{W_\rho}} \tag{1.44}$$

for all $g \in G$.

The following proposition is an immediate consequence of the previous results.

Proposition 1.5.17

(i) *The set* $\{e_j^\rho : \rho \in \widehat{G}, j = 1, 2, \dots, d_\rho\}$ *is a (vector space) basis for* $\mathcal{B}$;

(ii) $e_j^\rho * e_i^\sigma = \delta_{\rho,\sigma}\delta_{j,i}e_j^\rho$, *for all* $\rho, \sigma \in \widehat{G}$, $j = 1, 2, \dots, d_\rho$ *and* $i = 1, 2, \dots d_\sigma$;

(iii) $\sigma(e_j^\rho)v_i^\sigma = \delta_{\rho,\sigma}\delta_{j,i}e_j^\rho$, *for all* $\rho, \sigma \in \widehat{G}$, $j = 1, 2, \dots, d_\rho$ *and* $i = 1, 2, \dots d_\sigma$; *in particular,* $\rho(e_j^\rho) : W_\rho \to W_\rho$ *is the orthogonal projection onto the* 1*-dimensional subspace spanned by* v_j^ρ;

(iv) *let* $f \in \mathcal{B}$ *and* $\rho(f)v_j^\rho = \lambda_j^\rho v_j^\rho$, *with* $\lambda_j^\rho \in \mathbb{C}$, $\rho \in \widehat{G}$ *and* $j = 1, 2, \dots, d_\rho$, *then*

$$f = \sum_{\rho\in\widehat{G}} \sum_{j=1}^{d_\rho} \lambda_j^\rho e_j^\rho \quad \textit{(Fourier inversion formula in } \mathcal{B}\textit{)}$$

and

$$f * e_j^\rho = \lambda_j^\rho e_j^\rho.$$

Exercise 1.5.18 (1) Show that a function $\psi \in L(G)$ belongs to $\mathcal{T}_j^\rho W_\rho$ (see Corollary 1.5.5) if and only if $\psi * e_j^\sigma = \delta_{\sigma\rho}\delta_{ij}\psi$, for all $\sigma \in \widehat{G}$ and $i = 1, 2, \dots, d_\sigma$.

(2) Show that a function $f \in L(G)$ belongs to $\mathcal{B}$ if and only if each subspace $\mathcal{T}_j^\rho W_\rho$ is an eigenspace for the associated convolution operator $T_f\psi = \psi * f$; moreover, if $f = \sum_{\rho\in G}\sum_{j=1}^{d_\rho} \lambda_j^\rho e_j^\rho$, then the eigenvalue of T_f associated with $\mathcal{T}_j^\rho W_\rho$ is precisely λ_j^ρ.

1.5.3 Algebras of bi-K-invariant functions

In this section we shall continue the analysis of the structure of the algebra $L(G)$. Now, we express some results in Section 1.4 in the group algebra setting.

Let G be a group acting transitively on a (finite) set X. Fix $x_0 \in X$ and denote by K the stabilizer of x_0. Then $X = G/K$.

Let S be a set of representatives for the double cosets $K\backslash G/K$ of K in G. This means that

$$G = \coprod_{s\in S} KsK,$$

with disjoint union. In other words, $\{KsK : s \in S\}$ are the equivalence classes for the relation

$$g \sim h \Leftrightarrow \exists k_1, k_2 \in K : g = k_1 h k_2$$

for $g, h \in G$.

For $s \in S$, set $\Omega_s = \{ksx_0 : k \in K\} \equiv Ksx_0 (\equiv KsKx_0) \subset X$ and $\Theta_s = \{(gx_0, gsx_0) : g \in G\} \equiv G(x_0, sx_0) \subset X \times X$.

Lemma 1.5.19 *With the previous notation, we have:*

(i) $X = \coprod_{s\in S} \Omega_s$ *is the decomposition of* X *into* K*-orbits.*
(ii) $X \times X = \coprod_{s\in S} \Theta_s$ *is the decomposition of* $X \times X$ *into the* G*-orbits under the diagonal action:* $g(x, y) = (gx, gy)$*, for* $x, y \in X$ *and* $g \in G$*.*

Proof (i) If $g \in G$, then $KgKx_0 \equiv Kgx_0$ is the K-orbit of gx_0. Moreover, if h is another element of G, we have

$$\begin{aligned} Kgx_0 = Khx_0 &\Leftrightarrow \exists k_1 \in K \text{ s.t. } gx_0 = k_1 h x_0 \\ &\Leftrightarrow \exists k_1, k_2 \in K \text{ s.t. } g = k_1 h k_2. \end{aligned}$$

Therefore, if S is a system of representatives for $K\backslash G/K$, then $\{ksx_0 : k \in K\}$, $s \in S$, are the K-orbits on X.

(ii) We first note that any G-orbit on $X \times X$ is of the form $G(x_0, gx_0)$ for a suitable $g \in G$. Indeed, if $x, y \in X$, say $x = g_0 x_0$ with $g_0 \in G$, and we take $g \in G$ such that $gx_0 = g_0^{-1} y$, then

$$G(x, y) = G(g_0 x_0, g_0 g_0^{-1} y) = G(x_0, gx_0).$$

Moreover, if $g, h \in G$, then

$$\begin{aligned} G(x_0, gx_0) = G(x_0, hx_0) &\Leftrightarrow \exists k_1 \in K \text{ s.t. } (x_0, gx_0) = (x_0, k_1 h x_0) \\ &\Leftrightarrow \exists k_1, k_2 \in K \text{ s.t. } g = k_1 h k_2. \end{aligned}$$

□

Exercise 1.5.20 For every orbit Ω of K on X, set $\Theta_\Omega = \{(gx_0, gx) : g \in G, x \in \Omega\}$. Show that the map $\Omega \mapsto \Theta_\Omega$ is a bijection between the K-orbits on X and the G-orbits on $X \times X$.

Definition 1.5.21 Let G be a finite group and let $K \leq G$ be a subgroup. A function $f \in L(G)$ is *right* (resp. *left*) *K-invariant* when $f(gk) = f(g)$ (resp. $f(kg) = f(g)$) for all $g \in G$ and $k \in K$. A function $f \in L(G)$ is *bi-K-invariant* if

$$f(k_1 g k_2) = f(g)$$

for all $k_1, k_2 \in K$ and $g \in G$.

We denote by $L(G/K)$ and by $L(K\backslash G/K)$ the subspaces of right K invariant and bi-K-invariant functions on G, respectively. Note that if f_1 is left K-invariant then $f_1 * f$ is left K-invariant and, symmetrically, if f_2 is right K-invariant, then $f * f_2$ is right K-invariant, for all $f \in L(G)$. In other words, $L(G/K)$ is a left-ideal in $L(G)$ and, cf. (1.32), it is invariant with respect to the left regular representation, while $L(K\backslash G/K)$ is a two-sided ideal.

Recall that $L(X \times X)^G$ denotes the set of all functions on $X \times X$ which are constant on the G-orbits.

Theorem 1.5.22

(i) *For every $f \in L(X)$ define the function $\widetilde{f} \in L(G)$ by setting*

$$\widetilde{f}(g) = f(gx_0)$$

for all $g \in G$. Then the map $f \mapsto \widetilde{f}$ is a linear isomorphism between $L(X)$ and $L(G/K)$. Moreover, it intertwines the permutation representation of G on $L(X)$ with the restriction to $L(G/K)$ of the left regular representation of G on $L(G)$.

(ii) *For every $F \in L(X \times X)^G$ define the function $\widetilde{F} \in L(G)$ by setting*

$$\widetilde{F}(g) = \frac{1}{|K|} F(x_0, gx_0)$$

for all $g \in G$. Then $\widetilde{F} \in L(K\backslash G/K)$ and the map

$$\begin{array}{ccc} L(X \times X)^G & \to & L(K\backslash G/K) \\ F & \mapsto & \widetilde{F} \end{array}$$

is an isomorphism of algebras.

Proof (i) This is trivial: X is the same thing as the space of all right cosets G/K of K in G.

(ii) It is easy to see that $\widetilde{F} \in L(K\backslash G/K)$ and that the map $F \mapsto \widetilde{F}$ is an isomorphism of vector spaces (cf. Lemma 1.5.19). Now suppose that $F_1, F_2 \in L(X \times X)^G$ and set

$$F(x, y) = \sum_{z \in X} F_1(x, z) F_2(z, y)$$

for all $x, y \in X$. Then, for all $g \in G$,

$$\begin{aligned} \widetilde{F}(g) &= \frac{1}{|K|^2} \sum_{h \in G} F_1(x_0, hx_0) F_2(hx_0, gx_0) \\ &= \frac{1}{|K|^2} \sum_{h \in G} F_1(x_0, hx_0) F_2(x_0, h^{-1} gx_0) \\ &= [\widetilde{F}_1 * \widetilde{F}_2](g). \end{aligned}$$

□

Denote by $L(X)^K$ the space of K-invariant functions on $L(X)$, that is, $f \in L(X)^K$ if and only if it is constant on the K-orbits on X.

Combining Lemma 1.5.19 and Theorem 1.5.22, we have the following isomorphisms:

$$L(X \times X)^G \cong L(K \backslash G/K) \cong L(X)^K.$$

As the first two spaces above are algebras, $L(X)^K$ can be endowed with a structure of an algebra as well. From Theorem 1.5.22, Corollary 1.2.17 and Proposition 1.4.1 we also have the following.

Corollary 1.5.23 *(G, K) is a Gelfand pair if and only if the algebra $L(K \backslash G/K)$ is commutative.* □

Note that the above is usually taken as the definition of a Gelfand pair. We end this section by presenting Gelfand's lemma (cf. Proposition 1.4.8) in its general form.

Proposition 1.5.24 (Gelfand's lemma) *Let G be a finite group and let $K \leq G$ be a subgroup. Suppose there exists an automorphism τ of G such that $g^{-1} \in K\tau(g)K$ for all $g \in G$. Then (G, K) is a Gelfand pair.*

Proof If $f \in L(K \backslash G/K)$ we have $f(\tau(g)) = f(g^{-1})$ for all $g \in G$. Then, for $f_1, f_2 \in L(K \backslash G/K)$ and $g \in G$ we have:

$$\begin{aligned}
[f_1 * f_2](\tau(g)) &= \sum_{h \in G} f_1(\tau(gh)) f_2(\tau(h^{-1})) \\
&= \sum_{h \in G} f_1((gh)^{-1}) f_2(h) \\
&= \sum_{h \in G} f_2(h) f_1(h^{-1}g^{-1}) \\
&= [f_2 * f_1](g^{-1}) \\
&= [f_2 * f_1](\tau(g))
\end{aligned}$$

and therefore $L(K \backslash G/K)$ is commutative. □

Exercise 1.5.25 Show that (G, K) is a symmetric Gelfand pair (see Definition 1.4.9) if and only if $g^{-1} \in KgK$ for all $g \in G$. Note that this corresponds to the case $\tau(g) = g$ for all $g \in G$ in the previous proposition.

We also say that (G, K) is a *weakly symmetric* Gelfand pair when it satisfies Gelfand's lemma (Proposition 1.5.24) for some automorphism τ.

Example 1.5.26 $G \times G$ acts on G as follows: $(g_1, g_2) \cdot g = g_1 g g_2^{-1}$, for all $g_1, g_2, g \in G$. The stabilizer of 1_G is clearly the diagonal subgroup

$\widetilde{G} = \{(g, g) : g \in G\} \leq G \times G$. Therefore, $G \equiv (G \times G)/\widetilde{G}$. We now show that $(G \times G, \widetilde{G})$ is a weakly symmetric Gelfand pair. Indeed, the flip automorphism defined by $\tau(g_1, g_2) = (g_2, g_1)$ gives

$$(g_1, g_2)^{-1} = (g_1^{-1}, g_2^{-1}) = (g_1^{-1}, g_1^{-1})(g_2, g_1)(g_2^{-1}, g_2^{-1}) \in \widetilde{G}\tau(g_1, g_2)\widetilde{G},$$

for all $g_1, g_2 \in G$.

A group G is *ambivalent* if g^{-1} is conjugate to g for every $g \in G$.

Exercise 1.5.27 Show that the Gelfand pair $(G \times G, \widetilde{G})$ is symmetric if and only if G is ambivalent.

Let (G, K) be a Gelfand pair and let $L(X) = \bigoplus_{\rho \in J} W_\rho$ be the (multiplicity free) decomposition of $L(X)$ into G-irreducible sub-representations. For each W_ρ, let v^ρ be a K-invariant vector in W_ρ such that $\|v^\rho\|_{W_\rho} = 1$ (recall that $\dim W_\rho^K = 1$; see Corollary 1.4.13). The *spherical function* associated with W_ρ is the bi-K-invariant matrix coefficient

$$\varphi_\rho(g) = \langle v^\rho, \rho(g)v^\rho \rangle_{W_\rho}.$$

Consider the isomorphism $L(X) \ni f \mapsto \widetilde{f} \in L(G/K)$ (see Theorem 1.5.22). Let $\phi_\rho \in L(X)$ denote the K-invariant function defined by setting $\phi_\rho(x) = \varphi_\rho(g)$ if $gx_0 = x$. Then $\varphi_\rho = \widetilde{\phi}_\rho$.

Exercise 1.5.28 (1) Show that the set $\{\varphi_\rho : \rho \in J\}$ is an orthogonal basis for $L(K\backslash G/K)$ and that $\|\varphi_\rho\|^2_{L(G)} = |G|/\dim W_\rho$.

(2) Let χ^ρ be the character of W_ρ and set $d_\rho = \dim W_\rho$. Show that

$$\varphi_\rho(g) = \frac{1}{|K|} \sum_{k \in K} \overline{\chi^\rho(gk)}$$

and that

$$\chi^\rho(g) = \frac{d_\rho}{|G|} \sum_{h \in G} \overline{\varphi_\rho(h^{-1}gh)}.$$

Hint. Take an orthonormal basis $w_1, w_2, \ldots, w_{d_\rho}$ in W_ρ such that $w_1 = v^\rho$, the K-invariant vector, and use Lemma 1.3.5 and the orthogonality relations in Lemma 1.5.7.

For more on spherical functions, the spherical Fourier transform, and their applications (in the finite setting), see our monograph [20] and Terras' book [117].

1.6 Induced representations

1.6.1 Definitions and examples

Let G be a finite group, K a subgroup of G and (ρ, V) a representation of K.

Definition 1.6.1 (Induced representation) The *representation induced* by (ρ, V) is the G-representation (σ, Z) defined by setting

$$Z = \{f : G \to V : f(gk) = \rho(k^{-1})f(g), \text{ for all } g \in G, k \in K\} \tag{1.45}$$

and

$$[\sigma(g_1)f](g_2) = f(g_1^{-1}g_2), \qquad \text{for all } g_1, g_2 \in G \text{ and } f \in Z. \tag{1.46}$$

Note that $\sigma(g)f \in Z$ for all $g \in G$ and $f \in Z$. One denotes σ and Z by $\mathrm{Ind}_K^G \rho$ and $\mathrm{Ind}_K^G V$, respectively.

We now give an alternative description of $\mathrm{Ind}_K^G V$. Suppose that $G = \coprod_{s \in S} sK$, that is, S is a system of representatives for the set G/K of right cosets of K in G. For every $v \in V$ define the function $f_v : G \to V$ by setting

$$f_v(g) = \begin{cases} \rho(g^{-1})v & \text{if } g \in K \\ 0 & \text{otherwise.} \end{cases} \tag{1.47}$$

Then $f_v \in Z$ and the space $V' = \{f_v : v \in V\}$ is K-invariant and K-isomorphic to V. Indeed, $\sigma(k)f_v = f_{\rho(k)v}$ for all $k \in K$. Moreover, we have the following direct decomposition:

$$Z = \bigoplus_{s \in S} \sigma(s)V'. \tag{1.48}$$

Indeed, for any $f \in Z$ and $s \in S$ set $v_s = f(s)$. It is immediate to check that

$$f = \sum_{s \in S} \sigma(s) f_{v_s} \tag{1.49}$$

and that such an expression is unique (this is just a rephrasing of the fact that $f \in Z$ is determined by its values on S as indicated in (1.45)). Moreover, if we fix $g \in G$, for every $s \in S$ we can define $t_s \in S$ and $k_s \in K$ as the unique elements such that $gs = t_s k_s$. We then have

$$\sigma(g)f = \sum_{s \in S} \sigma(gs) f_{v_s} = \sum_{s \in S} \sigma(t_s)\sigma(k_s) f_{v_s} = \sum_{s \in S} \sigma(t_s) f_{\rho(k_s)v_s}. \tag{1.50}$$

Conversely we have:

Lemma 1.6.2 *Let G, $K \le G$ and S as above and let (τ, W) be a representation of G. Suppose that $V \le W$ is a K-invariant subspace and that the direct* *decomposition*

$$W = \bigoplus_{s \in S} \tau(s)V \tag{1.51}$$

holds. Then the G*-representations* W *and* $\mathrm{Ind}_K^G V$ *are isomorphic.*

Proof Defining V' as above, from (1.48) we have that $Z = \mathrm{Ind}_K^G V$ and W are G-isomorphic. Indeed, if $w \in W$ and $w = \sum_{s \in S} \tau(s)v_s$, with $v_s \in V$, then, as in (1.50), we have $\tau(g)w = \sum_{s \in S} \tau(t_s)[\tau(k_s)v_s]$, that is, under the map $w \mapsto f$ we also have $\tau(g)w \mapsto \tau(g)f$. □

Exercise 1.6.3 Let K be a subgroup of G and let S be a set of the representatives of the right cosets of K in G, so that $G = \coprod_{s \in S} sK$. Let (π, W) be a representation of G, suppose that $V \leq W$ is K-invariant and denote by (ρ, V) the corresponding K-representation.

Show that if $W =< \pi(s)V : s \in S >$, then there exists a surjective map that intertwines π and $\mathrm{Ind}_K^G \rho$.

Hint. Setting $\sigma = \mathrm{Ind}_K^G \rho$, the required surjective intertwiner is the linear extension of $\sigma(s)v \mapsto \pi(s)v$, $s \in S$ and $v \in V$. Indeed, if $g \in G$ and $gs = tk$, with $s, t \in S$ and $k \in K$, then

$$\sigma(g)[\sigma(s)v] = \sigma(t)[\rho(k)v] \mapsto \pi(t)[\pi(k)v] = \pi(g)[\pi(s)v].$$

We observe that the dimension of the induced representation is given by

$$\dim(\mathrm{Ind}_K^G V) = [G : K]\dim(V) \tag{1.52}$$

as immediately follows from the previous lemma and from $|S| = [G : K] \equiv |G/K|$.

Example 1.6.4 (Permutation representation) Suppose that G acts transitively on a finite set X. Fix a point $x_0 \in X$ and denote by $K = \{g \in G : gx_0 = x_0\}$ its stabilizer; thus $X = G/K$ is the set of right cosets of K. Denote by $(\iota, \mathbb{C})$ the *trivial* (one-dimensional) representation of K. Then $\mathrm{Ind}_K^G \mathbb{C}$ is nothing but the space $\{f : G \to \mathbb{C} : f(gk) = f(g), \forall g \in G, k \in K\}$ of all right K-invariant functions on G. As the latter coincides with $L(X)$ (see (i) in Theorem 1.5.22) we have:

Proposition 1.6.5 *Let* G *be a group,* $K \leq G$ *be a subgroup and let* $X = G/K$*. The permutation representation of* G *on* $L(X)$ *coincides with the representation obtained by inducing the trivial representation of the subgroup* K*.*

1.6.2 First properties of induced representations

One of the first important properties of induction is transitivity.

Proposition 1.6.6 (Induction in stages) *Let G be a finite group, $K \leq H \leq G$ be subgroups and (ρ, V) a representation of K. Then*

$$\mathrm{Ind}_H^G(\mathrm{Ind}_K^H V) \cong \mathrm{Ind}_K^G V \tag{1.53}$$

as G-representations.

Proof By definition we have

$$\mathrm{Ind}_K^H V = \{f' : H \to V : f'(hk) = \rho(k^{-1})f'(h),\ \forall h \in H, k \in K\}$$

and, setting $\sigma = \mathrm{Ind}_K^H \rho$,

$$\mathrm{Ind}_H^G(\mathrm{Ind}_K^H V) = \{f : G \to \mathrm{Ind}_K^H V : f(gh) = \sigma(h^{-1})f(g), \forall g \in G, h \in H\}.$$

Thus $\mathrm{Ind}_H^G(\mathrm{Ind}_K^H V)$ may be identified with the function space

$$\begin{aligned} U = \{F : G \times H \to V : F(gh, h'k) \\ = \rho(k^{-1})F(g, hh'),\ \forall g \in G, h, h' \in H, k \in K\} \end{aligned}$$

by setting $F(g, h) = [f(g)](h)$ for all $g \in G$ and $h \in H$. If $F \in U$, then $F(g, h) = F(gh, 1_G)$, so that F is uniquely determined by its values on $G \times \{1_G\}$. Thus, setting $\widetilde{F}(g) = F(g, 1_G)$ we have that $\widetilde{F} : G \to V$ and

$$\widetilde{F}(gk) = F(gk, 1_G) = F(g, k) = \rho(k^{-1})F(g, 1_G) = \rho(k^{-1})\widetilde{F}(g)$$

and $\{\widetilde{F} : F \in U\}$ is G-isomorphic to $\mathrm{Ind}_K^G V$. This completes the proof: the isomorphism is $F \mapsto \widetilde{F}$. □

Theorem 1.6.7 (Frobenius' character formula for induced representations) *Let (ρ, W) be a representation of K and denote by χ^ρ its character. Then the character $\chi^{\mathrm{Ind}_K^G(\rho)}$ of the induced representation is given by*

$$\chi^{\mathrm{Ind}_K^G(\rho)}(g) = \sum_{\substack{s \in S: \\ s^{-1}gs \in K}} \chi^\rho(s^{-1}gs), \tag{1.54}$$

where S is any system of representatives for G/K.

Proof Let $\{e_1, e_2, \ldots, e_n\}$ be an orthonormal basis for W and set $f_{s,j} = \sigma(s)f_{e_j}$, where f_{e_j} is as in (1.47), $\sigma = \mathrm{Ind}_K^G \rho$ and $G = \coprod_{s \in S} sK$ (as in (1.48)). Then $\{f_{s,j} : s \in S, j = 1, 2, \ldots, n\}$ is a basis for $\mathrm{Ind}_K^G W$.

Now suppose that $g, g_0 \in G$, with $g_0 = sk$ where $s \in S$ and $k \in K$. Then there exist $s_1 \in S$ and $\tilde{k} \in K$ such that $g^{-1}s = s_1\tilde{k}$ and therefore

$$\begin{aligned}[\sigma(g)f_{s,j}](g_0) &= f_{s,j}(g^{-1}g_0) = \\ &= f_{s,j}(s_1\tilde{k}k) = \\ &= \delta_{s,s_1}\rho(k^{-1})\rho(\tilde{k}^{-1})e_j.\end{aligned}$$

But $\tilde{k} = s_1^{-1}g^{-1}s$ and we have $s = s_1$ if and only if $g^{-1}s \in sK$, that is, if and only if $s^{-1}gs \in K$. Hence we can write

$$[\sigma(g)f_{s,j}](g_0) = \begin{cases} \rho(k^{-1})\rho(s^{-1}gs)e_j & \text{if } s^{-1}gs \in K \\ 0 & \text{otherwise.} \end{cases} \tag{1.55}$$

The coefficient of $f_{s,j}$ in $\sigma(g)f_{s,j}$ is equal to the coefficient of $f_{s,j}(g_0)$ in $[\sigma(g)f_{s,j}](g_0)$. In particular, from (1.55) it follows that it is equal to zero if $s^{-1}gs \notin K$.

Moreover, if $\widetilde{k} = s^{-1}g^{-1}s \in K$ and

$$\rho(\widetilde{k}^{-1})e_j = \sum_{r=1}^{n} \alpha_{r,j}(\widetilde{k})e_r$$

then

$$\begin{aligned}[\sigma(g)f_{s,j}](g_0) &= \sum_{r=1}^{n} \alpha_{r,j}(\widetilde{k})\rho(k^{-1})e_r \\ (f_{s,r}(g_0) = f_{e_r}(k)) \quad &= \sum_{r=1}^{n} \alpha_{r,j}(\widetilde{k})f_{s,r}(g_0)\end{aligned}$$

and the coefficient of $f_{s,j}(g_0)$ in $[\sigma(g)f_{s,j}](g_0)$ equals $\alpha_{j,j}(\widetilde{k})$.

Since $\chi^\rho(\widetilde{k}^{-1}) = \sum_{j=1}^{n} \alpha_{j,j}(\widetilde{k})$ and $\widetilde{k}^{-1} = s^{-1}gs$, we have

$$\chi^\sigma(g) = \sum_{\substack{s\in S:\\ s^{-1}gs\in K}} \sum_{j=1}^{n} \alpha_{j,j}(\widetilde{k}) = \sum_{\substack{s\in S:\\ s^{-1}gs\in K}} \chi^\rho(s^{-1}gs).$$

□

Exercise 1.6.8 Prove that the right-hand side of (1.54) does not depend on the choice of S.

The following lemma is a prelude on the connection between induction and restriction.

Lemma 1.6.9 *Let (θ, V) be a representation of a group G and (σ, W) a representation of a subgroup $H \leq G$. Then we have the following isomorphism of G-representations*

$$V \otimes \mathrm{Ind}_H^G W \cong \mathrm{Ind}_H^G[(\mathrm{Res}_H^G V) \otimes W]. \tag{1.56}$$

Proof First of all, the right-hand side in (1.56) is made up of all functions $F : G \to V \otimes W$ such that $F(gh) = [\theta(h^{-1}) \otimes \sigma(h^{-1})]F(g)$, for all $h \in H$ and $g \in G$. Moreover, the left-hand side is spanned by all products $v \otimes f$ where $v \in V$ and $f : G \to W$ satisfies $f(gh) = \sigma(h^{-1})f(g)$, for all $h \in H$ and $g \in G$.

It is easy to see that the map $\phi : V \otimes \mathrm{Ind}_H^G W \to \mathrm{Ind}_H^G[(\mathrm{Res}_H^G V) \otimes W]$ defined by setting $[\phi(v \otimes f)](g) = \theta(g^{-1})v \otimes f(g)$ is a G-equivariant linear isomorphism. □

Corollary 1.6.10 *If (θ, V) is a representation of a group G, H is a subgroup of G and $X = G/H$ then*

$$\mathrm{Ind}_H^G \mathrm{Res}_H^G V \cong V \otimes L(X) \tag{1.57}$$

as G-representations.

Proof Apply Lemma 1.6.9 with $\sigma = \iota_H$, the trivial representation of H. In this case, $\mathrm{Ind}_H^G W = L(X)$ and $(\mathrm{Res}_H^G V) \otimes W = (\mathrm{Res}_H^G V) \otimes \mathbb{C} \cong \mathrm{Res}_H^G V$. □

1.6.3 Frobenius reciprocity

The following fundamental result of Frobenius provides another relation between the operations of induction and restriction.

Theorem 1.6.11 (Frobenius reciprocity) *Let G be a finite group, $K \leq G$ a subgroup, (σ, W) a representation of G and (ρ, V) a representation of K. For every $T \in \mathrm{Hom}_G(W, \mathrm{Ind}_K^G V)$ define $\widehat{T} : W \to V$ by setting, for every $w \in W$,*

$$\widehat{T}w = [Tw](1_G). \tag{1.58}$$

Then $\widehat{T} \in \mathrm{Hom}_K(\mathrm{Res}_K^G W, V)$ and the map

$$\begin{aligned} \mathrm{Hom}_G(W, \mathrm{Ind}_K^G V) &\to \mathrm{Hom}_K(\mathrm{Res}_K^G W, V) \\ T &\mapsto \widehat{T} \end{aligned}$$

is a linear isomorphism.

Proof We first check that $\widehat{T} \in \mathrm{Hom}_K(\mathrm{Res}^G_K W, V)$. For $k \in K$ and $w \in W$ we have:

$$\begin{aligned}\widehat{T}\sigma(k)w &= [T(\sigma(k)w)](1_G)\\ &=_{*)} [\tau(k)(Tw)](1_G)\\ &=_{**)} [Tw](k^{-1})\\ &=_{***)} \rho(k)[Tw(1_G)]\\ &= \rho(k)\widehat{T}w\end{aligned} \tag{1.59}$$

where $\tau = \mathrm{Ind}^G_K\rho$ and the starred equalities depend on the following facts: $*$) because T is an intertwiner; $**$) by the definition of the action of G on $\mathrm{Ind}^G_K V$; $***$) by the defining property of the elements in $\mathrm{Ind}^G_K V$. On the other hand, for $U \in \mathrm{Hom}_K(\mathrm{Res}^G_K W, V)$ we can define $\check{U} : W \to \mathrm{Ind}^G_K V$ by setting, for every $w \in W$ and $g \in G$

$$[\check{U}w](g) = U\sigma(g^{-1})w.$$

Again, it is easy to check that $\check{U} \in \mathrm{Hom}_G(W, \mathrm{Ind}^G_K V)$. Moreover from

$$[Tw](g) = \left[\tau(g^{-1})(Tw)\right](1_G) = [T\sigma(g^{-1})w](1_G) = \widehat{T}\sigma(g^{-1})w \tag{1.60}$$

one deduces that the maps $T \to \widehat{T}$ and $U \to \check{U}$ are one inverse to the other and therefore we have established the required isomorphism. □

In particular, from Lemma 1.2.5 we deduce:

Corollary 1.6.12 *In the case that the representations W and V are irreducible we have that the multiplicity of W in $\mathrm{Ind}^G_K V$ equals the multiplicity of V in $\mathrm{Res}^G_K W$.* □

In the case of the trivial representation versus the permutation representation, namely Example 1.6.4, $\widehat{T} \in \mathrm{Hom}_K(W, \mathbb{C})$ and it can be expressed in the form

$$\widehat{T}w = [Tw](x_0),$$

where $x_0 \in X$ is the point stabilized by K. In this case, any element $\theta \in \mathrm{Hom}_K(W, \mathbb{C})$ may be uniquely expressed in the form

$$\theta(w) = \langle w, w_0\rangle_W$$

where w_0 is a K-invariant vector in W. Therefore, if $T \in \mathrm{Hom}_G(W, L(X))$ and w_0 is the K-invariant vector corresponding to $\widehat{T}$, we have $\widehat{T}w = \langle w, w_0\rangle_W$ and

$$[Tw](gx_0) = \langle w, \sigma(g)w_0\rangle_W \qquad \forall g \in G, w \in W, \tag{1.61}$$

that is, in the notation of (1.24), we have

$$T \equiv \sqrt{\frac{|X|}{d_\sigma}}\mathcal{T}_{w_0}.$$

1.6.4 Mackey's lemma and the intertwining number theorem

Let G be a finite group, $H, K \leq G$ two subgroups and (ρ, W) a representation of K. Let S be a system of representatives for the double cosets in $H\backslash G/K$ so that

$$G = \coprod_{s\in S} HsK.$$

For all $s \in S$, consider the subgroup $G_s = sKs^{-1} \cap H$ and define a representation (ρ_s, W_s) of G_s, by taking $W_s = W$ and setting $\rho_s(t)w = \rho(s^{-1}ts)w$, for all $t \in G_s$ and $w \in W_s$.

Exercise 1.6.13 Identify $H\backslash G/K$ with the set of H-orbits on $X = G/K$ and prove that the subgroup G_s is the stabilizer in H of the point $x_s = sK$ (compare with Lemma 1.5.19).

Theorem 1.6.14 (Mackey's lemma) *The representation* $\mathrm{Res}^G_H \mathrm{Ind}^G_K \rho$ *is isomorphic to the direct sum of the representations* $\mathrm{Ind}^H_{G_s} \rho_s$, $s \in S$, *i.e.*

$$\mathrm{Res}^G_H \mathrm{Ind}^G_K \rho \equiv \bigoplus_{s\in S} \mathrm{Ind}^H_{G_s} \rho_s. \tag{1.62}$$

Proof Let $Z_s = \{F : G \to W \text{ s.t. } F(hs'k) = \delta_{s,s'}\rho(k^{-1})F(hs), \ \forall h \in H, k \in K \text{ and } s' \in S\}$. Then, recalling (1.45), Z_s coincides with the subspace of functions in $Z = \mathrm{Ind}^G_K W$ which vanish outside HsK. Clearly

$$Z = \bigoplus_{s\in S} Z_s. \tag{1.63}$$

For $F \in Z_s$, we can define $f_s : H \to W$ by setting $f_s(h) = F(hs)$, for all $h \in H$. If $t \in G_s$, then we have $s^{-1}ts \in K$ and therefore

$$f_s(ht) = F(hts) = \rho[s^{-1}t^{-1}s]F(hs) = \rho_s(t^{-1})f_s(h)$$

so that $f_s \in \mathrm{Ind}^H_{G_s} W_s$.

Vice versa, given $f \in \mathrm{Ind}^H_{G_s} W_s$ we can define $F_s : G \to W$ by setting $F_s(hs'k) = \delta_{s,s'}\rho(k^{-1})f(h)$, for $k \in K, h \in H$ and $s' \in S$. F_s is well defined:

indeed if $hsk = h_1sk_1$, then $t := skk_1^{-1}s^{-1} = h^{-1}h_1 \in G_s$ and thus

$$\begin{aligned}\rho(k_1^{-1})f(h_1) &= \rho(k^{-1})[\rho(s^{-1}h^{-1}h_1s)f(h_1)]\\ &= \rho(k^{-1})[\rho_s(t)f(h_1)]\\ &= \rho(k^{-1})f(h_1t^{-1})\\ &= \rho(k^{-1})f(h).\end{aligned}$$

Also with $g = hs'k'$ we have $F_s(gk) = F_s(hs'k'k) = \delta_{s,s'}\rho(k^{-1}k'^{-1})f(h) = \rho(k^{-1})F_s(hs'k') = \rho(k^{-1})F_s(g)$ so that $F_s \in Z_s$.

This shows that the representation of H on Z_s is isomorphic to the representation obtained by inducing ρ_s from G_s to H. Combining this with (1.63), we end the proof. □

We now express Mackey's lemma in the setting of a permutation representation. Let H and K be two subgroups of G. Denote by $X = G/K$ the homogeneous space corresponding to K. Let $x_0 \in X$ be a point stabilized by K and S a set of representatives for the double cosets of $H\backslash G/K$. As in Exercise 1.6.13, for $s \in S$ we have that G_s is the stabilizer of sx_0 and, setting $\Omega_s = \{hsx_0 : h \in H\} \equiv Hsx_0$, then $X = \coprod_{s\in S}\Omega_s$ is the decomposition of X into H-orbits.

Let now $\rho = \iota_K$ be the trivial representation of K. Then ρ_s is the trivial representation of G_s and $\mathrm{Ind}_{G_s}^H\rho_s$ is the permutation representation of H on $L(\Omega_s)$.

Therefore, Mackey's lemma reduces to the decomposition

$$\mathrm{Res}_H^G L(X) = \bigoplus_{s\in S} L(\Omega_s). \tag{1.64}$$

The following is a very important application of Mackey's lemma.

Theorem 1.6.15 (Intertwining number theorem) *In the same hypotheses of Mackey's lemma, assume that* (σ, W) *is a representation of* H*. Then*

$$\dim\mathrm{Hom}_G(\mathrm{Ind}_H^G\sigma, \mathrm{Ind}_K^G\rho) = \sum_{s\in S}\dim\mathrm{Hom}_{G_s}(\mathrm{Res}_{G_s}^H\sigma, \rho_s).$$

Proof We have

$$\begin{aligned}\dim\mathrm{Hom}_G(\mathrm{Ind}_H^G\sigma, \mathrm{Ind}_K^G\rho) &=_{(*)} \dim\mathrm{Hom}_H(\sigma, \mathrm{Res}_H^G\mathrm{Ind}_K^G\rho)\\ &= \sum_{s\in S}\dim\mathrm{Hom}_H(\sigma, \mathrm{Ind}_{G_s}^H\rho_s)\\ &=_{(*)} \sum_{s\in S}\dim\mathrm{Hom}_{G_s}(\mathrm{Res}_{G_s}^H\sigma, \rho_s)\end{aligned}$$

where $=_{(*)}$ follows from Frobenius' reciprocity (Theorem 1.6.11) and the remaining one from Mackey's lemma. □

2
The theory of Gelfand–Tsetlin bases

In this chapter we develop the theory of Gelfand–Tsetlin bases for group algebras and permutations representations. A peculiarity of our approach is the description of the algebra of conjugacy invariant functions in terms of permutation representations, so that a commutative algebra may be related to a suitable Gelfand pair.

2.1 Algebras of conjugacy invariant functions

2.1.1 Conjugacy invariant functions

The algebras of conjugacy invariant functions in this section were considered in [10], [118] and by A. A.-A. Jucys [71] and E. P. Wigner in [124] and [125]. A generalization to the compact case was presented in [74].

Let G be a finite group and let $H \leq G$ be a subgroup. A function $f \in L(G)$ is H*-conjugacy invariant* if

$$f(h^{-1}gh) = f(g) \quad \text{for all } h \in H \text{ and } g \in G.$$

We denote by $\mathcal{C}(G, H)$ the set of all H-conjugacy invariant functions defined on G. It is an algebra under convolution: if $f_1, f_2 \in \mathcal{C}(G, H)$ then $f_1 * f_2 \in \mathcal{C}(G, H)$. Indeed, for $g \in G$ and $h \in H$ we have

$$\begin{aligned} [f_1 * f_2](h^{-1}gh) &= \sum_{s \in G} f_1(h^{-1}ghs) f_2(s^{-1}) \\ (t = hsh^{-1}) \quad &= \sum_{t \in G} f_1(gt) f_2(t^{-1}) \\ &= [f_1 * f_2](g). \end{aligned}$$

If $H = G$, then $\mathcal{C}(G, H)$ is the usual algebra of central functions on G.

We now consider the action of $G \times H$ on G defined by

$$(g, h) \cdot g_0 = g g_0 h^{-1} \tag{2.1}$$

for all $g, g_0 \in G$ and $h \in H$. The associated permutation representation, that we denote by η, is given by

$$[\eta(g, h) f](g_0) = f(g^{-1} g_0 h), \tag{2.2}$$

for all $f \in L(G)$, $g, g_0 \in G$ and $h \in H$.

Lemma 2.1.1

(i) *The stabilizer of* 1_G *under the action (2.1) is the* $(G \times H)$*-subgroup* $\widetilde{H} = \{(h, h) : h \in H\}$.

(ii) *Denote by* $L(\widetilde{H} \backslash (G \times H)/\widetilde{H})$ *the algebra of bi-*$\widetilde{H}$*-invariant functions on* $G \times H$. *Then the map*

$$\Phi : L(\widetilde{H} \backslash (G \times H)/\widetilde{H}) \to \mathcal{C}(G, H)$$

given by

$$[\Phi(F)](g) = |H| F(g, 1_G)$$

for all $F \in L(\widetilde{H} \backslash (G \times H)/\widetilde{H})$ *and* $g \in G$, *is a linear isomorphism of algebras. Moreover,*

$$\|\Phi(F)\|_{L(G)} = \sqrt{|H|} \|F\|_{L(G \times H)}. \tag{2.3}$$

Proof (i) It is obvious that $(g, h) \cdot 1_G = 1_G$ if and only if $g = h \in H$.

(ii) Observe that $F \in L(G \times H)$ is bi-$\widetilde{H}$-invariant if and only if

$$F(h_1 g h_2, h_1 h h_2) = F(g, h)$$

for all $g \in G$ and $h, h_1, h_2 \in H$. Therefore, if $F \in L(\widetilde{H} \backslash (G \times H)/\widetilde{H})$ and $f = \Phi(F)$, we have

$$f(h^{-1} g h) = |H| F(h^{-1} g h, h^{-1} h) = |H| F(g, 1_G) = f(g)$$

showing that $f \in \mathcal{C}(G, H)$. Moreover,

$$F(g, h) = F(g h^{-1} h, h) = F(g h^{-1}, 1_G) = \frac{1}{|H|} f(g h^{-1}),$$

and F is uniquely determined by f.

Conversely, if $f \in \mathcal{C}(G, H)$, then the function $F \in L(G \times H)$ defined by setting $F(g, h) = \frac{1}{|H|} f(g h^{-1})$, is clearly bi-$\widetilde{H}$-invariant.

This shows that Φ is a bijection. Linearity is obvious and (2.3) is left to the reader.

We are only left to show that Φ is multiplicative. Let $F_1, F_2 \in L(\widetilde{H}\backslash(G \times H)/\widetilde{H})$. Then

$$\begin{aligned}
[\Phi(F_1 * F_2)](g) &= |H|[F_1 * F_2](g, 1_G) \\
&= |H| \sum_{s \in G} \sum_{h \in H} F_1(gs, h) F_2(s^{-1}, h^{-1}) \\
&= |H| \sum_{s \in G} \sum_{h \in H} F_1(gsh^{-1}, 1_G) F_2(hs^{-1}, 1_G) \\
(sh^{-1} = t) \quad &= |H|^2 \sum_{t \in G} F_1(gt, 1_G) F_2(t^{-1}, 1_G) \\
&= [\Phi(F_1) * \Phi(F_2)](g).
\end{aligned}$$

□

We now decompose the permutation representation η (cf. (2.2)) into $(G \times H)$-irreducible sub-representations. Recall (see Theorem 1.3.17) that any irreducible representation of $G \times H$ is of the form $\sigma \boxtimes \rho$ for some irreducible representations σ and ρ of G and H, respectively. Note also that the adjoint of $\mathrm{Res}^G_H \sigma$ is precisely $\mathrm{Res}^G_H \sigma'$.

Theorem 2.1.2 *Let $(\sigma, V) \in \widehat{G}$ and $(\rho, W) \in \widehat{H}$ and denote by $(\rho', W') \in \widehat{H}$ the adjoint representation of (ρ, W). Given $T \in \mathrm{Hom}_{G \times H}(\sigma \boxtimes \rho, \eta)$, define $\widetilde{T} : W \to V'$ by setting*

$$[\widetilde{T} w](v) = [T(v \otimes w)](1_G) \tag{2.4}$$

for all $v \in V$ and $w \in W$. Then $\widetilde{T} \in \mathrm{Hom}_H(\rho, \mathrm{Res}^G_H \sigma')$ and the map

$$\begin{aligned}
\mathrm{Hom}_{G \times H}(\sigma \boxtimes \rho, \eta) &\to \mathrm{Hom}_H(\rho, \mathrm{Res}^G_H \sigma') \\
T &\mapsto \widetilde{T}
\end{aligned}$$

is a linear isomorphism.

Proof First of all, note that a linear map $T : V \otimes W \to L(G)$ belongs to $\mathrm{Hom}_{G \times H}(\sigma \boxtimes \rho, \eta)$ if and only if

$$\{T[(\sigma(g)v) \otimes (\rho(h)w)]\}(g_0) = [T(v \otimes w)](g^{-1} g_0 h) \tag{2.5}$$

for all $g, g_0 \in G$, $h \in H$, $v \in V$ and $w \in W$.

Therefore, if this is the case, and $\widetilde{T}$ is defined as in (2.4), we have

$$\begin{aligned}
[\widetilde{T} \rho(h) w](v) &= \{T[v \otimes \rho(h) w]\}(1_G) \\
(\text{by } (2.5)) \quad &= [T(v \otimes w)](h) \\
(\text{again by } (2.5)) \quad &= \{T[\sigma(h^{-1}) v \otimes w]\}(1_G) \\
&= (\widetilde{T} w)[\sigma(h^{-1}) v] \\
(\text{by } (1.6)) \quad &= [\sigma'(h) \widetilde{T} w](v)
\end{aligned}$$

for all $h \in H$, $v \in V$ and $w \in W$. This means that, for all $h \in H$, $\sigma'(h)\widetilde{T} = \widetilde{T}\rho(h)$, that is $\widetilde{T} \in \mathrm{Hom}_H(\rho, \mathrm{Res}^G_H \sigma')$. Note also that again by (2.5), we have

$$[T(v \otimes w)](g) = \{T[\sigma(g^{-1})v \otimes w]\}(1_G) = [\widetilde{T}w](\sigma(g^{-1})v), \tag{2.6}$$

so that T is uniquely determined by $\widetilde{T}$. This shows that the map $T \mapsto \widetilde{T}$ is injective.

It remains to show that this map is also surjective. This follows from the fact that (2.6) is its inversion formula, that is, given $\widetilde{T} \in \mathrm{Hom}_H(\rho, \mathrm{Res}^G_H \sigma')$, if we define $T \in \mathrm{Hom}(V \otimes W, L(G))$ by setting

$$[T(v \otimes w)](g_0) = [\widetilde{T}w](\sigma(g_0^{-1})v) \tag{2.7}$$

for $g_0 \in G$, $v \in V$ and $w \in W$, then $T \in \mathrm{Hom}_{G\times H}(\sigma \boxtimes \rho, \eta)$.

Indeed, if (2.7) holds, we have

$$\begin{aligned} \{T[(\sigma(g)v) \otimes (\rho(h)w)]\}(g_0) &= [\widetilde{T}\rho(h)w](\sigma(g_0^{-1}g)v) \\ (\text{since } \widetilde{T} \in \mathrm{Hom}_H(\rho, \mathrm{Res}^G_H \sigma')) \quad &= [\sigma'(h)\widetilde{T}w](\sigma(g_0^{-1}g)v) \\ &= [\widetilde{T}w](\sigma(h^{-1}g_0^{-1}g)v) \\ &= [T(v \otimes w)](g^{-1}g_0h) \end{aligned}$$

for all $g \in G$, $h \in H$, $v \in V$ and $w \in W$, that is (2.5) is satisfied and therefore $T \in \mathrm{Hom}_{G\times H}(\sigma \boxtimes \rho, \eta)$. □

From Lemma 1.2.5 we get:

Corollary 2.1.3 *The multiplicity of* $\sigma \boxtimes \rho$ *in* η *is equal to the multiplicity of* ρ *in* $\mathrm{Res}^G_H \sigma'$.

Given $\sigma \in \widehat{G}$ and $\rho \in \widehat{H}$, denote by $m_{\rho,\sigma}$ the multiplicity of ρ in $\mathrm{Res}^G_H \sigma'$ (that is, $m_{\rho,\sigma} = \dim\mathrm{Hom}_H(\rho, \mathrm{Res}^G_H \sigma')$).

Corollary 2.1.4 *The decomposition of* η *into irreducible sub-representations is given by*

$$\eta = \bigoplus_{\sigma\in\widehat{G}} \bigoplus_{\rho\in\widehat{H}} m_{\rho,\sigma}(\sigma \boxtimes \rho).$$

Suppose that $H = G$. Observe that now $(\eta, L(G))$ is a representation of $G \times G$. We already know that $(G \times G, \widetilde{G})$ is a Gelfand pair (see Example 1.5.26). We now present the corresponding decomposition into irreducible sub-representations.

Remark 2.1.5 We can give to (2.7) a more explicit form. Let $w_1, w_2, \ldots, w_m$ be a basis in W and take $v_j \in V$ such that $\theta_{v_j} = \widetilde{T}w_j$, $j = 1, 2, \ldots, m$, where

θ is the Riesz map (as in (1.4)). Then (2.7) becomes

$$[\widetilde{T}w_j](\sigma(g_0^{-1})v) = \theta_{v_j}(\sigma(g_0^{-1})v) = \langle v, \sigma(g_0)v_j\rangle.$$

That is, the set of all matrix coefficients of this form spans a subspace of $L(G)$ isomorphic to $\sigma \boxtimes \rho$. See also Corollary 2.1.6 and Exercise 2.1.8.

Corollary 2.1.6 ([40]) *Suppose that $H = G$. Then*

$$\eta = \bigoplus_{\rho\in\widehat{G}} \rho' \boxtimes \rho.$$

Moreover, the representation space of $\rho' \boxtimes \rho$ is M_ρ, the subspace of $L(G)$ spanned by all matrix coefficients $\varphi(g) = \langle \rho(g)v, w\rangle_{W_\rho}$, with $v, w \in W_\rho$. In other words, we have $L(G) = \bigoplus_{\rho\in\widehat{G}} M_\rho$. In particular, this gives another proof that $(G \times G, \widetilde{G})$ is a Gelfand pair (see Example 1.5.26).

Proof In our setting we have

$$m_{\rho,\sigma} = \dim\mathrm{Hom}_G(\rho, \sigma') = \begin{cases} 1 & \text{if } \rho \sim \sigma' \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, taking $\sigma = \rho'$ and $\widetilde{T} = I_{W_\rho}$ in (2.6), we get $T \in \mathrm{Hom}_{G\times G}(\rho' \boxtimes \rho, \eta)$. Also,

$$\begin{aligned}[T(\theta_v \otimes w)](g) &= [\rho'(g^{-1})\theta_v](w)\\ &= \theta_v[\rho(g)w]\\ &= \langle \rho(g)w, v\rangle\end{aligned}$$

for all $v, w \in W_\rho$ and $g \in G$, so that $T(\theta_v \otimes w) \in M_\rho$.

Since $\dim M_\rho = d_\rho^2$ (a basis for M_ρ is given by the matrix coefficients $\varphi_{i,j}^\rho$, $1 \le i, j \le d_\rho$, see Corollary 1.5.8), then $T(W_\rho' \otimes W_\rho) = M_\rho$. □

Exercise 2.1.7 Show that the spherical functions of the Gelfand pair $(G \times G, \widetilde{G})$ are the normalized characters, namely $\frac{1}{d_\rho}\chi^\rho$, $\rho \in \widehat{G}$.

Exercise 2.1.8 (1) Give a direct proof, by using the properties of the matrix coefficients, that $M_\rho \cong W_\rho' \otimes W_\rho$ and deduce Corollary 2.1.6.

(2) From (1) deduce Corollary 2.1.4.

Hint. (1) Expand $\varphi_{i,j}(g^{-1}g_0)$ and $\varphi_{i,j}(g_0 g)$ by means of Lemma 1.1.9.

(2) Analyze $\mathrm{Res}^{G\times G}_{H\times G}(\rho' \boxtimes \rho)$.

2.1.2 Multiplicity-free subgroups

Definition 2.1.9 Let H be a subgroup of G. We say that H is a *multiplicity-free subgroup* of G if for every $\sigma \in \widehat{G}$ the restriction $\mathrm{Res}^G_H \sigma$ is multiplicity-free, equivalently, if $\dim\mathrm{Hom}_H(\rho, \mathrm{Res}^G_H \sigma) \leq 1$ for all $\rho \in \widehat{H}$ and $\sigma \in \widehat{G}$.

Theorem 2.1.10 *The following conditions are equivalent:*

(i) *The algebra $\mathcal{C}(G, H)$ is commutative.*
(ii) *$(G \times H, \widetilde{H})$ is a Gelfand pair.*
(iii) *H is a multiplicity-free subgroup of G.*

Proof The equivalence between (ii) and (iii) follows from Corollary 2.1.4 and from the definition of Gelfand pair. The equivalence between (i) and (ii) follows from Lemma 2.1.1 and Corollary 1.5.23. □

The equivalence between (i) and (iii) was estabilished by Wigner in [124]. Another proof of this equivalence will be presented at the end of Section 7.4.6.

Remark 2.1.11 (a) When $H = G$ the conditions in the previous theorem always hold true. Indeed $\mathcal{C}(G, G)$ is the space of central functions, that is, the center of the algebra $L(G)$ (cf. Remark 1.5.1) and therefore it is commutative. (ii) corresponds to Example 1.5.26, while (iii), namely that G is a multiplicity-free subgroup of itself, is trivially satisfied.

(b) On the other hand, when $H = \{1_G\}$, the conditions in the previous theorem are equivalent to G being Abelian. Indeed $L(G)$ is commutative if and only if G is commutative. Moreover, one has $\mathcal{C}(G, \{1_G\}) = L(G)$ and $L(\{(1_G, 1_G)\}\backslash(G \times \{1_G\})/\{(1_G, 1_G)\}) \cong L(G)$. Finally, (iii) amounts to say that all irreducible representations of G are one-dimensional; this is clearly equivalent to the algebra $\mathcal{A}(G)$ (and therefore, by Theorem 1.5.11, to $L(G)$) being commutative.

Proposition 2.1.12 *$(G \times H, \widetilde{H})$ is a symmetric Gelfand pair if and only if for every $g \in G$ there exists $h \in H$ such that $hgh^{-1} = g^{-1}$ (that is, every $g \in G$ is H-conjugate to its inverse g^{-1}). Moreover, if this is the case, then $\mathcal{C}(G, H)$ is commutative and H is a multiplicity-free subgroup of G.*

Proof The pair $(G \times H, \widetilde{H})$ is symmetric if and only if for all $(g, h) \in G \times H$ there exist $h_1, h_2 \in H$ such that

$$\begin{cases} h^{-1} = h_1 h h_2 \\ g^{-1} = h_1 g h_2. \end{cases} \tag{2.8}$$

Taking $h = 1_G$ we get $h_2 = h_1^{-1}$ and therefore $g^{-1} = h_1 g h_1^{-1}$.

Conversely, suppose that every $g \in G$ is H-conjugate to g^{-1}. Then, for $(g, h) \in G \times H$, let $t \in H$ be such that $(gh^{-1})^{-1} = t(gh^{-1})t^{-1}$. We can solve the system (2.8) by setting $h_1 = h^{-1}t$ and $h_2 = h^{-1}t^{-1}$. □

Exercise 2.1.13 Show that if $(G \times H, \widetilde{H})$ is a Gelfand pair, then the spherical function associated with the representation $\sigma \boxtimes \rho$ is given by

$$\phi_{\sigma,\rho}(g) = \frac{1}{|H|} \sum_{h \in H} \overline{\chi^\sigma(gh)} \cdot \overline{\chi^\rho(h)}$$

(we identify the double cosets in $\widetilde{H}\backslash(G \times H)/\widetilde{H}$ with the H-conjugacy classes in G; note that the right hand side does not depend on the particular element g but only on its H-conjugacy class).

[*Hint.* Use Exercise 1.5.28.]

2.1.3 Greenhalgebras

In this section, we present a generalization of the theory of subgroup-conjugacy-invariant algebras and the related Gelfand pairs. Following Diaconis [27], we call them Greenhalgebras, since they were studied in A. Greenhalgh's thesis [53]. They were also considered by Brender [11] (but Greenhalgh considered a more general case).

Definition 2.1.14 Let G be a finite group and suppose that K and H are two subgroups of G, with $K \trianglelefteq H \leq G$. The *Greenhalgebra* associated with G, H and K is

$$\mathcal{G}(G, H, K) = \{f \in L(G) : f(h^{-1}gh) = f(g) \text{ and } f(k_1 g k_2) = f(g), \\ \forall g \in G,\ h \in H,\ k_1, k_2 \in K\}.$$

In other words, $\mathcal{G}(G, H, K)$ is the set of all functions defined on G that are both H-conjugacy invariant and bi-K-invariant.

Set $X = G/K$. Then the group $G \times H$ acts on X by the following rule:

$$(g, h) \cdot g_0 K = g g_0 h^{-1} K \tag{2.9}$$

(note that, if $g, g_0 \in G$, $h \in H$ and $k \in K$, then

$$(g, h) \cdot g_0 k K = g g_0 h^{-1} h k h^{-1} K = g g_0 h^{-1} K,$$

for all $g \in G$ and $h \in H$, because $K \trianglelefteq H$ and therefore (2.9) does not depend on the choice of $g_0 \in g_0K$). This clearly generalizes the case $K = \{1_G\}$ considered in Section 2.1.1. We still denote by η the associated permutation representation:

$$[\eta(g,h)f](g_0K) = f(g^{-1}g_0hK)$$

with $f \in L(G/K)$, $g, g_0 \in G$ and $h \in H$.

Also, Lemma 2.1.1 generalizes as follows:

Lemma 2.1.15

(i) *The stabilizer of* $K \in G/K$ *under the action of* $G \times H$ *is the group* $B = (K \times \{1_G\})\widetilde{H} \equiv \{(kh, h) : k \in K, h \in H\}$.

(ii) *The map*

$$\Phi : L(B\backslash(G \times H)/B) \to \mathcal{G}(G, H, K)$$

given by

$$[\Phi(F)](g) = |H|F(g, 1_G)$$

for all $F \in L(B\backslash(G \times H)/B)$ *and* $g \in G$, *is a linear isomorphism of algebras. Moreover,*

$$\|\Phi(F)\|_{L(G)} = \sqrt{|H|}\|F\|_{L(G\times H)}. \tag{2.10}$$

Proof (i) We have $(g,h)K = K$ if and only if $gh^{-1} \in K$, that is if and only if $(g,h) = (kh, h)$ for some $k \in K$.

(ii) The proof is the same as that in (ii) of Lemma 2.1.1 with the same correspondence Φ. Just note that if $f = \Phi(F)$ then F satisfies $F(k_1gk_2, h) = F(g,h)$ if and only if f satisfies $f(k_1gk_2) = f(g)$, for all $g \in G$, $h \in H$ and $k_1, k_2 \in K$. □

Now we determine the multiplicity of an irreducible representation $\sigma \boxtimes \theta \in \widehat{G \times H} \equiv \widehat{G} \times \widehat{H}$ (see Theorem 1.3.17) in the permutation representation η. First we examine the case $G = H$. To this end, we introduce a new notion: if γ is a representation of the quotient group H/K, its *extension* to H is the representation $\overline{\gamma}$ given by:

$$\overline{\gamma}(h) = \gamma(hK)$$

for all $h \in H$. In other words, we have composed γ with the projection $\pi : H \to H/K$:

$$\overline{\gamma} = \gamma\pi : H \xrightarrow{\pi} H/K \xrightarrow{\gamma} GL(V_\gamma).$$

Clearly, $\overline{\gamma}$ is irreducible if and only if γ is irreducible. Moreover, if we set $\widehat{H}_K = \{\rho \in \widehat{H} : \mathrm{Res}^H_K \rho = (\dim\rho)\iota_K\}$ (where ι_K is the trivial representation of K), then

$$\begin{array}{ccc} \widehat{H/K} & \to & \widehat{H}_K \\ \gamma & \mapsto & \overline{\gamma} \end{array}$$

is clearly a bijection.

Proposition 2.1.16 *Denote by α the permutation representation η in the case $G = H$. Then the decomposition of α into irreducible $(H \times H)$-representations is given by*

$$\alpha = \bigoplus_{\rho \in \widehat{H}_K} (\rho \boxtimes \rho').$$

Proof Start with Corollary 2.1.6 applied to the group H/K: if β is the permutation representation of $(H/K) \times (H/K)$ on $L(H/K)$ defined by setting

$$[\beta(h_1K, h_2K)f](hK) = f(h_1^{-1}hh_2K)$$

for all $h_1, h_2, h \in H$ and $f \in L(H/K)$, then

$$\beta = \bigoplus_{\gamma \in \widehat{H/K}} (\gamma \boxtimes \gamma')$$

is its decomposition into irreducible representations (note that for this decomposition one may give a proof that does not use Theorem 2.1.2; see Exercise 2.1.8). The extension of β to H coincides exactly with α, and this yields immediately the proof. □

Now we are in position to deal with the general case $G \geq H$. We recall that given a representation ρ we denote by ρ' the corresponding adjoint representation (see Section 1.1.5).

Theorem 2.1.17 *For every $\sigma \in \widehat{G}$ and $\rho \in \widehat{H}_K$, denote by $m_{\rho,\sigma}$ the multiplicity of ρ' in $\mathrm{Res}^G_H\sigma$. Then the decomposition of η into irreducible representations is*

$$\eta = \bigoplus_{\sigma \in \widehat{G}} \bigoplus_{\rho \in \widehat{H}_K} m_{\rho,\sigma}(\sigma \boxtimes \rho).$$

Proof First note that the stabilizer of $K \in H/K$ under the action of $H \times H$ coincides with the stabilizer of $K \in G/K$ under the action of $G \times H$: in both cases it is $B \leq H \times H \leq G \times H$. Then applying transitivity of induction (Proposition 1.6.6) and Example 1.6.4 we can write

$$\eta = \mathrm{Ind}_B^{G\times H}\iota_B = \mathrm{Ind}_{H\times H}^{G\times H}\mathrm{Ind}_B^{H\times H}\iota_B = \mathrm{Ind}_{H\times H}^{G\times H}\alpha.$$

Note also that, by Frobenius reciprocity (Theorem 1.6.11), we have

$$\mathrm{Ind}_H^G \rho' = \bigoplus_{\sigma \in \widehat{G}} m_{\rho,\sigma} \sigma$$

for all $\rho \in \widehat{H}_K$. Switching ρ with ρ' ($\rho \in \widehat{H}_K \Leftrightarrow \rho' \in \widehat{H}_K$) we can write the decomposition in Proposition 2.1.16 in the form

$$\alpha = \bigoplus_{\rho \in \widehat{H}_K} (\rho' \boxtimes \rho)$$

and therefore

$$\eta = \bigoplus_{\sigma \in \widehat{G}} \bigoplus_{\rho \in \widehat{H}_K} m_{\rho,\sigma} (\sigma \boxtimes \rho). \qquad \square$$

Exercise 2.1.18 Give a proof of Theorem 2.1.17 along the lines of Theorem 2.1.2 and Corollary 2.1.3.

Theorem 2.1.19 *The following conditions are equivalent:*

(i) *The algebra $\mathcal{G}(G, H, K)$ is commutative;*
(ii) *$(G \times H, B)$ is a Gelfand pair;*
(iii) *for every $\sigma \in \widehat{G}$ and $\rho \in \widehat{H}_K$, the multiplicity of ρ in $\mathrm{Res}_H^G \sigma$ is ≤ 1;*
(iv) *for every $\sigma \in \widehat{G}$ and $\rho \in \widehat{H}_K$, the multiplicity of σ in $\mathrm{Ind}_H^G \rho$ is ≤ 1.*

Proof The equivalence of (ii) and (iii) follows from Theorem 2.1.17 (noting again that $\rho \in \widehat{H}_K$ if and only if $\rho' \in \widehat{H}_K$). The equivalence of (i) and (ii) follows from Lemma 2.1.15. Finally (iii) and (iv) are equivalent by virtue of Frobenius reciprocity (Theorem 1.6.11). $\square$

Exercise 2.1.20 Suppose that $\mathcal{G}(G, H, K)$ is commutative. Show that the spherical function associated with the representation $\sigma \boxtimes \rho$ has the following expression:

$$\varphi_{\sigma,\rho}(g, h) = \frac{1}{|H| \cdot |K|} \sum_{t \in H} \sum_{k \in K} \overline{\chi^\sigma(gkt)} \cdot \overline{\chi^\rho(ht)}.$$

Compare with Exercise 2.1.13.

We end this section by examining the symmetric Gelfand lemma (see Proposition 1.4.8) in the present context.

Proposition 2.1.21 *$(G \times H, B)$ is a symmetric Gelfand pair if and only if for any $g \in G$ there exist $k_1, k_2 \in K$ and $h_1 \in H$ such that*

$$g^{-1} = k_1 h_1 g h_1^{-1} k_2.$$

Proof The proof is the same of that of Proposition 2.1.12: just replace $g^{-1} = h_1 g h_2$ with $g^{-1} = k_1 h_1 g k_2 h_2$, t with $k_1 h_1$ and t^{-1} with $h_1^{-1} k_2$. □

2.2 Gelfand–Tsetlin bases

2.2.1 Branching graphs and Gelfand–Tsetlin bases

Let G be a group. A chain

$$G_1 = \{1_G\} \leq G_2 \leq \cdots \leq G_{n-1} \leq G_n \leq \cdots \tag{2.11}$$

of subgroups of G is said to be *multiplicity-free* if G_{n-1} is a multiplicity-free subgroup of G_n for all $n \geq 2$. Note that, by Remark 2.1.11.(b), if (2.11) is multiplicity-free, then G_2 is necessarily Abelian.

The *branching graph* of a multiplicity-free chain (2.11) is the oriented graph whose vertex set is $\widehat{G_1} \coprod \widehat{G_2} \coprod \cdots \coprod \widehat{G_{n-1}} \coprod \widehat{G_n} \coprod \cdots$ and whose edge set is

$$\{(\rho, \sigma) : \rho \in \widehat{G_n}, \sigma \in \widehat{G_{n-1}} \text{ s.t. } \sigma \preceq \mathrm{Res}^{G_n}_{G_{n-1}} \rho, n = 2, 3, \ldots\}.$$

We shall write $\rho \to \sigma$ if (ρ, σ) is an edge of the branching graph.

From Theorem 2.1.10 we have that (2.11) is multiplicity-free if and only if the algebra of G_{n-1}-conjugacy invariant functions on G_n is commutative. Let $(\rho, V_\rho) \in \widehat{G_n}$. If (2.11) is multiplicity free then

$$\mathrm{Res}^{G_n}_{G_{n-1}} V_\rho = \bigoplus_{\substack{\sigma \in \widehat{G_{n-1}}: \\ \rho \to \sigma}} V_\sigma$$

is an orthogonal (by multiplicity freeness) decomposition. Iterating this decomposition we obtain that if $\sigma \in \widehat{G_{n-1}}$ then the decomposition

$$\mathrm{Res}^{G_{n-1}}_{G_{n-2}} V_\sigma = \bigoplus_{\substack{\theta \in \widehat{G_{n-2}}: \\ \sigma \to \theta}} V_\theta$$

is again orthogonal.

Continuing this way, after $n-1$ steps (the last one corresponding to restricting from G_2 to G_1), we get sums of one-dimensional trivial representations. To formalize this, denote by $\mathcal{T}(\rho)$ the set of all paths T in the branching graph of the form:

$$T = (\rho = \rho_n \to \rho_{n-1} \to \rho_{n-2} \to \cdots \to \rho_2 \to \rho_1) \tag{2.12}$$

where $\rho_k \in \widehat{G_k}$ for $k = 1, 2, \ldots, n-1$. Then we can write

$$V_\rho = \bigoplus_{\substack{\rho_{n-1}\in\widehat{G_{n-1}}:\\ \rho\to\rho_{n-1}}} V_{\rho_{n-1}} = \bigoplus_{\substack{\rho_{n-1}\in\widehat{G_{n-1}}:\\ \rho\to\rho_{n-1}}} \bigoplus_{\substack{\rho_{n-2}\in\widehat{G_{n-2}}:\\ \rho_{n-1}\to\rho_{n-2}}} V_{\rho_{n-2}} = \cdots = \bigoplus_{T\in\mathcal{T}(\rho)} V_{\rho_1}. \quad (2.13)$$

In the last term of (2.13) each space V_{ρ_1} is one dimensional. We thus choose, for each $T \in \mathcal{T}(\rho)$, a vector v_T in the corresponding space V_{ρ_1} with $\|v_T\| = 1$ (note that v_T is defined up to a scalar factor of modulus one). Thus, (2.13) may be rewritten in the form

$$V_\rho = \bigoplus_{T\in\mathcal{T}(\rho)} < v_T >, \quad (2.14)$$

that is, $\{v_T : T \in \mathcal{T}(\rho)\}$ is an orthonormal basis of V_ρ. It is called a *Gelfand–Tsetlin basis* of V_ρ with respect to the multiplicity-free chain (2.11). Note that if $\theta \in \widehat{G_k}$, $1 \le k \le n-1$, then the multiplicity of θ in $\mathrm{Res}^{G_n}_{G_k}\rho$ equals the number of paths from ρ to θ in the branching graph. Moreover, the procedure in (2.13) gives an effective way to decompose the V_θ-isotypic component in $\mathrm{Res}^{G_n}_{G_k}\rho$ into orthogonal G_k-irreducible sub-representations (each isomorphic to V_θ). Indeed, with each path from ρ to θ one associates a unique component V_θ of V_ρ; moreover, for distinct paths, the corresponding components are mutually orthogonal.

For $j = 1, 2, \ldots, n$ we denote by $\mathcal{T}_j(\rho)$ the set of all paths S in the branching graph of the form:

$$S = (\rho = \sigma_n \to \sigma_{n-1} \to \sigma_{n-2} \to \cdots \to \sigma_{j+1} \to \sigma_j) \quad (2.15)$$

where $\sigma_k \in \widehat{G_k}$ for $k = j, j+1, \ldots, n-1$. In particular, $\mathcal{T}_1(\rho) = \mathcal{T}(\rho)$. For $T \in \mathcal{T}(\rho)$ of the form 2.12 we denote by $T_j \in \mathcal{T}_j(\rho)$ the path

$$T_j = (\rho = \rho_n \to \rho_{n-1} \to \rho_{n-2} \to \cdots \to \rho_{j+1} \to \rho_j). \quad (2.16)$$

We call T_j the jth truncation of T.

For $1 \le j \le n$ and $S \in \mathcal{T}_j(\rho)$ we set

$$V_S = \bigoplus_{\substack{T\in\mathcal{T}(\rho):\\ T_j=S}} V_{\rho_1}. \quad (2.17)$$

Note that if we set

$$\rho_S = [\mathrm{Res}^G_{G_j}\rho]|_{V_S} \quad (2.18)$$

then (ρ_S, V_S) is G_j-irreducible and, in fact $\rho_S \sim \rho_j$. Also, $V_T = \mathbb{C}v_T$, for all $T \in \mathcal{T}(\rho)$. Finally, we have

$$V_\rho = \bigoplus_{S \in \mathcal{T}_j(\rho)} V_S \tag{2.19}$$

and (2.13) becomes

$$V_\rho = \bigoplus_{T \in \mathcal{T}(\rho)} V_T = \bigoplus_{T \in \mathcal{T}(\rho)} \mathbb{C}v_T. \tag{2.20}$$

Let $j \in \{1, 2, \ldots, n\}$ and let $S \in \mathcal{T}_j(\rho)$ and $T \in \mathcal{T}(\rho)$. Then $S = T_j$ if and only if $v_T \in V_S$.

2.2.2 Gelfand–Tsetlin algebras

Let H be a subgroup of G. For $f \in L(H)$ we denote by $f_H^G \in L(G)$ the function defined by

$$f_H^G(g) = \begin{cases} f(g) & \text{if } g \in H \\ 0 & \text{otherwise.} \end{cases} \tag{2.21}$$

Note that if $H \le K \le G$ then $(f_H^K)_K^G = f_H^G$, $(f_1 * f_2)_H^G = (f_1)_H^G * (f_2)_H^G$ and $(\alpha_1 f_1 + \alpha_2 f_2)_H^G = \alpha_1 (f_1)_H^G + \alpha_2 (f_2)_H^G$ for all $f, f_1, f_2 \in L(H)$ and α_1, $\alpha_2 \in \mathbb{C}$.

Moreover,

$$\rho(f_H^G) = \sum_{g \in G} f_H^G(g)\rho(g) = \sum_{h \in H} f(h)\rho(h) = \sum_{h \in H} f(h)\mathrm{Res}_H^G \rho(h) = (\mathrm{Res}_H^G \rho)(f) \tag{2.22}$$

for all $\rho \in \widehat{G}$ and $f \in L(H)$. This way, we regard $L(H)$ as a subalgebra of $L(G)$. By abuse of notation, unless otherwise needed, we shall still denote by f its extension f_H^G.

Definition 2.2.1 We denote by $Z(n)$ the center of the group algebra $L(G_n)$, that is, the subalgebra of central functions on G_n. The *Gelfand–Tsetlin algebra $GZ(n)$* associated with the multiplicity-free chain (2.11) is the algebra generated by the subalgebras

$$Z(1), Z(2), \ldots, Z(n)$$

of $L(G_n)$.

Theorem 2.2.2 *The Gelfand–Tsetlin algebra $GZ(n)$ is a maximal Abelian subalgebra of $L(G_n)$. Moreover, it coincides with the subalgebra of functions*

$f \in L(G_n)$ whose Fourier transforms $\rho(f)$, $\rho \in \widehat{G_n}$, are diagonalized by the Gelfand–Tsetlin basis of V_ρ. In formulas:

$$GZ(n) = \{f \in L(G) : \ \rho(f)v_T \in \mathbb{C}v_T, \text{for all } \rho \in \widehat{G_n} \text{ and } T \in \mathcal{T}(\rho)\}. \tag{2.23}$$

Proof First of all we have that $GZ(n)$ is commutative and it is spanned by the products

$$f_1 * f_2 * \cdots * f_n$$

where $f_k \in Z(k)$ for all $k = 1, 2, \ldots, n$. Indeed, if $f_i \in Z(i)$ and $f_j \in Z(j)$, with $i < j$, then $f_i * f_j = f_j * f_i$ as $f_i, \in L(G_i) \subset L(G_j)$ and f_j is in the center of $L(G_j)$.

Denote by $\mathcal{A}$ the right-hand side of (2.23). From the multiplicativity property of the Fourier transform (cf. (1.38)) it immediately follows that $\mathcal{A}$ is an algebra.

For $f_j \in Z(j)$, $\rho \in \widehat{G_n}$ and $T \in \mathcal{T}(\rho)$, set $S = T_j$. Then

$$\begin{aligned} \rho(f_j)v_T &= \sum_{g\in G} f_j(g)\rho(g)v_T \\ \text{(because } f_j \in L(G_j)) \quad &= \sum_{g\in G_j} f_j(g)[\mathrm{Res}^G_{G_j}\rho](g)v_T \\ \text{(because } v_T \in V_S \text{ and by (2.18))} \quad &= \sum_{g\in G_j} f_j(g)\rho_S(g)v_T \\ &= \rho_S(f_j)v_T \\ \text{(by Lemma 1.3.13)} \quad &= \alpha v_T, \end{aligned} \tag{2.24}$$

where $\alpha = \alpha_{S,f_j} \in \mathbb{C}$. This shows that $Z(j) \subseteq \mathcal{A}$ so that $GZ(n) \subseteq \mathcal{A}$.

To end the proof it suffices to show that $\mathcal{A} \subseteq GZ(n)$. Let $\rho \in \widehat{G_n}$ and let $T \in \mathcal{T}(\rho)$, say $T = (\rho = \rho_n \to \rho_{n-1} \to \cdots \to \rho_1)$. By virtue of Theorem 1.5.11 we can choose $f_j \in L(G_j)$, $j = 1, 2, \ldots, n$, such that

$$\sigma(f_j) = \begin{cases} I_{V_{\rho_j}} & \text{if } \sigma = \rho_j \\ 0 & \text{otherwise} \end{cases}$$

for all $\sigma \in \widehat{G_j}$. Arguing as in (2.24), we have that the function

$$F_T := f_1 * f_2 * \cdots * f_n$$

satisfies

$$\rho(F_T)v_S = \begin{cases} v_T & \text{if } S = T \\ 0 & \text{otherwise} \end{cases} \tag{2.25}$$

for all $S \in \mathcal{T}(\rho)$.

It follows that $\{F_T : T \in \mathcal{T}(\rho), \rho \in \widehat{G_n}\}$ is a basis for $\mathcal{A}$, and therefore $\mathcal{A} \subseteq GZ(n)$.

In view of Example 1.2.12, $\mathcal{A}$ is a maximal Abelian subalgebra of $L(G_n)$ because (Theorem 1.5.11)

$$L(G_n) \cong \bigoplus_{\rho \in \widehat{G_n}} \mathrm{Hom}(V_\rho, V_\rho) \cong \bigoplus_{\rho \in \widehat{G_n}} M_{d_\rho, d_\rho}(\mathbb{C}).$$

□

Corollary 2.2.3 *Every element v_T, $T \in \mathcal{T}(\rho)$, in the Gelfand–Tsetlin basis of V_ρ (see (2.14)) is a common eigenvector for all operators $\rho(f)$ with $f \in GZ(n)$. In particular, it is uniquely determined, up to a scalar factor, by the corresponding eigenvalues.*

Proof The last statement follows immediately from the existence of the functions F_T in $GZ(n)$ such that (2.25) holds. □

Denote by $\mathcal{T}_{v_T} : V_\rho \to L(G)$ the intertwining operator associated with the vector v_T, as in Theorem 1.5.4. Using Exercise 1.5.18, prove the following alternative characterization of $GZ(n)$.

Exercise 2.2.4 The Gelfand–Tsetlin algebra $GZ(n)$ is isomorphic to the algebra of all convolution operators Ψ_f, $f \in L(G)$, such that for all $\rho \in \widehat{G_n}$ and $T \in \mathcal{T}(\rho)$, $\mathcal{T}_{v_T}(V_\rho)$ is an eigenspace of Ψ_f (where $\Psi_f \phi = \phi * f$ for all $\phi \in L(G)$).

Now suppose that $f_1, f_2, \dots, f_n$ belong to $GZ(n)$, and denote by X_i the convolution operator associated with f_i: $X_i \phi = \phi * f_i$ for all $\phi \in L(G)$ and $i = 1, 2, \dots, n$. Clearly, for all $\rho \in \widehat{G_n}$, $T \in \mathcal{T}(\rho)$ and $1 \le i \le n$, there exists $\alpha_{\rho,T,i} \in \mathbb{C}$ such that $\rho(f_i)v_T^\rho = \alpha_{\rho,T,i} v_T^\rho$, where v_T^ρ is the GZ-vector associated with the path $T \in \mathcal{T}(\rho)$. From Exercise 2.2.4 and Exercise 1.5.18.(2), we deduce that this is equivalent to the following condition: $\alpha_{\rho,T,i}$ is the eigenvalue of X_i relative to $\mathcal{T}_{v_T}(V_\rho)$.

When the map

$$(\rho, T) \mapsto (\alpha_{\rho,T,1}, \alpha_{\rho,T,2}, \dots, \alpha_{\rho,T,n})$$

with $\rho \in \widehat{G_n}$ and $T \in \mathcal{T}(\rho)$ is injective (this means that the values $(\alpha_{\rho,T,i})_{i=1}^n$ uniquely determine ρ and T), we say that $f_1, f_2, \dots, f_n$ *separate* the vectors of the GZ-bases $\{v_T^\rho : \rho \in \widehat{G_n}, T \in \mathcal{T}(\rho)\}$, or that they separate the subspaces in the decomposition into irreducibles

$$L(G) = \bigoplus_{\rho \in \widehat{G_n}, T \in \mathcal{T}(\rho)} \mathcal{T}_{v_T^\rho}(V_\rho).$$

Exercise 2.2.5 Prove that if the functions $f_1, f_2, \dots, f_n \in GZ(n)$ separate the vectors of the GZ-bases then the set $\{\delta_{1_G}, f_1, f_2, \dots, f_n\}$ generates $GZ(n)$ as an algebra.

Hint: for $\rho \in \widehat{G_n}$ and $T \in \mathcal{T}(\rho)$, we define $F_{\rho,T}$ as the convolution of all functions of the form:

$$\frac{f_i - \alpha_{\sigma,S,i}\delta_{1_G}}{\alpha_{\rho,T,i} - \alpha_{\sigma,S,i}},$$

for all $\sigma \in \widehat{G_n}$, $S \in \mathcal{T}(\sigma)$ such that $(\sigma, S) \neq (\rho, T)$ and all $1 \leq i \leq n$ such that $\alpha_{\sigma,S,i} \neq \alpha_{\rho,T,i}$. Show that $F_{\rho,T}$ is as in (2.25).

Remark 2.2.6 In general, we must add δ_{1_G} to $f_1, f_2, \dots, f_n$ in order to get a generating set. For instance, we may have a pair (ρ, T) such that $\alpha_{\rho,T,i} = 0$, for $i = 1, 2, \dots, n$, and in this case no polynomial in $f_1, f_2, \dots, f_n$ (without δ_{1_G}) could give a function with the properties of $F_{\rho,T}$.

We end this section by showing that the Gelfand–Tsetlin algebra $GZ(n)$ of a multiplicity-free chain (2.11) contains the algebra of G_{n-1}-conjugacy invariant functions on G_n.

Proposition 2.2.7 *Let $G_1 \leq G_2 \leq \cdots \leq G_{n-1} \leq G_n$ be a multiplicity-free chain of groups. Then $GZ(n) \supseteq \mathcal{C}(G_n, G_{n-1})$.*

Proof First note that $f \in \mathcal{C}(G_n, G_{n-1})$ if and only if $f * \delta_h = \delta_h * f$ for all $h \in G_{n-1}$. This implies that if $\rho \in \widehat{G_n}$, then

$$\rho(h)\rho(f) = \rho(\delta_h * f) = \rho(f * \delta_h) = \rho(f)\rho(h),$$

that is,

$$\rho(f) \in \mathrm{Hom}_{G_{n-1}}(\mathrm{Res}^{G_n}_{G_{n-1}}\rho, \mathrm{Res}^{G_n}_{G_{n-1}}\rho).$$

Since $\mathrm{Res}^{G_n}_{G_{n-1}}\rho$ is multiplicity-free,

$$\rho(f)V_\sigma \subseteq V_\sigma \tag{2.26}$$

for all $f \in \mathcal{C}(G_n, G_{n-1})$ and $\sigma \in \widehat{G_{n-1}}$, with $V_\sigma \leq V_\rho$.

Observe now that if $f \in \mathcal{C}(G_n, G_{n-1})$, then $f \in \mathcal{C}(G_n, G_k)$ for all $k = n-2, n-3, \dots, 1$. Therefore, iterating the above argument, the relation (2.26), when $\sigma \in \widehat{G_1}$, implies that every vector v_T of the Gelfand–Tsetlin basis is an eigenvector of $\rho(f)$. This shows $\mathcal{C}(G_n, G_{n-1}) \subseteq \mathcal{A}$ (the right-hand side of (2.23)) and by the previous theorem this completes the proof. □

2.2.3 Gelfand–Tsetlin bases for permutation representations

Let G be a finite group, $K \leq G$ a subgroup and set $X = G/K$. Denote by ι_K the trivial representation of K, so that $\mathrm{Ind}_K^G \iota_K$ is the permutation representation of G on $L(X)$.

We say that a subgroup H with $K \leq H \leq G$ is *multiplicity-free* for the pair (G, K) if for every irreducible representation $\theta \in \widehat{H}$ contained in $\mathrm{Ind}_K^H \iota_K$, the induced representation $\mathrm{Ind}_H^G \theta$ is multiplicity-free. By Frobenius reciprocity (Theorem 1.6.11), this condition on θ is equivalent to the following: for every $\sigma \in \widehat{G}$, the multiplicity of θ in $\mathrm{Res}_H^G \sigma$ is ≤ 1 (and it suffices to check this condition for those $\sigma \in \widehat{G}$ that contain non-trivial K-invariant vectors).

For $H = K$ we have that K is multiplicity-free for (G, K) if and only if the latter is a Gelfand pair; for $K = \{1_G\}$, H is multiplicity-free for $(G, \{1_G\})$ if and only if H is a multiplicity-free subgroup of G.

A *multiplicity-free chain* for the pair (G, K) is a chain of subgroups

$$G = H_m \geq H_{m-1} \geq \cdots \geq H_2 \geq H_1 = K \tag{2.27}$$

such that H_j is a multiplicity-free subgroup for the pair (H_{j+1}, K), $j = 1, 2, \ldots, m-1$. In particular (H_2, K) is a Gelfand pair.

Denote by $\mathcal{H}_j$ the subset of $\widehat{H_j}$ formed by those irreducible representations containing non-trivial K-invariant vectors (with $\mathcal{H}_1 \equiv \{\iota_K\}$). The *Bratteli diagram* associated with the chain (2.27) is the oriented graph whose vertex set is the disjoint union $\coprod_{j=1}^m \mathcal{H}_j$ and whose edge set is

$$\coprod_{j=1}^{m-1} \{(\sigma, \theta) \in \mathcal{H}_{j+1} \times \mathcal{H}_j : \theta \preceq \mathrm{Res}_{H_j}^{H_{j+1}} \sigma\}.$$

We also write $\sigma \to \theta$ to indicate that (σ, θ) is an edge. For every $\rho \in \mathcal{H}_m$ (this means that $\rho \in \widehat{G}$ and it is contained in $\mathrm{Ind}_K^G \iota_K$) let $\mathcal{T}(\rho)$ be the set of all paths of the form

$$T = (\rho = \rho_m \to \rho_{m-1} \to \rho_{m-1} \to \rho_{m-2} \cdots \to \rho_2 \to \rho_1 = \{\iota_K\}).$$

By transitivity of induction, we may consider the following chain:

$$\mathrm{Ind}_K^G \iota_K = \mathrm{Ind}_{H_{m-1}}^{H_m} \mathrm{Ind}_{H_{m-2}}^{H_{m-1}} \cdots \mathrm{Ind}_{H_1}^{H_2} \iota_K. \tag{2.28}$$

At each stage, the induction of a single irreducible representation is multiplicity-free. Then, for $\rho \in \mathcal{H}_m$ and $T \in \mathcal{T}_\rho$, we can construct the following sequence of spaces: $V_{\rho_1, T} \cong \mathbb{C}$ is the space of ι_K; $V_{\rho_{j+1}, T}$ is the subspace of $\mathrm{Ind}_{H_j}^{H_{j+1}} V_{\rho_j, T}$ that corresponds to ρ_j, $j = 1, 2, , \ldots, m-1$. We set $V_T := V_{\rho_m, T}$. Clearly, if $T_1, T_2 \in \mathcal{T}(\rho)$ and $T_1 \neq T_2$, then V_{T_1} is orthogonal to V_{T_2} (at some stage of

(2.28), they come from inequivalent orthogonal representations) and

$$\bigoplus_{T\in\mathcal{T}(\rho)} V_T \tag{2.29}$$

is the orthogonal decomposition of the ρ-isotypic component of $L(X)$ (with $X = G/K$).

On the other hand we have the reciprocal chain:

$$\mathrm{Res}^G_K V_\rho = \mathrm{Res}^{H_2}_{H_1}\mathrm{Res}^{H_3}_{H_2}\cdots\mathrm{Res}^{H_m}_{H_{m-1}} V_\rho,$$

where V_ρ is the space of ρ (here we think of V_ρ as an abstract space, it is not contained in a homogeneous space).

Again, if $T \in \mathcal{T}(\rho)$, we can construct a chain of subspaces of V_ρ

$$V_m = V_\rho \supseteq V_{m-1} \supseteq V_{m-2} \supseteq \cdots \supseteq V_2 \supseteq V_1 \tag{2.30}$$

by requiring that V_j is the subspace of $\mathrm{Res}^{H_{j+1}}_{H_j} V_{j+1}$ corresponding to ρ_j, $j = m-1, m-2, \ldots, 1$. Then we can choose a unitary vector $w_T \in V_1$ (which is one-dimensional) and this is clearly a K-invariant vector in V_ρ. Moreover the set $\{w_T : T \in \mathcal{T}(\rho)\}$ is an orthogonal basis for V^K_ρ (the subspace of K-invariant vectors in V_ρ). It is called the *Gelfand–Tsetlin basis* (or *adapted basis*) for V^K_ρ associated with the multiplicity-free chain (2.27). The associated decomposition of $L(X)$ (see Corollary 1.4.14) is the *Gelfand–Tsetlin decomposition* of $L(X)$.

Clearly, the Gelfand–Tsetlin basis is strictly connected with the decomposition (2.30); to obtain the connection we need a preliminary lemma. For the moment, suppose that $G \geq H \geq K$. Set $X = G/K$ and identify $Z = H/K$ with a subset of X; this way, if $x_0 \in X$ is a point stabilized by K then $x_0 \in Z$ and K is also the stabilizer of x_0 in H. Suppose that S is a system of representatives for H in G, so that $G = \coprod_{s\in S} sH$ (and therefore $X = \coprod_{s\in S} sZ$). Note also that $L(X) = \mathrm{Ind}^G_H L(Z)$, by transitivity of induction.

Let (σ, V) be an irreducible representation of G and W an H-invariant, irreducible subspace of V. Denote by (θ, W) the irreducible H-representation $\mathrm{Res}^G_H\sigma$ restricted to the space W. If $w_0 \in W$ is K-invariant, then, by means of (1.24), we can form two distinct intertwining operators: $\mathcal{S}_{w_0} : W \to L(Z)$ (H-invariant) and $\mathcal{T}_{w_0} : V \to L(X)$ (G-invariant). In other words,

$$(\mathcal{S}_{w_0} w)(hx_0) = \sqrt{\frac{d_\theta}{|Z|}}\langle w, \theta(h)w_0\rangle_W$$

for all $h \in H$ and $w \in W$ and

$$(\mathcal{T}_{w_0} w)(gx_0) = \sqrt{\frac{d_\sigma}{|X|}} \langle v, \sigma(g)w_0 \rangle_V$$

for all $g \in G$ and $v \in V$.

By Lemma 1.6.2, $\mathrm{Ind}_H^G \mathcal{S}_{w_0}(W)$ is a well-defined subspace of $L(X) \equiv \mathrm{Ind}_H^G L(Z)$. Namely, if λ is the permutation representation of G on $L(X)$, then we have $L(X) = \coprod_{s \in S} \lambda(s) L(Z)$ and

$$\mathrm{Ind}_H^G \mathcal{S}_{w_0}(W) = \bigoplus_{s \in S} \lambda(s) \mathcal{S}_{w_0}(W). \tag{2.31}$$

Lemma 2.2.8 *The operator $\mathcal{T}_{w_0}$ intertwines V with the subspace $\mathrm{Ind}_H^G \mathcal{S}_{w_0}(W)$. In particular, $\mathcal{T}_{w_0}(V)$ is contained in $\mathrm{Ind}_H^G \mathcal{S}_{w_0}(W)$.*

Proof Denote by $P_W : V \to W$ the orthogonal projection onto W. Suppose that $x \in X$ and $x = shx_0$, with $s \in S$ and $h \in H$. By (1.24), for any $v \in V$ we have

$$\begin{aligned}
(\mathcal{T}_{w_0} v)(x) = (\mathcal{T}_{w_0} v)(shx_0) &= \sqrt{\frac{d_\sigma}{|X|}} \langle v, \sigma(s)\theta(h)w_0 \rangle_V \\
&= \sqrt{\frac{d_\sigma}{|X|}} \langle \sigma(s^{-1})v, \theta(h)w_0 \rangle_V \\
&= \sqrt{\frac{d_\sigma}{|X|}} \langle P_W \sigma(s^{-1})v, \theta(h)w_0 \rangle_W \\
&= \sqrt{\frac{|Z| d_\sigma}{|X| d_\theta}} \left\{ \mathcal{S}_{w_0}[P_W \sigma(s^{-1})v] \right\} (hx_0) \\
(x = shx_0) \qquad &= \sqrt{\frac{|Z| d_\sigma}{|X| d_\theta}} \cdot \left\{ \lambda(s) \left[\mathcal{S}_{w_0}(P_W \sigma(s^{-1})v) \right] \right\} (x),
\end{aligned}$$

that is, $\mathcal{T}_{w_0} v \in \lambda(s) \mathcal{S}_{w_0}(W)$. By (2.31), this shows that $\mathcal{T}_{w_0} v \in \mathrm{Ind}_H^G \mathcal{S}_{w_0}(W)$. □

Lemma 2.2.8 may be summarized by the following diagram (where the symbol $\hookrightarrow$ means inclusion); the content of the Lemma is precisely the inclusion

$\mathcal{T}_{w_0}(V) \hookrightarrow \mathrm{Ind}_H^G \mathcal{S}_{w_0}(W)$:

$$\begin{array}{ccccccc} & & V & \xrightarrow{\mathcal{T}_{w_0}} \mathcal{T}_{w_0}(V) \hookrightarrow & \mathrm{Ind}_H^G \mathcal{S}_{w_0}(W) & \hookrightarrow & L(X) \\ & & \big\uparrow & & \big\uparrow \mathrm{Ind}_H^G & & \big\uparrow \mathrm{Ind}_H^G \\ w_0 \in W^K & \hookrightarrow & W & \xrightarrow{\qquad \mathcal{S}_{w_0} \qquad} & \mathcal{S}_{w_0}(W) & \hookrightarrow & L(Z). \end{array}$$

Returning to the multiplicity-free chain (2.27), we have:

Theorem 2.2.9 *For $T \in \mathcal{T}(\rho)$ let $\mathcal{S}_T$ be the intertwining operator associated with the K-invariant vector w_T. Then $\mathcal{S}_T(V_\rho) = V_T$, where V_T is as in (2.29).*

Proof This may be proved by a repeated application of Lemma 2.2.8. Let $V_m \supseteq V_{m-1} \supseteq \cdots \supseteq V_1$ be as in (2.30). Denote by $\mathcal{S}_T^{(j)} : V_j \to L(H_j/K)$ the intertwining operator associated with w_T (which is a K-invariant vector in all subspaces in (2.30)), $j = 1, 2, \ldots, m$. Then Lemma 2.2.8 ensures that $\mathcal{S}_T^{(j+1)}(V_{j+1}) \subseteq \mathrm{Ind}_{H_j}^{H_{j+1}} \mathcal{S}_T^{(j)}(V_j)$. But this is precisely the definition of $V_{\rho_{j+1},T}$. Certainly, at the first step we have $\mathcal{S}_T^{(1)} V_1 \equiv V_{\rho_1,T}$ and therefore $\mathcal{S}_T^{(j)} V_1 \equiv V_{\rho_j,T}$ for $j = 2, 3, \ldots, m$. The case $j = m$ is exactly what we had to prove. □

3
The Okounkov–Vershik approach

This chapter is based on the papers by Okounkov and Vershik [99, 100] and Vershik [120]. For a nice short presentation we refer to [104]. We have also benefited greatly from the book by Kleshchev [73]. Our exposition is just a slightly more detailed treatement of the orginal sources, though we have also added a new derivation of Pieri's rule (Corollary 3.5.14) and Young's rule (Section 3.7.2 and Section 3.7.3).

3.1 The Young poset

In this section, we introduce some basic algebraic and combinatorial tools necessary for the representation theory of the symmetric group.

3.1.1 Partitions and conjugacy classes in $\mathfrak{S}_n$

Let $\mathfrak{S}_n$ be the symmetric group of degree n, that is, the group of all permutations of the set $\{1, 2, \dots, n\}$.

We recall some basic facts on $\mathfrak{S}_n$ and its conjugacy classes (see the books by Herstein [59], Hungerford [61] and Lang [76]).

A permutation $\gamma \in S_n$ is called a *cycle* of length t, and we denote it by $\gamma = (a_1, a_2, \dots, a_t)$, with $1 \leq a_i \neq a_j \leq n$ for $1 \leq i \neq j \leq t$, if

$$\gamma(a_1) = a_2,\ \gamma(a_2) = a_3, \dots, \gamma(a_{t-1}) = a_t,\ \gamma(a_t) = a_1$$

and

$$\gamma(b) = b \ \text{ if } b \in \{1, 2, \dots, n\} \setminus \{a_1, a_2, \dots, a_t\}.$$

Two cycles $\gamma = (a_1, a_2, \dots, a_t)$ and $\theta = (b_1, b_2, \dots, b_{t'})$ with $\{a_1, a_2, \dots, a_t\} \cap \{b_1, b_2, \dots, b_{t'}\} = \emptyset$ are said to be *disjoint*. It is clear that two disjoint cycles γ and θ commute: $\gamma\theta = \theta\gamma$.

Remark 3.1.1 Another useful notation for the cycle $(a_1, a_2, \dots, a_t)$ is

$$(a_1 \to a_2 \to \cdots \to a_t \to a_1).$$

It will be widely used in Section 3.2.2.

A *transposition* is a cycle of length 2.

Every $\pi \in \mathfrak{S}_n$ may be written as a product of disjoint cycles:

$$\pi = (a_1, a_2, \dots, a_{\mu_1})(b_1, b_2, \dots, b_{\mu_2}) \cdots (c_1, c_2, \dots, c_{\mu_k})$$

where the numbers $a_1, a_2, \dots, a_{\mu_1}, b_1, b_2, \dots, b_{\mu_2}, \dots, c_1, c_2, \dots, c_{\mu_k}$ form a permutation of $1, 2, \dots, n$. We may suppose that $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_k > 0$ (and clearly $\mu_1 + \mu_2 + \cdots + \mu_k = n$).

Given a permutation $\pi \in \mathfrak{S}_n$, an element $i \in \{1, 2, \dots, n\}$ such that $\pi(i) = i$ is called a *fixed point* for π. Note that if i is a fixed point for π, then in the cycle decomposition of π, i appears in a cycle of length one.

If σ is another permutation in $\mathfrak{S}_n$ then

$$\sigma\pi\sigma^{-1} = (\sigma(a_1), \sigma(a_2), \dots, \sigma(a_{\mu_1}))(\sigma(b_1), \sigma(b_2), \dots, \sigma(b_{\mu_2})) \cdots \\ \cdots (\sigma(c_1), \sigma(c_2), \dots, \sigma(c_{\mu_k})). \quad (3.1)$$

We think of permutations as bijective functions $\pi, \sigma : \{1, 2, \dots, n\} \to \{1, 2, \dots, n\}$, so that the $\sigma\pi\sigma^{-1}$-image of a is $\sigma\pi\sigma^{-1}(a)$.

From (3.1) it follows that two elements $\pi, \pi' \in \mathfrak{S}_n$ are conjugate if and only if they have the same cycle structure, that is, if and only if

$$\pi' = (a'_1, a'_2, \dots, a'_{\lambda_1})(b'_1, b'_2, \dots, b'_{\lambda_2}) \cdots (c'_1, c'_2, \dots, c'_{\lambda_h})$$

with $h = k$ and $\lambda_i = \mu_i$ for all $i = 1, 2, \dots, k$.

Definition 3.1.2 Let n be a positive integer. A *partition* of n is a sequence $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_h)$ of positive integers such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_h$ and $\lambda_1 + \lambda_2 + \cdots + \lambda_h = n$. We then write $\lambda \vdash n$.

From the above discussion we have:

Proposition 3.1.3 *The conjugacy classes of $\mathfrak{S}_n$ may be parameterized by the partitions of n: if $\lambda \vdash n$, then the conjugacy class associated with λ consists of all permutations $\pi \in \mathfrak{S}_n$ whose cycle decomposition is of the form*

$$\pi = (a_1, a_2, \dots, a_{\lambda_1})(b_1, b_2, \dots, b_{\lambda_2}) \cdots (c_1, c_2, \dots, c_{\lambda_h}).$$

3.1.2 Young frames

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_h)$ be a partition of n. The *Young frame* associated with λ, also called the *Young frame of shape* λ, is the array formed by n boxes with h left-justified rows, the ith row containing exactly λ_i boxes for all $i = 1, 2, \ldots, h$. In particular, such a Young frame has exactly λ_1 columns.

For example, the Young frame associated with the partition $\lambda = (4, 3, 1) \vdash 8$ is shown in Figure 3.1.

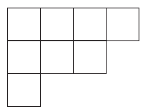


Figure 3.1

The rows and the columns are numbered from top to bottom and from left to right, like the rows and the columns of a matrix, respectively. This way, we have a system of coordinates for the boxes as in Figure 3.2.

(1,1)	(1,2)	(1,3)	(1,4)
(2,1)	(2,2)	(2,3)	
(3,1)			

Figure 3.2

In the Young frame associated with the partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_h) \vdash n$, the box of coordinates (i, j) is said to be *removable* if in the positions $(i+1, j)$ and $(i, j+1)$ there is no box; equivalently, either $i < h$ and $j = \lambda_i > \lambda_{i+1}$, or $i = h$ and $j = \lambda_h$. This means that, removing such a box, the corresponding reduced frame is still a Young frame, associated with a partition $\lambda' \vdash n - 1$.

Similarly, we say that the position (i, j) is *addable* if $\lambda_i = j - 1 < \lambda_{i-1}$ or $i = h + 1$ and $j = 1$. This means that if we add a box in position (i, j), then we obtain a Young frame associated with a partition of $n + 1$ (see Figure 3.3).

3.1.3 Young tableaux

Let $\lambda \vdash n$ be a partition of n. A (*bijective*) *Young tableau* of shape λ is a bijection between the boxes of the Young frame of shape λ and the set $\{1, 2, \ldots, n\}$. It

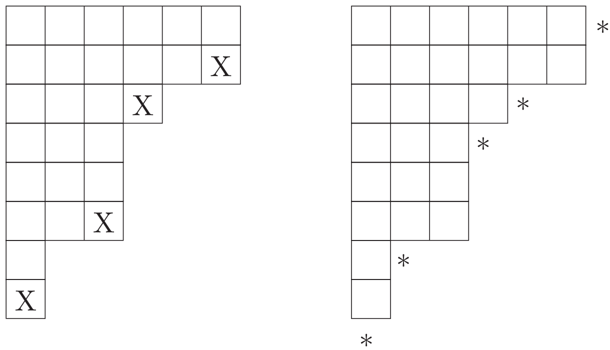


Figure 3.3 The X denote the removable boxes, the $*$ the addable boxes.

is represented by filling the above boxes with the numbers $1, 2, \ldots, n$, each number in exactly one box. For instance, in Figure 3.4 we present a Young tableau of shape $(4, 3, 1)$.

3	5	2	6
4	8	1	
7			

Figure 3.4

A Young tableau is *standard* if the numbers filled into the boxes are increasing both along the rows (from left to right) and along the columns (from top to bottom). The Young tableau in Figure 3.4 is not standard, while the one in Figure 3.5 is.

1	2	5	7
3	4	6	
8			

Figure 3.5

We observe that in a standard Young tableau, the number 1 is always in position $(1, 1)$, while n is always in a removable box.

For any partition $\lambda \vdash n$, we denote by $\mathrm{Tab}(\lambda)$ the set of all standard tableaux of shape λ. Finally we set

$$\mathrm{Tab}(n) = \bigcup_{\lambda \vdash n} \mathrm{Tab}(\lambda).$$

3.1.4 Coxeter generators

A distinguished set of generators for the symmetric group $\mathfrak{S}_n$ consists of the *adjacent transpositions*

$$s_i = (i, i+1) \quad i = 1, 2, \ldots, n-1. \tag{3.2}$$

These are also called the *Coxeter generators* of $\mathfrak{S}_n$; the fact that they generate $\mathfrak{S}_n$ will be proved in Proposition 3.1.4.

Let T be a Young tableau of shape λ and take $\pi \in \mathfrak{S}_n$. Then, we denote by πT the tableau obtained by replacing i with $\pi(i)$ for all $i = 1, 2, \ldots, n$. As an example, if $\pi = (316784)(25) \in \mathfrak{S}_8$, then, denoting by T the Young tableau in Figure 3.4, we have that πT is the (standard) Young tableau in Figure 3.5.

If T is standard, we say that an adjacent transposition s_i is *admissible* for T if $s_i T$ is still standard. It is easy to see that s_i is admissible for T if and only if i and $i+1$ belong neither to the same row, nor to the same column of T.

Given $\pi \in \mathfrak{S}_n$, an *inversion* for π is a pair (i, j) with $i, j \in \{1, 2, \ldots, n\}$ such that $i < j$ and $\pi(i) > \pi(j)$. We denote by $\mathcal{I}(\pi)$ the set of all inversions in π and by

$$\ell(\pi) = |\mathcal{I}(\pi)|$$

the number of inversions of π.

The *Coxeter length* of π is the smallest integer k such that π can be written as a product of k Coxeter generators, that is, $\pi = s_{i_1} s_{i_2} \cdots s_{i_k}$.

Proposition 3.1.4 *The Coxeter length of $\pi \in \mathfrak{S}_n$ equals $\ell(\pi)$.*

Proof We first observe that for any $\pi \in \mathfrak{S}_n$ one has

$$\ell(\pi s_i) = \begin{cases} \ell(\pi) - 1 & \text{if } (i, i+1) \in \mathcal{I}(\pi) \\ \ell(\pi) + 1 & \text{if } (i, i+1) \notin \mathcal{I}(\pi). \end{cases} \tag{3.3}$$

For, if $k \in \{1, 2, \ldots, n\}$ and $k \neq i, i+1$, then there are three possibilities:

- $\pi(k) < \min\{\pi(i), \pi(i+1)\} =: i_-$
- $\pi(k) > \max\{\pi(i), \pi(i+1)\} =: i_+$
- $i_- < \pi(k) < i_+$.

Therefore each of the pairs (k, i), $(k, i+1)$, (i, k), and $(i+1, k)$ is an inversion for π if and only if it is an inversion for πs_i. On the other hand, it is obvious that $(i, i+1) \in \mathcal{I}(\pi)$ if and only if $(i, i+1) \notin \mathcal{I}(\pi s_i)$.

This shows that the Coxeter length is bounded from below by $\ell(\pi)$. Indeed, if $\pi = s_{i_1} s_{i_2} \cdots s_{i_k}$ is a minimal representation of π as a product of Coxeter elements, then by (3.3)

$$\ell(\pi) = \ell(s_{i_1} s_{i_2} \cdots s_{i_{k-1}}) \pm 1 \leq \ell(s_{i_1} s_{i_2} \cdots s_{i_{k-1}}) + 1 \leq \cdots \leq k.$$

We now show that π can always be expressed as the product of $\ell(\pi)$ adjacent transpositions.

Let $j_n \in \{1, 2, \dots, n\}$ be such that $\pi(j_n) = n$. Set

$$\pi_n = \pi s_{j_n} s_{j_n+1} \cdots s_{n-1}$$

so that $\pi_n(n) = n$. We have that

$$\ell(\pi_n) = \ell(\pi) - (n - j_n). \tag{3.4}$$

Indeed, by (3.3), $\ell(\pi_n s_{n-1}) = \ell(\pi_n) + 1$. For the same reason, $\ell(\pi_n s_{n-1} s_{n-2}) = \ell(\pi_n s_{n-1}) + 1 = \ell(\pi_n) + 2$. Continuing this way, one finally gets (3.4).

Now observe that denoting by ℓ_k the number of inversions relative to $\mathfrak{S}_k$ (acting on $\{1, 2, \dots, k\}$) we have that $\ell_n(\sigma) = \ell_k(\sigma)$ for all $\sigma \in \mathfrak{S}_n$ such that: $\sigma(i) = i$ for $i = k+1, k+2, \dots, n$.

By induction, we have that π_n can be expressed as a product of $\ell_{n-1}(\pi_n)$ adjacent transpositions and by (3.4) we are done. □

For $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k) \vdash n$ we denote by T^λ the standard tableau of shape λ, shown in Figure 3.6,

1	2	3	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	λ_1
λ_1+1	λ_1+2	$\cdots$	$\cdots$	$\cdots$	$\lambda_1+\lambda_2$			
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$			
$\overline{\lambda}_{k-1}$	$\cdots$	n						

Figure 3.6

where $\overline{\lambda}_{k-1} = \lambda_1 + \lambda_2 + \cdots + \lambda_{k-1} + 1$.

If $T \in \mathrm{Tab}(\lambda)$ we denote by $\pi_T \in \mathfrak{S}_n$ the unique permutation such that $\pi_T T = T^\lambda$.

Theorem 3.1.5 *Let $T \in \mathrm{Tab}(\lambda)$ and set $\ell = \ell(\pi_T)$. Then there exists a sequence of ℓ admissible transpositions which transforms T into T^λ.*

Proof Let j denote the number in the rightmost box of the last row of T. If $j = n$, then, as this box is removable, we can consider the standard tableau T' of shape $\lambda' = (\lambda_1, \lambda_2, \ldots, \lambda_k - 1) \vdash (n-1)$ obtained by removing that box. Then applying induction to T' one finds a sequence of $\ell' = \ell(\pi_{T'})$ admissible transformations which transform T' into $T^{\lambda'}$. It is clear that the same sequence transforms T into T^λ and that $\ell' = \ell$. Suppose now that $j \neq n$. Clearly, s_j is admissible for T. Similarly, s_{j+1} is admissible for $s_j T$,..., s_{n-1} is admissible for $s_{n-2} \cdots s_{j+1} s_j T$. Now, $s_{n-1} s_{n-2} \cdots s_j T$ contains n in the rightmost box of the last row of T and one reduces to the previous case. □

Corollary 3.1.6 *Let $T, S \in \mathrm{Tab}(\lambda)$. Then S may be obtained from T by applying a sequence of admissible adjacent transpositions.*

Remark 3.1.7 In the proof of the above theorem, we have obtained a *standard* procedure to decompose π_T as a product of $\ell(\pi_T)$ admissible adjacent transpositions. We shall use it in what follows.

3.1.5 The content of a tableau

Let T be a Young tableau of shape $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \vdash n$. We denote by $i : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, k\}$ and $j : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, \lambda_1\}$ the functions defined by setting $i(t)$ and $j(t)$ to be the row and the column of T containing t, respectively. For instance, for the tableau in Figure 3.4 we have $i(6) = 1$ and $j(6) = 4$, while for the tableau in Figure 3.5 we have $i(6) = 2$ and $j(6) = 3$.

The *content* of T is the vector in $\mathbb{Z}^n$ given by

$$C(T) = (j(1) - i(1), j(2) - i(2), \ldots, j(n) - i(n)). \tag{3.5}$$

For instance, for the tableau T in Figure 3.4, $C(T) = (1, 2, 0, -1, 1, 3, -2, 0)$ while, for the tableau T' in Figure 3.5, $C(T') = (0, 1, -1, 0, 2, 1, 3, -2)$. Note that, in both examples, the components of the respective contents are the same, modulo an exchange of positions.

In other words, given a box of the Young frame with coordinates (i, j) we define the quantity $c(i, j) := j - i$. Also, we denote by $(i(k), j(k))$ the coordinates of the box of the tableau T containing the number k. Then we have

$$C(T) = (c(i(1), j(1)), c(i(2), j(2)), \ldots, c(i(n), j(n))).$$

For instance, the partition $\lambda = (4, 3, 1)$ determines the numbers $c(i, j)$ shown in Figure 3.7.

0	1	2	3
−1	0	1	
−2			

Figure 3.7

The choice of a particular tableau T of shape λ determines the order in which the numbers $c(i, j)$ appear in the vector $C(T)$. Note also that the Young frame of shape $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ may be divided into *diagonals*, numbered

$$-k + 1, -k + 2, \ldots, 0, 1, \ldots, \lambda_1 - 1.$$

The diagonal numbered h consists of those boxes with coordinates (i, j) such that $c(i, j) = h$.

Definition 3.1.8 Let $\mathrm{Cont}(n)$ be the set of all vectors $\alpha = (a_1, a_2, \ldots, a_n) \in \mathbb{C}^n$ such that

$$\begin{aligned}
&(1)\ a_1 = 0;\\
&(2)\ \{a_q + 1, a_q - 1\} \cap \{a_1, a_2, \ldots, a_{q-1}\} \neq \emptyset, \text{ for all } q > 1;\\
&(3)\ \text{if } a_p = a_q \text{ for some } p < q \text{ then } \{a_q - 1, a_q + 1\} \subseteq \{a_{p+1}, a_{p+2}, \ldots, a_{q-1}\}.
\end{aligned} \tag{3.6}$$

Observe that $\mathrm{Cont}(n) \subseteq \mathbb{Z}^n$. For instance, $\mathrm{Cont}(1) = \{0\}$ and

$$\mathrm{Cont}(2) = \{(0, 1), (0, -1)\}. \tag{3.7}$$

Given $\alpha, \beta \in \mathrm{Cont}(n)$ we write $\alpha \approx \beta$ if β can be obtained from α by permuting its entries, i.e. there exists $\pi \in \mathfrak{S}_n$ such that $\pi\beta = \alpha$. Clearly, $\approx$ is an equivalence relation in $\mathrm{Cont}(n)$. Note however that given $\alpha \in \mathrm{Cont}(n)$ there are permutations $\pi \in \mathfrak{S}_n$ such that $\pi\alpha \notin \mathrm{Cont}(n)$. For example, if $\alpha = (0, 1) \in \mathrm{Cont}(2)$ (see (3.7)) and $\pi = (1, 2) \in \mathfrak{S}_2$, then $\pi\alpha = (1, 0) \notin \mathrm{Cont}(2)$.

We now show that conditions (2) and (3) in (3.6) have immediate stronger consequences.

Proposition 3.1.9 *Let* $\alpha = (a_1, a_2, \ldots, a_n) \in \mathrm{Cont}(n)$. *Then, for* $p, q \in \{1, 2, \ldots, n\}$ *we have the following:*

(i) *if* $a_q > 0$ *then* $a_q - 1 \in \{a_1, a_2, \ldots, a_{q-1}\}$*; if* $a_q < 0$ *then* $a_q + 1 \in \{a_1, a_2, \ldots, a_{q-1}\}$*;*

(ii) *if* $p < q$*,* $a_p = a_q$ *and* $a_r \neq a_q$ *for all* $r = p+1, p+2, \ldots, q-1$*, then there exist unique* $s_-, s_+ \in \{p+1, p+2, \ldots, q-1\}$ *such that* $a_{s_-} = a_q - 1$ *and* $a_{s_+} = a_q + 1$.

Proof (i) Suppose that, for instance, $a_q > 0$. Then we can use (1) and (2) in (3.6) to construct a sequence $a_{s_0} = a_q, a_{s_1}, \ldots, a_{s_k} = 0$ such that $s_0 = q > s_1 > \cdots > s_k \geq 1$, with $a_{s_h} > 0$ and $|a_{s_h} - a_{s_{h+1}}| = 1$ for all $h = 0, 1, \ldots, k-1$. Then, as h varies, a_{s_h} attains all integer values between 0 and a_q; in particular it attains the value $a_q - 1$. For $a_q < 0$ the argument is analogous.

(ii) The existence of s_- and s_+ is guaranteed by (3) in (3.6). Their uniqueness follows from the fact that if there is another s'_- such that $a_{s'_-} = a_q - 1$, say with $s_- < s'_-$, then, again by condition (3) in (3.6) there exists s between s_- and s'_- such that $a_s = (a_q - 1) + 1 = a_q$, contradicting the assumptions. □

Theorem 3.1.10 *For any* $T \in \mathrm{Tab}(n)$ *we have* $C(T) \in \mathrm{Cont}(n)$ *and the map*

$$\begin{aligned} \mathrm{Tab}(n) &\to \mathrm{Cont}(n) \\ T &\mapsto C(T) \end{aligned}$$

is a bijection. Moreover, if $\alpha, \beta \in \mathrm{Cont}(n)$*, say* $\alpha = C(T)$ *and* $\beta = C(S)$*, with* $T, S \in \mathrm{Tab}(n)$*, then* $\alpha \approx \beta$ *if and only if* T *and* S *are tableaux of the same shape.*

Proof Let T be a standard tableau and let $C(T) = (a_1, a_2, \ldots, a_n)$ be its content. Clearly, $a_1 = 0$ because $i(1) = j(1) = 1$. If $q \in \{2, 3, \ldots, n\}$ is placed in position (i, j), so that $a_q = j - i$, then we have $i > 1$ or $j > 1$. In the first case, consider the number p in the box of coordinates $(i-1, j)$ (namely the next upper box). We then have $p < q$, as T is standard, and $a_p = j - i + 1 = a_q + 1$. Similarly, if $j > 1$ we consider the number p' in the box of coordinates $(i, j-1)$ (namely the next left box). We then have $p' < q$, as T is standard, and $a_{p'} = j - 1 - i = a_q - 1$. Therefore (2) in (3.6) is satisfied.

Now suppose that $a_p = a_q$ with $p < q$. This means that p and q are placed in the same diagonal. Thus, if (i, j) are the coordinates of the box containing q, then $i, j > 1$ and denoting by q_- and q_+ the numbers (in $\{p+1, p+2, \ldots, q-1\}$ because T is standard) placed in the boxes of coordinates $(i-1, j)$ and $(i, j-1)$, by the same argument above, we have $a_{q_+} = a_q - 1$ and $a_{q_-} = a_q + 1$ as in Figure 3.8.

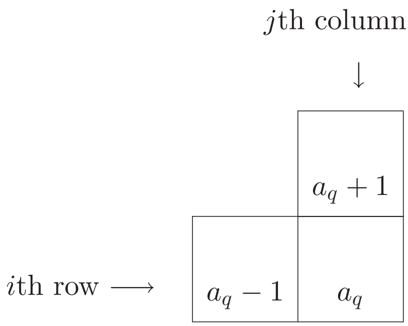


Figure 3.8

This proves that (3) in (3.6) is satisfied and therefore $C(T) \in \mathrm{Cont}(n)$.

We now prove that the map $T \mapsto C(T)$ is injective. Indeed, if $C(T) = (a_1, a_2, \ldots, a_n)$, then the diagonal h in T is filled with the numbers $q \in \{1, 2, \ldots, n\}$ such that $a_q = h$ from up-left to down-right, as in Figure 3.9.

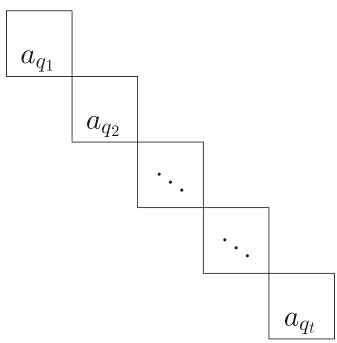


Figure 3.9

Here $q_1 < q_2 < \cdots < q_t$, $a_{q_1} = a_{q_2} = \cdots = a_{q_t} = h$ and $a_q \neq h$ if $q \notin \{q_1, q_2, \ldots, q_t\}$. Thus, if $T_1, T_2 \in \mathrm{Tab}(n)$ have the same content, namely $C(T_1) = C(T_2)$, then they have the same diagonals and, therefore, must coincide.

It remains to show that the map $T \mapsto C(T)$ is surjective. We prove it by induction on n. For $n = 1$ and 2 this is trivial. Suppose that the map $\mathrm{Tab}(n-1) \to \mathrm{Cont}(n-1)$ is surjective. Let $\alpha = (a_1, a_2, \ldots, a_n) \in \mathrm{Cont}(n)$. Then $\alpha' = (a_1, a_2, \ldots, a_{n-1}) \in \mathrm{Cont}(n-1)$ and by the inductive hypothesis there exists $T' \in \mathrm{Tab}(n-1)$ such that $C(T') = \alpha'$. We now show that adding the lower-rightmost-diagonal box in the diagonal a_n of T' and placing n in this box, yields a tableau $T \in \mathrm{Tab}(n)$ such that $C(T) = \alpha$.

If $a_n \notin \{a_1, a_2, \dots, a_{n-1}\}$, then we add a box on the first row (if $a_n - 1 \in \{a_1, a_2, \dots, a_{n-1}\}$) or on the first column (if $a_n + 1 \in \{a_1, a_2, \dots, a_{n-1}\}$).

If $a_n \in \{a_1, a_2, \dots, a_{n-1}\}$ and p is the largest index $\leq n-1$ such that $a_p = a_n$, then if the coordinates of the box containing p are (i, j), we place n in the new box of coordinates $(i+1, j+1)$. This is an addable box: indeed, (ii) in Proposition 3.1.9 ensures the existence (and uniqueness) of $r, s \in \{p+1, p+2, \dots, n\}$ such that $a_r = a_n + 1$ and $a_s = a_n - 1$, as Figure 3.10 shows.

p	r
s	n

Figure 3.10

Finally, if $\alpha = C(T)$ and $\beta = C(S)$, then β may be obtained from α by permuting its entries if and only if T and S have the same shape. Indeed the shape of a standard tableau is uniquely determined by the lengths of its diagonals. □

Given $\alpha \in \mathrm{Cont}(n)$ we say that an adjacent transposition s_i is *admissible* for α if it is admissible for the (unique) $T \in \mathrm{Tab}(n)$ such that $\alpha = C(T)$. This is equivalent to the following condition: $a_{i+1} \neq a_i \pm 1$. From Corollary 3.1.6 we get:

Corollary 3.1.11 *Given $\alpha, \beta \in C(T)$ we have that $\alpha \approx \beta$ if and only if there exists a sequence of admissible transpositions which transforms α into β.*

Corollary 3.1.12 *The cardinality of the quotient set* $\mathrm{Cont}(n)/\approx$ *equals $p(n) = |\{\lambda : \lambda \vdash n\}|$, the number of partitions of n.*

3.1.6 The Young poset

Denote by $\mathbb{Y} = \{\lambda : \lambda \vdash n, n \in \mathbb{N}\}$ the set of all partitions. Alternatively, we can regard $\mathbb{Y}$ as the set of all Young frames. We endow $\mathbb{Y}$ with a structure of a *poset* (partially ordered set) by setting, for $\mu = (\mu_1, \mu_2, \dots, \mu_k) \vdash n$ and $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_h) \vdash m$,

$$\mu \preceq \lambda$$

if $m \geq n$, $h \geq k$ and $\lambda_j \geq \mu_j$ for all $j = 1, 2, \dots, k$.

Equivalently, $\mu \preceq \lambda$ if the Young frame of μ is contained in the Young frame of λ (that is, if the Young frame of μ contains a box in position (i, j), so does the Young frame of λ).

For instance, if $\lambda = (4, 3, 1)$ and $\mu = (3, 2, 1)$, then $\mu \preceq \lambda$.

If $\mu \preceq \lambda$ we denote by λ/μ the array obtained by removing from the Young frame of λ the boxes of the Young frames of μ as in Figure 3.11.

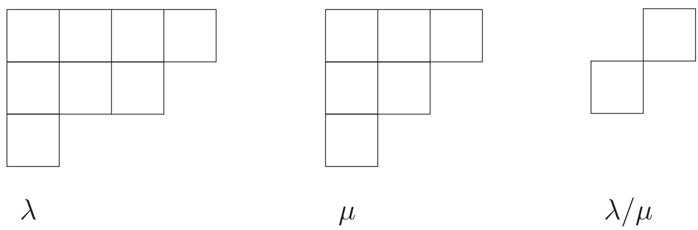


Figure 3.11

For $\mu, \lambda \in \mathbb{Y}$ we say that λ *covers* μ, or μ is *covered* by λ if $\mu \preceq \lambda$ and

$$\mu \preceq \nu \preceq \lambda, \ \nu \in \mathbb{Y} \Longrightarrow \nu = \mu \text{ or } \nu = \lambda.$$

Clearly, λ covers μ if and only of $\mu \preceq \lambda$ and λ/μ consists of a single box. We write $\lambda \to \mu$ to denote that λ covers μ.

The *Hasse diagram* of $\mathbb{Y}$ (which we also call the *Young (branching) graph*) is the oriented graph with vertex set $\mathbb{Y}$ and an arrow from λ to μ if and only if λ covers μ. Figure 3.12 is the bottom of the Hasse diagram of $\mathbb{Y}$.

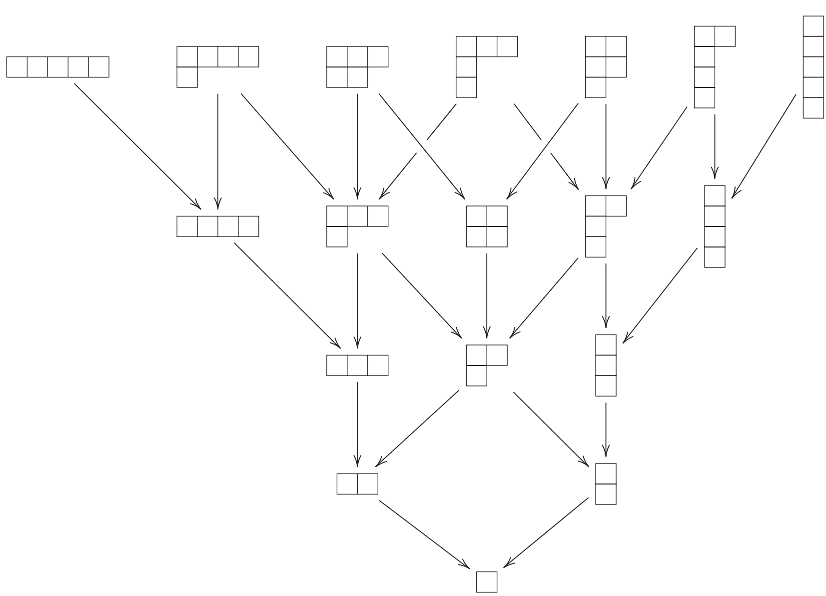


Figure 3.12 The bottom of the Young (branching) graph $\mathbb{Y}$

A *path* in the Young graph is a sequence $p = (\lambda^{(n)} \to \lambda^{(n-1)} \to \cdots \to \lambda^{(1)})$ of partitions $\lambda^{(k)} \vdash k$ such that $\lambda^{(k)}$ covers $\lambda^{(k-1)}$ for $k = 2, 3, \ldots, n$ (so that a path always ends at the trivial partition $\lambda^{(1)} = (1) \vdash 1$). The integer number $\ell(p) = n$ is called the *length* of the path p. We denote by $\Pi_n(\mathbb{Y})$ the set of all paths of length n in the Young graph and we set

$$\Pi(\mathbb{Y}) = \bigcup_{n=1}^{\infty} \Pi_n(\mathbb{Y}).$$

With a partition $\lambda \vdash n$ and a path $\lambda = \lambda^{(n)} \to \lambda^{(n-1)} \to \cdots \to \lambda^{(1)}$ we associate the standard tableau T of shape λ obtained by placing the integer $k \in \{1, 2, \ldots, n\}$ in the box $\lambda^{(k)}/\lambda^{(k-1)}$.

For instance, the standard tableau in Figure 3.5 is associated with the path

$$(4, 3, 1) \to (4, 3) \to (3, 3) \to (3, 2) \to (2, 2) \to (2, 1) \to (2) \to (1).$$

This way, we have established a natural bijection

$$\Pi_n(\mathbb{Y}) \leftrightarrow \mathrm{Tab}(n) \tag{3.8}$$

between the set of all paths in $\mathbb{Y}$ of length n and $\mathrm{Tab}(n)$ which extends to a bijection

$$\Pi(\mathbb{Y}) \leftrightarrow \bigcup_{n=1}^{\infty} \mathrm{Tab}(n). \tag{3.9}$$

By combining (3.8) with the bijection in Theorem 3.1.10 we obtain a bijection

$$\Pi_n(\mathbb{Y}) \leftrightarrow \mathrm{Cont}(n) \tag{3.10}$$

between the set of all paths of length n in $\mathbb{Y}$ and the set of all contents of n.

Finally, from Theorem 3.1.10, we deduce the following.

Proposition 3.1.13 *Let $\alpha, \beta \in \mathrm{Cont}(n)$. Suppose they correspond to the paths $\lambda^{(n)} \to \lambda^{(n-1)} \to \cdots \to \lambda^{(1)}$ and $\mu^{(n)} \to \mu^{(n-1)} \to \cdots \to \mu^{(1)}$, respectively. Then, $\alpha \approx \beta$ if and only if $\lambda^{(n)} = \mu^{(n)}$.*

3.2 The Young–Jucys–Murphy elements and a Gelfand–Tsetlin basis for $\mathfrak{S}_n$

In this section we prove that the chain

$$\mathfrak{S}_1 \le \mathfrak{S}_2 \le \cdots \le \mathfrak{S}_n \le \mathfrak{S}_{n+1} \le \cdots$$

is multiplicity-free (see Section 2.2) and we study the associated Gelfand–Tsetlin algebra. The main tool is represented by the Young–Jucys–Murphy elements and a related theorem of G. I. Olshanskii (Theorem 3.2.6).

3.2.1 The Young–Jucys–Murphy elements

Here and in the sequel, in order to simplify the notation, an element f of the group algebra $L(\mathfrak{S}_n)$ will be written as the formal sum $f = \sum_{\pi\in\mathfrak{S}_n} f(\pi)\pi$. In other words, we identify a group element $\pi \in \mathfrak{S}_n$ with the Dirac function centered at π. Analogously, if $A \subseteq \mathfrak{S}_n$, the characteristic function of A will be simply denoted by A. This way,

$$A = \sum_{\pi\in A} \pi. \tag{3.11}$$

Also, the convolution of two functions $f_1, f_2 \in L(\mathfrak{S}_n)$ will be denoted by $f_1 \cdot f_2$ and expressed in the form

$$f_1 \cdot f_2 = \sum_{\pi\in\mathfrak{S}_n} \left[\sum_{\substack{\sigma,\theta\in\mathfrak{S}_n:\\ \sigma\theta=\pi}} f_1(\sigma) f_2(\theta) \right] \pi,$$

that is, as a product of formal sums. The *Young–Jucys–Murphy* (YJM) *elements* in $L(\mathfrak{S}_n)$ are defined by $X_1 = 0$ and,

$$X_k = (1, k) + (2, k) + \cdots + (k - 1, k)$$

for $k = 2, \ldots, n$. These elements were introduced, independently, by A.-A. A. Jucys [70] and G.E. Murphy [96]. However, they were implicitly used in the original papers of A. Young [127] in connection with his orthogonal and seminormal forms (see also [44]).

3.2.2 Marked permutations

Let $\ell, k \geq 1$. In what follows $\mathfrak{S}_{\ell+k}, \mathfrak{S}_\ell, \mathfrak{S}_k$ will denote the symmetric groups on $\{1, 2, \ldots, \ell + k\}$, $\{1, 2, \ldots, \ell\}$ and $\{\ell + 1, \ell + 2, \ldots, \ell + k\}$, respectively. In particular we have $\mathfrak{S}_\ell, \mathfrak{S}_k \leq \mathfrak{S}_{\ell+k}$ and $\mathfrak{S}_\ell \cap \mathfrak{S}_k = \{1\}$. Finally, in the notation of Section 2.1, we set

$$Z(\ell, k) = \mathcal{C}(\mathfrak{S}_{\ell+k}, \mathfrak{S}_\ell),$$

that is, $Z(\ell, k)$ is the algebra of all $\mathfrak{S}_\ell$-conjugacy invariant functions in $L(\mathfrak{S}_{\ell+k})$.

In order to analyze $Z(\ell, k)$, the first step consists in determining a parameterization of the orbits of the $\mathfrak{S}_\ell$-conjugacy action on $\mathfrak{S}_{\ell+k}$. Since $\mathfrak{S}_\ell$ acts on $\{1, 2, \ldots, \ell\}$, given $\pi \in \mathfrak{S}_\ell$ and $\theta \in \mathfrak{S}_{\ell+k}$, the cycle structure of $\pi\theta\pi^{-1}$ is obtained by replacing $1, 2, \ldots, \ell$ with $\pi(1), \pi(2), \ldots, \pi(\ell)$ in the cycle structure of θ (cf. (3.1)). Therefore, the $\mathfrak{S}_\ell$-orbit of θ is obtained by permuting in all possible ways the elements $1, 2, \ldots, \ell$, leaving the remaining elements $\ell+1, \ell+2, \ldots, \ell+k$ unchanged, in the cycle structure of θ.

Following [99] and [100], we introduce a nice pictorial way to describe these orbits.

Consider a permutation written using the notation of Remark 3.1.1:

$$\begin{aligned}&(a_1 \to a_2 \to \cdots \to a_i \to a_1)(b_1 \to b_2 \to \cdots \to b_j \to b_1)\\ &\quad\cdots(c_1 \to c_2 \to \cdots \to c_h \to c_1).\end{aligned} \tag{3.12}$$

Note that we compute the product from right to left, that is, the cycle $(c_1 \to c_2 \to \cdots \to c_h \to c_1)$ acts first. For instance we have $(1 \to 2 \to 1)(1 \to 3 \to 1) = (1 \to 3 \to 2 \to 1)$.

A *marked permutation* of the set $\{a_1, a_2, \ldots, a_i, b_1, b_2, \ldots, b_j, \ldots, c_1, c_2, \ldots, c_h\}$ is a permutation as in (3.12) together with a labeling of all the arrows by nonnegative integers, called *tags*, with some additional empty cycles also labelled with tags. For instance, the permutation (3.12) may be marked as follows.

$$\begin{aligned}&(a_1 \xrightarrow{u_1} a_2 \xrightarrow{u_2} \cdots \xrightarrow{u_{i-1}} a_i \xrightarrow{u_i} a_1)(b_1 \xrightarrow{v_1} b_2 \xrightarrow{v_2} \cdots \xrightarrow{v_{j-1}} b_j \xrightarrow{v_j} b_1)\cdots\\ &\qquad\cdots(c_1 \xrightarrow{w_1} c_2 \xrightarrow{w_2} \cdots \xrightarrow{w_{h-1}} c_h \xrightarrow{w_h} c_1)(\xrightarrow{z_1})(\xrightarrow{z_2})\cdots(\xrightarrow{z_t}).\end{aligned}$$

It is easy to see that the conjugacy orbits of $\mathfrak{S}_\ell$ on $\mathfrak{S}_{\ell+k}$ are in natural one-to-one correspondence with the set of all marked permutations of $\{\ell+1, \ell+2, \ldots, \ell+k\}$ such that the sum of all tags is equal to ℓ.

For instance, if we take

$$(\ell+1 \xrightarrow{1} \ell+5 \xrightarrow{2} \ell+3 \xrightarrow{0} \ell+1)(\ell+2 \xrightarrow{3} \ell+4 \xrightarrow{1} \ell+2)(\xrightarrow{v})(\xrightarrow{w}) \tag{3.13}$$

where $1+2+3+1+v+w = \ell$, then (3.13) represents the orbit made up of all permutations of the form

$$\begin{aligned}&(\ell+1 \to x_1 \to \ell+5 \to x_2 \to x_3 \to \ell+3 \to \ell+1)\cdot\\ &\qquad\cdot(\ell+2 \to x_4 \to x_5 \to x_6 \to \ell+4 \to x_7 \to \ell+2)\cdot\\ &\qquad\cdot(x_8 \to x_9 \to \cdots \to x_{7+v} \to x_8)\cdot\\ &\qquad\cdot(x_{8+v} \to x_{9+v} \to \cdots \to x_{7+v+w} \to x_{8+v})\end{aligned}$$

where $\{x_1, x_2, \ldots, x_{7+v+w}\} = \{1, 2, \ldots, \ell\}$.

When writing a marked permutation, we shall usually omit trivial cycles of the form $(a \stackrel{0}{\to} a)$ and $(\stackrel{1}{\to})$.

We now give an application of this simple fact which indeed constitutes the core of the whole theory developed in this chapter.

Theorem 3.2.1 *Let $\mathfrak{S}_{n-1}$ be the symmetric group on $\{1, 2, \ldots, n-1\}$. Then $(\mathfrak{S}_n \times \mathfrak{S}_{n-1}, \widetilde{\mathfrak{S}}_{n-1})$ is a symmetric Gelfand pair.*

Proof By virtue of Proposition 2.1.12, we have to show that every permutation $\pi \in \mathfrak{S}_n$ is $\mathfrak{S}_{n-1}$-conjugate to π^{-1}. Now, if π has the cycle decomposition

$$\pi = (n \to a_2 \to a_3 \to \cdots \to a_i \to n)(b_1 \to b_2 \to \cdots \to b_j \to b_1)\cdots \\ \cdots(c_1 \to c_2 \to \cdots \to c_h \to c_1)$$

then π belong to the S_{n-1}-conjugacy class associated with the marked permutation

$$(n \stackrel{i-1}{\to} n)(\stackrel{j}{\to})\cdots(\stackrel{h}{\to}). \tag{3.14}$$

It is then clear that

$$\pi^{-1} = (n \to a_i \to \cdots \to a_3 \to a_2 \to n)(b_1 \to b_j \to \cdots \to b_2 \to b_1)\cdots \\ \cdots(c_1 \to c_h \to \cdots \to c_2 \to c_1)$$

belongs to the same class of (3.14). □

Recalling Theorem 2.1.10, we immediately have the following:

Corollary 3.2.2 *The algebra $\mathcal{C}(\mathfrak{S}_n, \mathfrak{S}_{n-1})$ is commutative, $\mathfrak{S}_{n-1}$ is a multiplicity-free subgroup of $\mathfrak{S}_n$ and*

$$\mathfrak{S}_1 \le \mathfrak{S}_2 \le \cdots \le \mathfrak{S}_{n-1} \le \mathfrak{S}_n \le \cdots$$

is a multiplicity-free chain.

Incidentally, this also proves that $\mathfrak{S}_n$ is ambivalent (see Exercise 1.5.27).

We now present three other basic examples of marked permutations. Recall that $\mathfrak{S}_\ell, \mathfrak{S}_k \le \mathfrak{S}_{\ell+k}$ are the symmetric groups on $\{1, 2, \ldots, \ell\}$ and $\{\ell+1, \ell+2, \ldots, \ell+k\}$, respectively and that, for a subset $A \subseteq \mathfrak{S}_{\ell+k}$ (for instance an $\mathfrak{S}_\ell$-conjugacy class) we use the notation in (3.11) to denote its characteristic function.

Example 3.2.3

(i) For $j = 1, 2, \dots, k$, the YJM element $X_{\ell+j}$ may be represented in the form:

$$X_{\ell+j} = (\ell + j \xrightarrow{1} \ell + j) + \sum_{h=\ell+1}^{\ell+j-1} (\ell + j \xrightarrow{0} h \xrightarrow{0} \ell + j). \tag{3.15}$$

This shows, in particular, that $X_{\ell+1}, X_{\ell+2}, \dots, X_{\ell+k} \in Z(\ell, k)$.

(ii) Any $\sigma \in \mathfrak{S}_k$ forms a one-element orbit of $\mathfrak{S}_\ell$. The corresponding marked permutation is simply the cyclic representation of σ with all tags equal to zero (and ℓ-many omitted trivial cycles of the form $(\xrightarrow{1})$). In other words, viewing an element $\sigma \in \mathfrak{S}_k$ as an element in the group algebra $L(\mathfrak{S}_k)$, we have $\mathfrak{S}_k \subseteq Z(\ell, k)$.

(iii) Suppose that $\mathcal{C}_\lambda$ is the conjugacy class of $\mathfrak{S}_\ell$ corresponding to the partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_h) \vdash \ell$ (cf. Proposition 3.1.3). Then, it forms an $\mathfrak{S}_\ell$-orbit and it is represented by the marked permutation $(\xrightarrow{\lambda_1})(\xrightarrow{\lambda_2})\cdots(\xrightarrow{\lambda_h})$ (with k omitted trivial cycles of the form $(a \xrightarrow{0} a)$, where $a = \ell + 1, \ell + 2, \dots, \ell + k$). In other words, the center of $L(\mathfrak{S}_\ell)$, that will be denoted by $Z(\ell)$, is contained in $Z(\ell, k)$, that is, $Z(\ell) \subseteq Z(\ell, k)$.

A recent paper on the Gelfand pair $(\mathfrak{S}_n \times \mathfrak{S}_{n-1}, \widetilde{\mathfrak{S}}_{n-1})$ (see Theorem 3.2.1) is [116].

3.2.3 Olshanskii's theorem

Our next task is to prove a converse to the preceding example, namely that $Z(\ell, k)$ is generated by $X_{\ell+1}, X_{\ell+2}, \dots, X_{\ell+k}$, $\mathfrak{S}_k$ and $Z(\ell)$. We first need another definition and an elementary lemma.

We denote by $Z_h(\ell, k)$ the subspace of $Z(\ell, k)$ spanned by the $\mathfrak{S}_\ell$-conjugacy classes made up of all permutations with at least $\ell + k - h$ fixed points (i.e. that move at most h elements). Then

$$\mathbb{C}1 = Z_0(\ell, k) \subseteq Z_1(\ell, k) \subseteq \cdots \subseteq Z_{\ell+k-1}(\ell, k) \subseteq Z_{\ell+k}(\ell, k) \equiv Z(\ell, k).$$

For $f_1, f_2, f_3 \in Z(\ell, k)$ we write

$$f_1 \cdot f_2 = f_3 + \text{lower terms}$$

if there exists h such that $f_3 \in Z_h(\ell, k) \setminus Z_{h-1}(\ell, k)$ and $f_1 \cdot f_2 - f_3 \in Z_{h-1}(\ell, k)$.

We now present a particular "multiplication rule" for marked permutations.

Lemma 3.2.4 *Let $i, j \geq 1$. Let $a_1, a_2, \dots, a_i$ and $b_1, b_2, \dots, b_j$ be two sequences of distinct elements in $\{\ell + 1, \ell + 2, \dots, \ell + k\}$. Suppose that, as a* *product in* $\mathfrak{S}_k$,

$$\begin{aligned}&(a_1 \to a_2 \to \cdots \to a_i \to a_1)(b_1 \to b_2 \to \cdots \to b_j \to b_1)\\ &\quad = (c_1 \to c_2 \to \cdots \to c_h \to c_1)\end{aligned}$$

with $h = |\{a_1, a_2, \dots, a_i\} \cup \{b_1, b_2, \dots, b_j\}| \le i + j$ *(note that, if* $h > 1$ *then, in the product above, none of the elements* $a_1, a_2, \dots, a_i, b_1, b_2, \dots, b_j$ *becomes a fixed point). Take* $u_1, u_2, \dots, u_i, v_1, v_2, \dots, v_j \ge 0$ *such that* $u_1 + u_2 + \cdots + u_i + v_1 + v_2 + \cdots + v_j \le \ell$*. Then, in* $Z(\ell, k)$ *we have:*

$$\begin{aligned}&(a_1 \xrightarrow{u_1} a_2 \xrightarrow{u_2} \cdots \xrightarrow{u_{i-1}} a_i \xrightarrow{u_i} a_1)(b_1 \xrightarrow{v_1} b_2 \xrightarrow{v_2} \cdots \xrightarrow{v_{j-1}} b_j \xrightarrow{v_j} b_1)\\ &\qquad = (c_1 \xrightarrow{w_1} c_2 \xrightarrow{w_2} \cdots \xrightarrow{w_{h-1}} c_h \xrightarrow{w_h} c_1) + \textit{lower terms}\end{aligned}$$

where the numbers $w_1, w_2, \dots, w_h$ *are given by the following rule:*

$$w_s = \begin{cases} v_t & \text{if } c_s = b_t \text{ and } b_{t+1} \notin \{a_1, a_2, \dots, a_i\} \\ v_t + u_m & \text{if } c_s = b_t \text{ and } b_{t+1} = a_m \\ u_m & \text{if } c_s = a_m \notin \{b_1, b_2, \dots, b_j\} \end{cases}$$

for $s = 1, 2, \dots, h$.

Proof Consider a product of the form

$$\begin{aligned}&(a_1, x_1, x_2, \dots, x_{u_1}, a_2, \dots, a_i, y_1, y_2, \dots, y_{u_i}) \cdot\\ &\qquad \cdot (b_1, z_1, z_2 \dots, z_{v_1}, b_2, \dots, b_j, r_1, r_2, \dots, r_{v_j}). \quad (3.16)\end{aligned}$$

If the numbers $x_1, x_2, \dots, x_{u_1}, y_1, y_2, \dots, y_{u_i}, z_1, z_2 \dots, z_{v_1}$ and $r_1, r_2, \dots, r_{v_j}$ in $\{1, 2, \dots, \ell\}$ are all distinct (this is possible because $u_1 + u_2 + \cdots + u_i + v_1 + v_2 + \cdots + v_j \le \ell$) then (3.16) is equal to a permutation of the form

$$(c_1, x'_1, x'_2, \dots, x'_{w_1}, c_2, \dots, c_h, y'_1, y'_2, \dots, y'_{w_h}),$$

where $w_1, w_2, \dots, w_h$ are given by the above rule.

Otherwise, the product in (3.16) moves less than $h + w_1 + w_2 + \cdots + w_h$ elements. □

We now give some examples of applications of this rule.

Example 3.2.5

(i) For $a \in \{\ell + 1, \ell + 2, \dots, \ell + k\}$ and $0 \le u \le \ell$ we have

$$[(a \xrightarrow{1} a)]^u = (a \xrightarrow{u} a) + \text{ lower terms.}$$

(ii) For $a, b \in \{\ell + 1, \ell + 2, \dots, \ell + k\}$, $a \ne b$ and $0 \le u \le \ell$ we have

$$(b \xrightarrow{u} b)(a \xrightarrow{0} b \xrightarrow{0} a) = (a \xrightarrow{u} b \xrightarrow{0} a).$$

Note that in this case we do not have lower terms in the right-hand side.

(iii) For $a_1, a_2, \ldots, a_i \in \{\ell+1, \ell+2, \ldots, \ell+k\}$, all distinct, and for $u_1, u_2, \ldots, u_i \geq 0$ with $u_1 + u_2 + \cdots + u_i \leq \ell$, we have

$$(a_1 \xrightarrow{u_i} a_1)(a_1 \xrightarrow{u_{i-1}} a_i \xrightarrow{0} a_1) \cdots (a_1 \xrightarrow{u_2} a_3 \xrightarrow{0} a_1) \cdot (a_1 \xrightarrow{u_1} a_2 \xrightarrow{0} a_1)$$
$$= (a_1 \xrightarrow{u_1} a_2 \xrightarrow{u_2} a_3 \xrightarrow{u_3} \cdots \xrightarrow{u_{i-1}} a_i \xrightarrow{u_i} a_1) + \text{lower terms.}$$

We are now in position to prove the following:

Theorem 3.2.6 (G. I. Olshanskii, [101]) *The centralizer algebra $Z(\ell, k)$ is generated by the YJM elements $X_{\ell+1}, X_{\ell+2}, \ldots, X_{\ell+k}$, the subgroup $\mathfrak{S}_k$ and the center $Z(\ell)$ of $\mathfrak{S}_\ell$. In formulas:*

$$Z(\ell, k) =< X_{\ell+1}, X_{\ell+2}, \ldots, X_{\ell+k}, \mathfrak{S}_k, Z(\ell) > .$$

Proof Set $\mathcal{A} =< X_{\ell+1}, X_{\ell+2}, \ldots, X_{\ell+k}, \mathfrak{S}_k, Z(\ell) >$. We have already seen that $\mathcal{A} \subseteq Z(\ell, k)$.

We prove by induction that $Z_h(\ell, k) \subset \mathcal{A}$ for all $h = 0, 1, \ldots, \ell + k$. First observe that $Z_0(\ell, k) = \mathbb{C}1$ is trivially contained in $\mathcal{A}$.

Now note that, for any choice of $a, j \in \{\ell+1, \ell+2, \ldots, \ell+k\}$ with $a \neq j$, one has $(a \xrightarrow{0} j \xrightarrow{0} a) \in \mathfrak{S}_k$. As a consequence, as (cf. (3.15))

$$(a \xrightarrow{1} a) = X_a - \sum_{j=\ell+1}^{a-1} (a \xrightarrow{0} j \xrightarrow{0} a) \tag{3.17}$$

one deduces that $(a \xrightarrow{1} a) \in \mathcal{A}$.

By a repeated application of Lemma 3.2.4 (see also the examples following it) we have that if $a_1, a_2, \ldots, a_i, \ldots, b_1, b_2, \ldots, b_j$ are distinct numbers in $\{\ell + 1, \ell+2, \ldots, \ell+k\}$ and $u_1, u_2, \ldots, u_i, \ldots, v_1, v_2, \ldots, v_j, m_1, m_2, \ldots, m_t$ are nonnegative integers such that

$$u_1 + u_2 + \cdots + u_i + \cdots + v_1 + v_2 + \cdots + v_j + m_1 + m_2 + \cdots + m_t = h \leq \ell,$$

then

$$\begin{aligned}
&(a_1 \xrightarrow{1} a_1)^{u_i} \cdot (a_i \xrightarrow{1} a_i)^{u_{i-1}} \cdot (a_1 \xrightarrow{0} a_i \xrightarrow{0} a_1) \cdots \\
&\quad \cdots (a_3 \xrightarrow{1} a_3)^{u_2} \cdot (a_1 \xrightarrow{0} a_3 \xrightarrow{0} a_1)(a_2 \xrightarrow{1} a_2)^{u_1} \cdot (a_1 \xrightarrow{0} a_2 \xrightarrow{0} a_1) \cdots \\
&\quad \cdots (b_1 \xrightarrow{1} b_1)^{v_j} \cdot (b_j \xrightarrow{1} b_j)^{v_{j-1}} \cdot (b_1 \xrightarrow{0} b_j \xrightarrow{0} b_1) \cdots \\
&\quad \cdots (b_3 \xrightarrow{1} b_3)^{v_2} \cdot (b_1 \xrightarrow{0} b_3 \xrightarrow{0} b_1)(b_2 \xrightarrow{1} b_2)^{v_1} \cdot (b_1 \xrightarrow{0} b_2 \xrightarrow{0} b_1) \cdot \\
&\quad \cdot (\xrightarrow{m_1})(\xrightarrow{m_2}) \cdots (\xrightarrow{m_t})
\end{aligned} \tag{3.18}$$

equals, modulo lower terms,

$$(a_1 \stackrel{u_1}{\to} a_2 \stackrel{u_2}{\to} \cdots \stackrel{u_{i-1}}{\to} a_i \stackrel{u_i}{\to} a_1)\cdots(b_1 \stackrel{v_1}{\to} b_2 \stackrel{v_2}{\to} \cdots \stackrel{v_{j-1}}{\to} b_j \stackrel{v_j}{\to} b_1). \qquad (3.19)$$

Note that (3.18) belongs to $\mathcal{A}$: see Example 3.2.5 and Example 3.2.3. As (3.19) represents the typical orbit in $Z_h(\ell, k)$ (but not in $Z_{h-1}(\ell, k)$), by the inductive hypothesis we are done. □

Corollary 3.2.7 *The Gelfand–Tsetlin algebra $GZ(n)$ of the multiplicity-free chain $\mathfrak{S}_1 \leq \mathfrak{S}_2 \leq \cdots \leq \mathfrak{S}_n$ is generated by the YJM elements $X_1, X_2, \dots, X_n$.*

Proof For $2 \leq k \leq n$ denote by T_k the set of all transpositions in $\mathfrak{S}_k$. Then $T_k \in Z(k)$ and since $X_k = T_k - T_{k-1}$, we conclude that $X_1, X_2, \dots, X_n \in GZ(n)$.

On the other hand,

$$Z(n) \equiv \mathcal{C}(\mathfrak{S}_n, \mathfrak{S}_n) \subseteq \mathcal{C}(\mathfrak{S}_n, \mathfrak{S}_{n-1}) \equiv Z(n-1, 1) =< Z(n-1), X_n >$$

where the last equality follows from Theorem 3.2.6. Supposing, by induction, that $GZ(n-1) =< X_1, X_2, \dots, X_{n-1} >$ it follows that

$$GZ(n) =< GZ(n-1), Z(n) >=< X_1, X_2, \dots, X_{n-1}, X_n >$$

completing the proof. □

Remark 3.2.8 We can use Corollary 3.2.7 to give an alternative proof that $Z(n-1, 1) \equiv \mathcal{C}(\mathfrak{S}_n, \mathfrak{S}_{n-1})$ is commutative (see Corollary 3.2.2). Indeed, $Z(n-1, 1) =< Z(n-1), X_n >$ and X_n commutes with every element in $Z(n-1)$. To see this last fact observe that $X_n = T_n - T_{n-1}$ and that $T_n \in Z(n)$ and $T_{n-1} \in Z(n-1)$.

Exercise 3.2.9 Prove directly that $X_1, X_2, \dots, X_n$ commute.

Hint. By induction on n, using the identity $X_k T_n = T_n X_k$, $k = 1, 2, \dots, n$.

3.2.4 A characterization of the YJM elements

Regarding $\mathfrak{S}_{n-1}$ as the subgroup of $\mathfrak{S}_n$ which acts on $\{1, 2, \dots, n-1\}$, we define a map

$$\begin{array}{ccc} \mathfrak{S}_n & \to & \mathfrak{S}_{n-1} \\ \pi & \mapsto & \pi_n \end{array} \qquad (3.20)$$

by setting, for $k = 1, 2, \dots, n-1$,

$$\pi_n(k) = \begin{cases} \pi(k) & \text{if } \pi(k) \neq n \\ \pi(n) & \text{if } \pi(k) = n. \end{cases} \qquad (3.21)$$

In other words, π_n is the permutation obtained by removing from the cycle decomposition of π the term $\to n \to$: if the cycle decomposition of π contains the cycle

$$(a_1 \to a_2 \to \cdots \to a_h \to n \to a_1) \tag{3.22}$$

then π_n is obtained from π by replacing in its cycle decomposition the term (3.22) with $(a_1 \to a_2 \to \cdots \to a_h \to a_1)$.

Lemma 3.2.10 *The map (3.20) has the following properties:*
(1) $(1_{\mathfrak{S}_n})_n = 1_{\mathfrak{S}_{n-1}}$;
(2) $\sigma_n = \sigma$ *for all* $\sigma \in \mathfrak{S}_{n-1}$;
(3) $(\sigma\pi\theta)_n = \sigma\pi_n\theta$ *for all* $\pi \in \mathfrak{S}_n$ *and* $\sigma, \theta \in \mathfrak{S}_{n-1}$.

Proof (1) and (2) are obvious. To prove (3), just note that, for all $k = 1, 2, \ldots, n-1$, we have

$$(\sigma\pi\theta)_n(k) = \begin{cases} \sigma\pi\theta(k) & \text{if } \pi[\theta(k)] \neq n \\ \sigma\pi(n) & \text{if } \pi[\theta(k)] = n \end{cases} = \sigma\pi_n\theta(k). \qquad \square$$

Lemma 3.2.11 *Suppose that* $n \geq 4$. *Let* $\Phi : \mathfrak{S}_n \to \mathfrak{S}_{n-1}$ *be a map satisfying the condition*

$$\Phi(\sigma\pi\theta) = \sigma\Phi(\pi)\theta \tag{3.23}$$

for all $\pi \in \mathfrak{S}_n$ *and* $\sigma, \theta \in \mathfrak{S}_{n-1}$ *(this is the analogue of condition* (3) *in the preceding lemma). Then,* Φ *necessarily coincides with the map (3.20).*

Proof First of all, note that if $\sigma, \theta \in \mathfrak{S}_{n-1}$, then $\Phi(\pi) = \sigma$ if and only if $\Phi(\pi\sigma^{-1}\theta) \equiv \Phi[1_{\mathfrak{S}_{n-1}}\pi(\sigma^{-1}\theta)]$ equals θ. This implies that $|\{\pi \in \mathfrak{S}_n : \Phi(\pi) = \sigma\}|$ is constant and equal to n for every $\sigma \in \mathfrak{S}_{n-1}$.

Moreover, if $\Phi(\pi) = 1_{\mathfrak{S}_{n-1}}$ then, for all $\sigma \in \mathfrak{S}_{n-1}$ one has $\Phi(\sigma\pi\sigma^{-1}) = \sigma\Phi(\pi)\sigma^{-1} = 1_{\mathfrak{S}_{n-1}}$. This implies that

$$\{\pi \in \mathfrak{S}_n : \Phi(\pi) = 1_{\mathfrak{S}_{n-1}}\} = \{1_{\mathfrak{S}_n}\} \cup \{(a \to n \to a) : a = 1, 2, \ldots, n-1\}. \tag{3.24}$$

Indeed, the second set in the right-hand side is the unique $\mathfrak{S}_{n-1}$-conjugacy class of $\mathfrak{S}_n$ with $n-1$ elements (recall that $n \geq 4$). Since for any cycle (in $\mathfrak{S}_n$) which contains n we have

$$(a_1 \to a_2 \to \cdots \to a_h \to n \to a_1) = (a_1 \to a_2 \to \cdots \to a_h \to a_1)(a_h \to n \to a_h)$$

we deduce that

$$\begin{aligned}
\Phi(a_1 \to a_2 \to \cdots \to a_h \to n \to a_1) & \\
&= \Phi[(a_1 \to a_2 \to \cdots \to a_h \to a_1)(a_h \to n \to a_h)] \\
\text{(by (3.23))} \quad &= (a_1 \to a_2 \to \cdots \to a_h \to a_1)\Phi(a_h \to n \to a_h) \\
\text{(by (3.24))} \quad &= (a_1 \to a_2 \to \cdots \to a_h \to a_1) \\
&= (a_1 \to a_2 \to \cdots \to a_h \to n \to a_1)_n.
\end{aligned}$$

Finally, as in the cycle decomposition of a permutation in $\mathfrak{S}_n$ the number n appears at most one time, another application of (3) shows that $\Phi(\pi) = \pi_n$ for all $\pi \in \mathfrak{S}_n$. □

The map $\pi \mapsto \pi_n$ extends to a map

$$p_n : L(\mathfrak{S}_n) \to L(\mathfrak{S}_{n-1})$$

simply by setting

$$p_n\left(\sum_{\pi \in \mathfrak{S}_n} f(\pi)\pi\right) = \sum_{\pi \in \mathfrak{S}_n} f(\pi)\pi_n.$$

Now we can give a characterization of X_n in terms of the map p_n.

Proposition 3.2.12 *We have*

$$p_n^{-1}(< 1_{\mathfrak{S}_{n-1}} >) \cap Z(n-1, 1) =< X_n, 1_{\mathfrak{S}_n} > . \tag{3.25}$$

Proof Recalling (3.24) we have that the $\mathfrak{S}_{n-1}$-conjugacy orbits of $\{\pi \in \mathfrak{S}_n : \pi_n = 1_{\mathfrak{S}_{n-1}}\}$ are $\{1_{\mathfrak{S}_n}\}$ and $\{(a \to n \to a) : a = 1, 2, \ldots, n-1\}$. Passing to the group algebra we immediately deduce (3.25). □

3.3 The spectrum of the Young–Jucys–Murphy elements and the branching graph of $\mathfrak{S}_n$

In this section, we show that the spectrum of the YJM elements is described by the set Cont(n) introduced in Section 3.1.5 and prove that the branching graph of the multiplicity-free chain $\mathfrak{S}_1 \leq \mathfrak{S}_2 \leq \cdots \leq \mathfrak{S}_n \leq \cdots$ coincides with the Young graph.

3.3.1 The weight of a Young basis vector

Let $\rho \in \widehat{\mathfrak{S}_n}$ and consider the Gelfand–Tsetlin basis $\{v_T : T \in \mathcal{T}(\rho)\}$ associated with the multiplicity free chain $\mathfrak{S}_1 \leq \mathfrak{S}_2 \leq \cdots \leq \mathfrak{S}_n$ (cf. (2.14)).

In this setting, $\{v_T : T \in \mathcal{T}(\rho)\}$ is also called the *Young basis* for V_ρ. From Theorem 2.2.2 and Corollary 3.2.7 we have that every v_T is an eigenvector of $\rho(X_j)$ for all $j = 1, 2, \dots, n$. This suggests the following definition: for every vector v_T we set

$$\alpha(T) = (a_1, a_2, \dots, a_n) \tag{3.26}$$

where a_j is the eigenvalue of $\rho(X_j)$ corresponding to v_T, that is, $\rho(X_j)v_T = a_j v_T$, $j = 1, 2, \dots, n$.

Since $X_1, X_2, \dots, X_n$ generate the Gelfand–Tsetlin algebra $GZ(n)$, Corollary 2.2.3 ensures that v_T is determined (up to a scalar factor) by the eigenvalues a_j's. The vector $\alpha(T)$ is called the *weight* of v_T.

Note that $\check{X}_j = X_j$ and therefore $\rho(X_j)$ is self-adjoint for all X_j and all representations ρ of $\mathfrak{S}_n$ (see Corollary 1.5.13).

In the following proposition we study the action of the Coxeter generators (cf. 3.2) on the Young basis; we show that s_k changes only the kth level of the branching graph.

Proposition 3.3.1 *For every $\rho \in \widehat{\mathfrak{S}_n}$ and $T \equiv (\rho = \rho_n \to \rho_{n-1} \to \cdots \to \rho_2 \to \rho_1) \in \mathcal{T}(\rho)$, the vector $\rho(s_k)v_T$ is a linear combination of vectors $v_{T'}$ with $T' \equiv (\sigma = \sigma_n \to \sigma_{n-1} \to \cdots \to \sigma_2 \to \sigma_1) \in \mathcal{T}(\rho)$ such that $\sigma_i = \rho_i$ for $i \neq k$.*

Proof Let V_j denote the representation space of ρ_j, $j = 1, 2, \dots, n$. Note that

$$V_j \equiv \{\rho_j(f)v_T : f \in L(\mathfrak{S}_j)\}.$$

Indeed, the right-hand side is a $\mathfrak{S}_j$-invariant subspace.

For $j > k$ we have $s_k \in \mathfrak{S}_j$ and therefore $\rho_j(s_k)v_T \in V_j$. This implies that $\sigma_j = \rho_j$ for all $j = k+1, k+2, \dots, n$.

On the other hand, if $j < k$ then s_k and $\mathfrak{S}_j$ commute. Setting $W_j = \{\rho_j(f)\rho(s_k)v_T : f \in L(\mathfrak{S}_j)\} \equiv \rho(s_k)V_j$ we have that the map

$$\begin{array}{ccc} V_j & \to & W_j \\ \rho_j(f)v_T & \mapsto & \rho_j(f)\rho(s_k)v_T \end{array}$$

is an isomorphism of $\mathfrak{S}_j$-representations. It follows that $\rho(s_k)v_T$ belongs to the ρ_j-isotypic component of $\mathrm{Res}^{\mathfrak{S}_n}_{\mathfrak{S}_j}\rho$ and therefore $\sigma_j = \rho_j$ for all $j = 1, 2, \dots, k-1$. □

3.3.2 The spectrum of the YJM elements

In what follows we set

$$\mathrm{Spec}(n) = \{\alpha(T) : T \in \mathcal{T}(\rho), \rho \in \widehat{\mathfrak{S}_n}\}$$

where $\alpha(T)$ is the weight of v_T (as in (3.26)). In other words, $\mathrm{Spec}(n)$ is the *spectrum* of the YJM elements of $\mathfrak{S}_n$.

Since this spectrum determines the elements of the Young basis, we have

$$|\mathrm{Spec}(n)| = \sum_{\rho \in \widehat{\mathfrak{S}_n}} \dim V_\rho. \tag{3.27}$$

In other words, $\mathrm{Spec}(n)$ is in natural bijection with the set of all paths of the branching graph of $\mathfrak{S}_1 \leq \mathfrak{S}_2 \leq \cdots \leq \mathfrak{S}_n$. We denote by $\mathrm{Spec}(n) \ni \alpha \mapsto T_\alpha$ this correspondence. We also denote by v_α the Young basis vector corresponding to T_α.

We now introduce an equivalence relation $\sim$ on $\mathrm{Spec}(n)$ by setting, for α and $\beta \in \mathrm{Spec}(n)$, $\alpha \sim \beta$ if v_α and v_β belong to the same irreducible $\mathfrak{S}_n$-representation (in terms of the branching graph, this means that the corresponding paths have the same starting point).

Remark 3.3.2 As a consequence of (3.27) and the definition of $\sim$ we immediately have

$$|\mathrm{Spec}(n)/\sim| = |\widehat{\mathfrak{S}_n}|. \tag{3.28}$$

The next task is to obtain an explicit description of $\mathrm{Spec}(n)$ and $\sim$. This will be achieved by means of commutation relations between the YJM elements and the Coxeter generators of $\mathfrak{S}_n$. These are

$$s_i X_j = X_j s_i \qquad \text{for } j \neq i, i+1 \tag{3.29}$$

which is obvious, and

$$s_i X_i + 1 = X_{i+1} s_i \tag{3.30}$$

which is equivalent to $s_i X_i s_i + s_i = X_{i+1}$, and this is immediate. Another equivalent way to write (3.30) is

$$s_i X_{i+1} - 1 = X_i s_i. \tag{3.31}$$

Warning. In what follows, if v_α is a vector of the Young basis of an irreducible representation ρ of $\mathfrak{S}_n$, we denote $\rho(s_i)v_\alpha$ and $\rho(X_i)v_\alpha$ simply by $s_i v_\alpha$ and $X_i v_\alpha$, respectively.

Proposition 3.3.3 *Let* $\alpha = (a_1, a_2, \ldots, a_i, a_{i+1}, \ldots a_n) \in \mathrm{Spec}(n)$. *Then*

(i) $a_i \neq a_{i+1}$ *for* $i = 1, 2, \ldots, n-1$
(ii) $a_{i+1} = a_i \pm 1$ *if and only if* $s_i v_\alpha = \pm v_\alpha$
(iii) *If* $a_{i+1} \neq a_i \pm 1$ *then*

$$\alpha' := s_i\alpha \equiv (a_1, a_2, \ldots, a_{i-1}, a_{i+1}, a_i, a_{i+2}, \ldots a_n) \in \mathrm{Spec}(n),$$

$\alpha \sim \alpha'$ and the vector associated with α' is, up to a scalar factor,

$$v_{\alpha'} = s_i v_\alpha - \frac{1}{a_{i+1} - a_i} v_\alpha. \tag{3.32}$$

Moreover, the space $< v_\alpha, v_{\alpha'} >$ is invariant for X_i, X_{i+1} and s_i, and in the basis $\{v_\alpha, v_{\alpha'}\}$, these operators are represented by the matrices

$$\begin{pmatrix} a_i & 0 \\ 0 & a_{i+1} \end{pmatrix}, \quad \begin{pmatrix} a_{i+1} & 0 \\ 0 & a_i \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \frac{1}{a_{i+1}-a_i} & 1 - \frac{1}{(a_{i+1}-a_i)^2} \\ 1 & \frac{1}{a_i - a_{i+1}} \end{pmatrix},$$

respectively.

Proof Clearly, from the definitions of α and v_α, we have $X_i v_\alpha = a_i v_\alpha$ and $X_{i+1} v_\alpha = a_{i+1} v_\alpha$. Moreover, from (3.30) and (3.31) it follows that the space $< v_\alpha, s_i v_\alpha >$ is invariant for X_i and X_{i+1} (and, clearly, it is invariant for s_i).

Note now that if $s_i v_\alpha = \lambda v_\alpha$, then $s_i^2 = 1$ implies that $\lambda^2 = 1$, that is, $\lambda = \pm 1$. Moreover, from (3.30) we have

$$a_i s_i v_\alpha + v_\alpha = a_{i+1} s_i v_\alpha,$$

and therefore $s_i v_\alpha = \pm v_\alpha$ if and only if $a_{i+1} = a_i \pm 1$. This proves (ii).

Suppose now that $a_{i+1} \neq a_i \pm 1$. Then $\dim < v_\alpha, s_i v_\alpha >= 2$ and the restrictions of s_i, X_i and X_{i+1} to $< v_\alpha, s_i v_\alpha >$ are represented, with respect to the basis $\{v_\alpha, s_i v_\alpha\}$, by the matrices

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} a_i & -1 \\ 0 & a_{i+1} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} a_{i+1} & 1 \\ 0 & a_i \end{pmatrix},$$

respectively. Indeed, from (3.30) and (3.31) we deduce that $X_i s_i v_\alpha = -v_\alpha + a_{i+1} s_i v_\alpha$ and $X_{i+1} s_i v_\alpha = v_\alpha + a_i s_i v_\alpha$.

But a matrix of the form

$$\begin{pmatrix} a & \pm 1 \\ 0 & b \end{pmatrix}$$

is diagonalizable if and only if $a \neq b$ and, if this is the case, the eigenvalues are a with eigenvector $(1, 0)$ and b with eigenvector $(\pm 1/(b-a), 1)$. Applying this elementary fact to our context, we get that $v' := s_i v_\alpha - \frac{1}{a_{i+1}-a_i} v_\alpha$

is an eigenvector of X_i and X_{i+1} with eigenvalues a_{i+1} and a_i, respectively. Moreover, (3.29) implies that $X_j v' = a_j v'$ for $j \neq i, i+1$; therefore $\alpha' := (a_1, a_2, \ldots, a_{i-1}, a_{i+1}, a_i, a_{i+2}, \ldots, a_n) \in \mathrm{Spec}(n)$ and $v' \equiv v_{\alpha'}$ is a vector of the Young basis. To end the proof, it suffices to check the formula representing s_i in the basis $\{v_\alpha, v_{\alpha'}\}$. We leave this elementary task to the reader. □

3.3.3 Spec(n) = Cont(n)

Let $\alpha = (a_1, a_2, \ldots, a_n) \in \mathrm{Spec}(n)$. If $a_{i+1} \neq a_i \pm 1$, so that $(a_1, a_2, \ldots, a_{i+1}, a_i, \ldots, a_n) \in \mathrm{Spec}(n)$, we say that s_i is an *admissible transposition* for α (cf. Corollary 3.1.11).

The Coxeter generators $s_1, s_2, \ldots, s_{n-1}$ satisfy the following relations (the *Coxeter relations*):

$$\begin{aligned} &\text{(i)}\ \ s_i s_j = s_j s_i && \text{if } |i-j| \neq 1 \\ &\text{(ii)}\ \ s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} && \text{for } i = 1, 2, \ldots, n-1 \end{aligned} \tag{3.33}$$

whose proof is immediate.

Lemma 3.3.4 *Let* $\alpha = (a_1, a_2, \ldots, a_n) \in \mathbb{C}^n$. *If* $a_i = a_{i+2} = a_{i+1} - 1$ *for some* $i \in \{1, 2, \ldots, n-2\}$, *then* $\alpha \notin \mathrm{Spec}(n)$.

Proof Suppose, by contradiction, that $\alpha \in \mathrm{Spec}(n)$ and that $a_i = a_{i+2} = a_{i+1} - 1$. Then by (ii) in Proposition 3.3.3 we have

$$s_i v_\alpha = v_\alpha \quad \text{and} \quad s_{i+1} v_\alpha = -v_\alpha.$$

By the Coxeter relation (ii) in (3.33) we have $v_\alpha = s_{i+1} s_i s_{i+1} v_\alpha = s_i s_{1+1} s_i v_\alpha = -v_\alpha$, which is impossible. □

Lemma 3.3.5

(i) *For every* $(a_1, a_2, \ldots, a_n) \in \mathrm{Spec}(n)$ *we have* $a_1 = 0$
(ii) *If* $\alpha = (a_1, a_2, \ldots, a_n) \in \mathrm{Spec}(n)$, *then* $\alpha' = (a_1, a_2, \ldots, a_{n-1}) \in \mathrm{Spec}(n-1)$
(iii) $\mathrm{Spec}(2) = \{(0, 1), (0, -1)\}$.

Proof (i) This is obvious as $X_1 = 0$.

(ii) This follows from the fact that $X_1, X_2, \ldots, X_{n-1} \in L(\mathfrak{S}_{n-1})$ and $X_j v_\alpha = a_j v_\alpha$ for all $j = 1, 2, \ldots, n-1$.

(iii) The irreducible representations of $\mathfrak{S}_2$ are ι and ε (cf. Example 1.2.4). The branching graph of $\mathfrak{S}_1 \leq \mathfrak{S}_2$ is just as shown in Figure 3.13

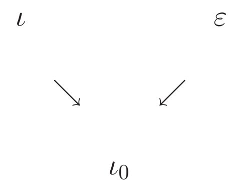


Figure 3.13

where ι_0 is the trivial representation of $\mathfrak{S}_1$. Now, $X_2 = (1 \to 2 \to 1)$ and if $v \in V_\iota$ then $X_2 v = v$, while if $w \in V_\varepsilon$ then $X_2 w = -w$. □

Lemma 3.3.6

(i) *For every* $n \geq 1$ *we have* $\mathrm{Spec}(n) \subseteq \mathrm{Cont}(n)$

(ii) *If* $\alpha \in \mathrm{Spec}(n)$*,* $\beta \in \mathrm{Cont}(n)$ *and* $\alpha \approx \beta$*, then* $\beta \in \mathrm{Spec}(n)$ *and* $\alpha \sim \beta$*.*

Proof (i) We show it by induction on n. For $n = 1$ this is trivial, while for $n = 2$ this follows from (iii) in Lemma 3.3.5 and (3.7).

Suppose that $\mathrm{Spec}(n-1) \subseteq \mathrm{Cont}(n-1)$. Let $\alpha = (a_1, a_2 \dots, a_n) \in \mathrm{Spec}(n)$.

By virtue of (i) in Lemma 3.3.5 we have $a_1 = 0$ and this corresponds to condition (1) in the definition of $\mathrm{Cont}(n)$ in (3.6).

By (ii) in Lemma 3.3.5, we only need to check that conditions (2) and (3) in (3.6) are satisfied just for $q = n$.

Suppose by contradiction that

$$\{a_n - 1, a_n + 1\} \cap \{a_1, a_2, \dots a_{n-1}\} = \emptyset. \tag{3.34}$$

By Proposition. 3.3.3.(iii), the transposition $(n-1 \to n \to n-1)$ is admissible for α, that is, $(a_1, a_2 \dots, a_{n-2}, a_n, a_{n-1}) \in \mathrm{Spec}(n)$. It follows that $(a_1, a_2 \dots, a_{n-2}, a_n) \in \mathrm{Spec}(n-1) = \mathrm{Cont}(n-1)$. From (3.34) we deduce that $\{a_n - 1, a_n + 1\} \cap \{a_1, a_2, \dots a_{n-2}\} = \emptyset$ which contradicts (2) in (3.6) for $\mathrm{Cont}(n-1)$.

Again by contradiction, suppose that α does not satisfy condition (3) in (3.6) for $q = n$, that is, $a_p = a_n = a$ for some $p < n$ and, for instance,

$$a - 1 \notin \{a_{p+1}, a_{p+2}, \dots, a_{n-1}\}.$$

We can also suppose that p is maximal, that is, also $a \notin \{a_{p+1}, a_{p+2}, \dots, a_{n-1}\}$.

Since $(a_1, a_2, \dots, a_{n-1}) \in \mathrm{Cont}(n-1)$ (by the inductive hypothesis), the number $a+1$ may appear in $\{a_{p+1}, a_{p+2}, \dots, a_{n-1}\}$ at most once (by the maximality of p). Suppose $a + 1 \notin \{a_{p+1}, a_{p+2}, \dots, a_{n-1}\}$. Then $(a_p, a_{p+1}, \dots, a_n) = (a, *, \dots, *, a)$, where every $*$ represents a number different from a, $a+1$ and $a-1$. In this case, by a sequence of $n - p - 1$

admissible transpositions we get

$$\alpha \sim \alpha' := (\ldots, a, a, \ldots) \in \mathrm{Spec}(n)$$

which contradicts (i) in Proposition 3.3.3.

Similarly, if $a+1 \in \{a_{p+1}, a_{p+2}, \ldots, a_{n-1}\}$, then $(a_p, a_{p+1}, \ldots, a_n) = (a, *, \ldots, *, a+1, *, \ldots, *, a)$, where again every $*$ represents a number different from $a, a+1$ and $a-1$. Now, by a sequence of admissible transpositions we get

$$\alpha \sim \alpha' := (\ldots, a, a+1, a \ldots) \in \mathrm{Spec}(n)$$

which is impossible by Lemma 3.3.4. Therefore, also (3) in (3.6) is satisfied and this ends the proof of (i).

(ii) This is an immediate consequence of (i), Corollary 3.1.11 and Proposition 3.3.3.(iii). □

Theorem 3.3.7 *We have* $\mathrm{Spec}(n) = \mathrm{Cont}(n)$. *Moreover, the equivalence relations* $\sim$ *and* $\approx$ *coincide. Finally the Young graph* $\mathbb{Y}$ *is isomorphic to the branching graph of the multiplicity-free chain* $\mathfrak{S}_1 \leq \mathfrak{S}_2 \leq \cdots \leq \mathfrak{S}_n \leq \mathfrak{S}_{n+1} \leq \cdots$

Proof First of all note that

$$|\mathrm{Cont}(n)/\approx| = |\mathrm{Spec}(n)/\sim|. \tag{3.35}$$

Indeed, by Corollary 3.1.12 we have that $|\mathrm{Cont}(n)/\approx|$ equals the number of partitions of n which (cf. Proposition 3.1.3) equals the number of conjugacy classes of $\mathfrak{S}_n$. The latter, by Corollary 1.3.16, equals the number of irreducible inequivalent representations of $\mathfrak{S}_n$ and, by (3.28), the equality in (3.35) follows.

Now, from Lemma 3.3.6 it follows that an equivalence class in $\mathrm{Cont}(n)/\approx$ either is disjoint from $\mathrm{Spec}(n)$ or it is contained in one equivalence class in $\mathrm{Spec}(n)/\sim$. In other words, the partition of $\mathrm{Spec}(n)$ induced by $\approx$ is finer than the partition of $\mathrm{Spec}(n)$ induced by $\sim$. Collecting all previous inequalities together with (3.35), we have

$$|\mathrm{Spec}(n)/\sim| \leq |\mathrm{Spec}(n)/\approx| \leq |\mathrm{Cont}(n)/\approx| = |\mathrm{Spec}(n)/\sim|,$$

which proves the first two statements of the theorem.

Note that this also gives a natural bijective correspondence between the set of all paths in the branching graph (parameterized by $\mathrm{Spec}(n)$) and the set of all paths in $\mathbb{Y}$ parameterized by $\mathrm{Cont}(n)$, see (3.10). This yields a bijective correspondence between the vertices of these graphs (by Proposition 3.1.13 and the definition of $\sim$). This correspondence is clearly a graph isomorphism. □

From the above theorem, we also get a natural correspondence between $\widehat{\mathfrak{S}_n}$ and the nth level of the branching graph $\mathbb{Y}$, that is, the set of all partitions of n.

Definition 3.3.8 Given a partition $\lambda \vdash n$, we denote by S^λ the irreducible representation of $\mathfrak{S}_n$ spanned by the vectors $\{v_\alpha\}$, with $\alpha \in \mathrm{Spec}(n) \equiv \mathrm{Cont}(n)$ corresponding to the standard tableau of shape λ.

Proposition 3.3.9 $\dim S^\lambda = |\mathrm{Tab}(\lambda)|$, *that is, the dimension of* S^λ *is equal to the number of standard* λ*-tableaux.*

As an immediate consequence of Theorem 3.3.7 we then have:

Corollary 3.3.10 *Let* $0 \le k < n$, $\lambda \vdash n$, *and* $\mu \vdash k$. *Then the multiplicity* $m_{\mu,\lambda}$ *of* S^μ *in* $\mathrm{Res}^{\mathfrak{S}_n}_{\mathfrak{S}_k} S^\lambda$ *is equal to zero if* $\mu \not\preceq \lambda$ *and it equals the number of paths in* $\mathbb{Y}$ *from* λ *to* μ*, otherwise. In any case,* $m_{\mu,\lambda} \le (n-k)!$ *and this estimate is sharp.*

Proof We clearly have

$$\mathrm{Res}^{\mathfrak{S}_n}_{\mathfrak{S}_k} S^\lambda = \mathrm{Res}^{\mathfrak{S}_{k+1}}_{\mathfrak{S}_k} \mathrm{Res}^{\mathfrak{S}_{k+2}}_{\mathfrak{S}_{k+1}} \cdots \mathrm{Res}^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}} S^\lambda$$

where at each step of the consecutive restrictions the decomposition is multiplicity-free and according to the branching graph $\mathbb{Y}$.

This way, the multiplicity of S^μ in $\mathrm{Res}^{\mathfrak{S}_n}_{\mathfrak{S}_k} S^\lambda$ is equal to the number of paths in $\mathbb{Y}$ that start at λ and end at μ. It is also equal to the number of ways in which we can obtain the diagram of λ from the diagram of μ by adding successively $n-k$ addable boxes to the diagram of μ (therefore at each step we have a diagram of a partition). In particular, this multiplicity is bounded above by $(n-k)!$; this estimate is sharp when the boxes can be added to different rows and columns: see Figure 3.14. □

Corollary 3.3.11 (Branching rule) *For every* $\lambda \vdash n$ *we have*

$$\mathrm{Res}^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}} S^\lambda = \bigoplus_{\substack{\mu \vdash n-1: \\ \lambda \to \mu}} S^\mu \tag{3.36}$$

that is, the sum runs over all partitions $\mu \vdash n-1$ *that may be obtained from* λ *by removing one box. Moreover, for every* $\mu \vdash n-1$,

$$\mathrm{Ind}^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}} S^\mu = \bigoplus_{\substack{\lambda \vdash n: \\ \lambda \to \mu}} S^\lambda. \tag{3.37}$$

Proof (3.36) is a particular case of Corollary 3.3.10. Finally, (3.37) is equivalent to (3.36) by Frobenius reciprocity (Theorem 1.6.11). □

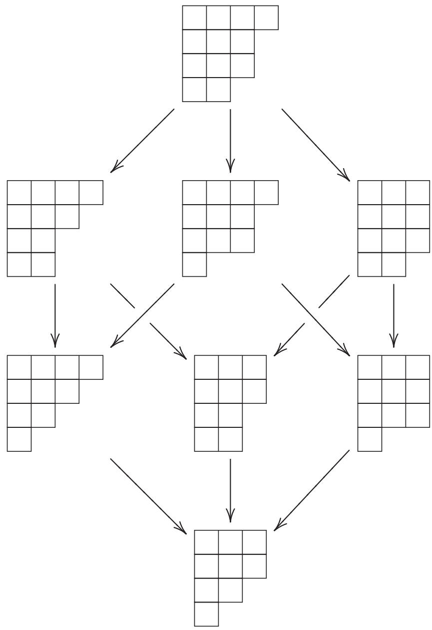


Figure 3.14 The $(12-9)! = 6$ paths from $(4,3,3,2) \vdash 12$ to $(3,3,2,1) \vdash 9$

In the following we give a characterization of the map $\lambda \mapsto S^\lambda$ (that associates with each partition of n an irreducible representation of $\mathfrak{S}_n$) by means of the branching rule.

Corollary 3.3.12 *For all $n \geq 1$, let $(V^\lambda)_{\lambda \vdash n}$ be a family of representations of $\mathfrak{S}_n$ such that*

(i) $V^{(1)} \sim S^{(1)}$ *(the trivial and unique representation of $\mathfrak{S}_1$);*
(ii) $V^{(2)}$ *and $V^{1,1}$ are the trivial and the alternating representations of $\mathfrak{S}_2$, respectively;*
(iii) $\mathrm{Ind}_{\mathfrak{S}_{n-1}}^{\mathfrak{S}_n} V^\mu = \bigoplus_{\substack{\lambda \vdash n:\\ \lambda \to \mu}} V^\lambda$ *for every $\mu \vdash n-1$ and $n \geq 2$.*

Then, V^λ is irreducible and isomorphic to S^λ, for every $\lambda \vdash n$, and $n \geq 1$.

Proof It is an easy inductive argument. Just note that $\lambda \vdash n$ is uniquely determined by the set $\{\mu \vdash n-1 : \lambda \to \mu\}$. □

Remark 3.3.13 It is worth examining the case $k = n-2$ of Corollary 3.3.10 as it illustrates, in this particular case, the statement of Proposition 3.3.3.

Let $\lambda \vdash n$ and $\mu \vdash n-2$. If $\mu \not\preceq \lambda$ then S^μ is not contained in $\mathrm{Res}_{\mathfrak{S}_{n-2}}^{\mathfrak{S}_n} S^\lambda$. If $\mu \preceq \lambda$, the multiplicity of S^μ in $\mathrm{Res}_{\mathfrak{S}_{n-2}}^{\mathfrak{S}_n} S^\lambda$ is at most 2. We have two cases.

(1) There is only one partition $\nu \vdash n-1$ such that $\mu \preceq \nu \preceq \lambda$. This means that in $\mathbb{Y}$ between μ and λ there is a chain as in Figure 3.15(a).

$$\begin{array}{c} \lambda \\ \downarrow \\ \nu \\ \downarrow \\ \mu \end{array}$$

Figure 3.15(a)

Clearly, in this case, the boxes of the skew diagram λ/μ are on the same row or on the same column. In other words, if

$$\lambda^{(n)} \equiv \lambda \to \lambda^{(n-1)} \equiv \nu \to \lambda^{(n-2)} \equiv \mu \to \lambda^{(n-3)} \to \cdots \to \lambda^{(1)} \tag{3.38}$$

is any path containing $\lambda \to \nu \to \mu$, then it corresponds to $\alpha = (a_1, a_2, \dots, a_n) \in \text{Spec}(n)$ with $a_n = a_{n-1} \pm 1$ (more precisely, $a_n = a_{n-1} + 1$ if the boxes of λ/μ are on the same row, $a_n = a_{n-1} - 1$ if they are on the same column). In particular, $s_{n-1} v_\alpha = \pm v_\alpha$ (cf. Proposition 3.3.3). This agrees with Proposition 3.3.1: s_{n-1} only affects the $(n-1)$st level of the branching graph and ν is the only partition between μ and λ.

(2) There are two partitions $\nu, \eta \vdash n-1$ such that $\mu \preceq \nu, \eta \preceq \lambda$. Now the boxes of λ/μ are on different rows and columns (as in Figure 3.11) and the branching graph from λ to μ is a square as in Figure 3.15(b).

$$\begin{array}{ccccc} & & \lambda & & \\ & \swarrow & & \searrow & \\ \nu & & & & \eta \\ & \searrow & & \swarrow & \\ & & \mu & & \end{array}$$

Figure 3.15(b)

Moreover, if $\alpha \in \mathrm{Spec}(n)$ corresponds to a path $\lambda \to \nu \to \mu \to \cdots$ as in (3.38), then $a_n \neq a_{n-1} \pm 1$ and $\alpha' = (a_1, a_2, \ldots, a_{n-2}, a_n, a_{n-1}) \in \mathrm{Spec}(n)$ corresponds to the path $\lambda \to \eta \to \mu \to \cdots$. Now, the action of s_{n-1} on v_α and $v_{\alpha'}$ is given by (iii) in Proposition 3.3.3 (see also the Young formula in the next section) and Proposition 3.3.1 is confirmed also in this case.

3.4 The irreducible representations of $\mathfrak{S}_n$

In this section, we prove some classical formulae for the matrix coefficients and the characters of the irreducible representations of $\mathfrak{S}_n$.

3.4.1 Young's seminormal form

Recall the standard tableau T^λ, defined in Figure 3.6, and Remark 3.1.7. Also recall that the chain $\mathfrak{S}_1 \leq \mathfrak{S}_2 \leq \cdots \leq \mathfrak{S}_n$ determines a decomposition of every irreducible representation of $\mathfrak{S}_n$ into one-dimensional subspaces and that the GZ-basis is obtained by choosing a non-trivial vector in each of these subspaces. If such vectors are normalized, we say that it is an *orthonormal* basis, otherwise we shall refer to it as an *orthogonal* basis. Note that, in both cases, the vectors are defined up to a *scalar factor* (of modulus one if they are normalized).

Theorem 3.3.7 ensures that we may parameterize the vectors of the GZ-basis with the standard tableaux: for $T \in \mathrm{Tab}(\lambda)$ we denote by v_T the corresponding vector in the GZ-basis.

Proposition 3.4.1 *It is possible to choose the scalar factors of the vectors* $\{v_T : T \in \mathrm{Tab}(n)\}$ *in such a way that, for every* $T \in \mathrm{Tab}(n)$*, one has*

$$\pi_T^{-1} v_{T^\lambda} = v_T + \sum_{\substack{R \in \mathrm{Tab}(\lambda): \\ \ell(\pi_R) < \ell(\pi_T)}} \gamma_R v_R,$$

where $\gamma_R \in \mathbb{C}$ *(actually, in Corollary 3.4.3, we will show that* $\gamma_R \in \mathbb{Q}$*) and* π_T *is as in Theorem 3.1.5.*

Proof We prove the statement by induction on $\ell(\pi_T)$. At each stage, we choose the scalar factor for all T with $\ell(\pi_T) = \ell$. If $\ell(\pi_T) = 1$, that is, π_T is an admissible transposition for T^λ, this follows from Proposition 3.3.3; in particular, we can use (3.32) to choose the scalar factor of v_T (that corresponds to $v_{\alpha'}$ in that formula).

Suppose now that $\pi_T = s_{i_1} s_{i_2} \cdots s_{i_{\ell-1}} s_j$ is the standard decomposition of π_T into the product of admissible transpositions (cf. Remark 3.1.7); we have set $\ell = \ell(\pi_T)$ and $j = i_\ell$ for simplicity of notation. Then $\pi_T = \pi_{T_1} s_j$, where $T_1 = s_j T$ is a standard tableau. Clearly, $\ell(\pi_{T_1}) = \ell(\pi_T) - 1$.

Therefore, by the inductive hypothesis, we may suppose that

$$\pi_{T_1}^{-1} v_{T^\lambda} = v_{T_1} + \sum_{\substack{R \in \mathrm{Tab}(\lambda): \\ \ell(\pi_R) < \ell(\pi_{T_1})}} \gamma_R^{(1)} v_R. \tag{3.39}$$

Since $T = s_j T_1$, the formula (3.32) in Proposition 3.3.3 ensures that we can choose the scalar factor of v_T so that

$$s_j v_{T_1} = v_T + \frac{1}{a_{j+1} - a_j} v_{T_1} \tag{3.40}$$

where $(a_1, a_2, \ldots, a_n) = C(T_1)$ is the content of T_1 (see (3.5)).

The formula in the statement follows from (3.39) and (3.40), keeping in mind again Proposition 3.3.3 for the computation of $s_j v_R$ for $R \in \mathrm{Tab}(\lambda)$ with $\ell(\pi_R) < \ell(\pi_{T_1})$. □

Theorem 3.4.2 (Young's seminormal form) *Choose the vectors of the GZ-basis of $\mathfrak{S}_n$ according to Proposition 3.4.1. If $T \in \mathrm{Tab}(\lambda)$ and $C(T) = (a_1, a_2, \ldots, a_n)$ is its content, then the adjacent transposition s_j acts on v_T as follows*

(i) *if $a_{j+1} = a_j \pm 1$ then $s_j v_T = \pm v_T$*
(ii) *if $a_{j+1} \neq a_j \pm 1$, setting $T' = s_j T$, then*

$$s_j v_T = \begin{cases} \frac{1}{a_{j+1}-a_j} v_T + v_{T'} & \text{if } \ell(\pi_{T'}) > \ell(\pi_T) \\ \frac{1}{a_{j+1}-a_j} v_T + [1 - \frac{1}{(a_{j+1}-a_j)^2}] v_{T'} & \text{if } \ell(\pi_{T'}) < \ell(\pi_T). \end{cases} \tag{3.41}$$

Proof (i) It is an immediate consequence of (ii) in Proposition 3.3.3.

(ii) These formulas again follow from Proposition 3.3.3. We only have to check that, with respect to the choice made in Proposition 3.4.1, $s_j v_T$ has exactly that expression.

Suppose that $\ell(\pi_{T'}) > \ell(\pi_T)$. We have

$$\pi_{T'} = \pi_T s_j \tag{3.42}$$

and

$$\pi_T^{-1} v_{T^\lambda} = v_T + \sum_{\substack{R \in \mathrm{Tab}(\lambda): \\ \ell(\pi_R) < \ell(\pi_T)}} \gamma_R v_R. \tag{3.43}$$

From (3.42) and (3.43) we deduce

$$\begin{aligned}\pi_{T'}^{-1} v_{T^\lambda} &= v_{T'} + \sum_{\substack{R' \in \mathrm{Tab}(\lambda): \\ \ell(\pi_{R'}) < \ell(\pi_{T'})}} \gamma'_{R'} v_{R'} \\ &= s_j v_T + s_j \sum_{\substack{R \in \mathrm{Tab}(\lambda): \\ \ell(\pi_R) < \ell(\pi_T)}} \gamma_R v_R\end{aligned}$$

and therefore (3.32) holds exactly in that form (the coefficient of $v_{T'}$ in $s_j v_T$ is 1).

The case $\ell(\pi_{T'}) < \ell(\pi_T)$ is analogous (starting from $\pi_T = \pi_{T'} s_j$ and taking $\alpha = C(T')$ when we apply Proposition 3.3.3). □

Corollary 3.4.3 *In the* orthogonal *bases of Proposition 3.4.1 (and Theorem 3.4.2) the matrix coefficients of the irreducible representations of* $\mathfrak{S}_n$ *are* rational *numbers. In particular, the coefficients* γ_R *in Proposition 3.4.1 are rational numbers.* □

3.4.2 Young's orthogonal form

Note that one can normalize the basis $\{v_T : T \in \mathrm{Tab}(\lambda)\}$ of S^λ by taking

$$w_T = \frac{v_T}{\|v_T\|_{S^\lambda}} \tag{3.44}$$

where the norm $\|\cdot\|_{S^\lambda}$ is associated with an arbitrary invariant scalar product that makes S^λ a unitary representation of $\mathfrak{S}_n$.

Let T be a standard tableau and let $C(T) = (a_1, a_2, \ldots, a_n)$ be its content. For $i, j \in \{1, 2, \ldots, n\}$, the *axial distance* from j to i in T is the integer $a_j - a_i$. It has a clear geometrical meaning. Suppose that we move from j to i, each step to the left or downwards being counted $+1$, while each step to the right or upwards being counted -1. Then the resulting integer is exactly $a_j - a_i$ and it is independent of the chosen path (Figure 3.16(a)(b)).

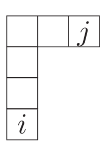


Figure 3.16(a) The axial distance is $a_j - a_i = 5$.

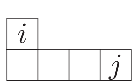


Figure 3.16(b) The axial distance is $a_j - a_i = 2$.

Theorem 3.4.4 (Young's orthogonal form) *Given the* orthonormal *basis* $\{w_T : T \in \mathrm{Tab}(n)\}$ *(cf.* (3.44)*) we have*

$$s_j w_T = \frac{1}{r} w_T + \sqrt{1 - \frac{1}{r^2}}\, w_{s_j T}$$

where, for $C(T) = (a_1, a_2, \dots, a_n)$*, the quantity* $r = a_{j+1} - a_j$ *is the axial distance from* $j + 1$ *to* j*.*

In particular, if $a_{j+1} = a_j \pm 1$ *we have* $r = \pm 1$ *and* $s_j w_T = \pm w_T$*.*

Proof Set $T' = s_j T$ and suppose $\ell(\pi_{T'}) > \ell(\pi_T)$. Then, from (ii) in Theorem 3.4.2 it follows that

$$\begin{aligned}
\|v_{T'}\|^2 &= \|s_j v_T - \frac{1}{r} v_T\|^2 \\
&= \|v_T\|^2 - \frac{1}{r}\langle s_j v_T, v_T\rangle - \frac{1}{r}\langle v_T, s_j v_T\rangle + \frac{1}{r^2}\|v_T\|^2 \\
&= \left(1 - \frac{1}{r^2}\right)\|v_T\|^2,
\end{aligned}$$

where the second equality follows from the fact that s_j is unitary and the third one from the fact that $s_j v_T = \frac{1}{r} v_T + v_{T'}$ and $v_T \perp v_{T'}$.

Then, in the orthonormal basis $\{\frac{v_T}{\|v_T\|}, \frac{v_{T'}}{\sqrt{1-\frac{1}{r^2}}\|v_T\|}\}$ the first line of (3.41) becomes

$$s_j w_T = \frac{1}{r} w_T + \sqrt{1 - \frac{1}{r^2}}\, w_{T'}.$$

In the case $\ell(\pi_{T'}) < \ell(\pi_T)$ the proof is similar. □

The following is the *Young orthogonal form*:

$$\begin{cases}
s_j w_T &= \frac{1}{r} w_T + \sqrt{1 - \frac{1}{r^2}}\, w_{s_j T} \\
s_j w_{s_j T} &= -\frac{1}{r} w_{s_j T} + \sqrt{1 - \frac{1}{r^2}}\, w_T
\end{cases}$$

where, if $C(T) = (a_1, a_2, \dots, a_n)$, we posed, as before, $r = a_{j+1} - a_j$. In other words, in the basis $\{w_T, w_{s_j T}\}$, the linear operator s_j is represented by the orthogonal matrix

$$s_j \equiv \begin{pmatrix} \frac{1}{r} & \sqrt{1 - \frac{1}{r^2}} \\ \sqrt{1 - \frac{1}{r^2}} & -\frac{1}{r} \end{pmatrix}.$$

Example 3.4.5 Let $\lambda = (n) \vdash n$ be the trivial partition. Then there exists only one standard λ-tableau T, namely that in Figure 3.17.

1	2					n

Figure 3.17

The corresponding content is $C(T) = (0, 1, \ldots, n-1)$ and, clearly, $s_j w_T = w_T$ for $j = 1, 2, \ldots, n-1$ (we have $a_{j+1} = a_j + 1$). It follows that $S^{(n)}$ is the trivial representation of $\mathfrak{S}_n$.

Example 3.4.6 Let $\lambda = (1, 1, \ldots, 1) \vdash n$. Then, again, there exists only one standard λ-tableau T, namely that in Figure 3.18.

1
2
n

Figure 3.18

The corresponding content is $C(T) = (0, -1, \ldots, -n+1)$ and, clearly, $s_j w_T = -w_T$ for $j = 1, 2, \ldots, n-1$ (now $a_{j+1} = a_j - 1$) and therefore $S^{1,1,\ldots,1}$ is the alternating representation of $\mathfrak{S}_n$.

Example 3.4.7 Consider the representation $S^{n-1,1}$. The standard $(n-1, 1)$-tableaux are of the form shown in Figure 3.19.

1	2				$j-1$	$j+1$			n
j									

Figure 3.19 $T_j \in \mathrm{Tab}(n-1, 1)$, $j = 2, 3, \ldots, n$.

The corresponding content is

$$C(T) = (0, 1, \ldots, j-2, -1, j-1, j, \ldots, n-1), \tag{3.45}$$

where -1 is in jth position. Denoting by w_j the normalized Young basis vector corresponding to T_j, then the Young orthogonal form becomes

$$s_j w_j = \frac{1}{j} w_j + \sqrt{1 - \frac{1}{j^2}}\, w_{j+1} \tag{3.46}$$

$$s_{j-1} w_j = -\frac{1}{j-1} w_j + \sqrt{1 - \frac{1}{(j-1)^2}}\, w_{j-1} \tag{3.47}$$

and

$$s_k w_j = w_j, \qquad \text{for } k \neq j-1, j. \tag{3.48}$$

Moreover, the branching rule yields

$$\mathrm{Res}^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}} S^{n-1,1} = S^{(n-1)} \oplus S^{n-2,1}. \tag{3.49}$$

In particular, $S^{n-1,1}$ coincides with the subspace V_1 in Example 1.4.5, namely the space of mean zero functions on $L(X)$, where $X = \mathfrak{S}_n/\mathfrak{S}_{n-1} \equiv \{1, 2, \ldots, n\}$. Indeed, $S^{n-1,1}$ is non-trivial and, by (3.49), it contains non-trivial $\mathfrak{S}_{n-1}$-invariant vectors. These facts determine V_1: the latter is the unique non-trivial irreducible representation in $L(X)$.

Now we want to express explicitly the vectors w_j as functions on X. Set

$$\widetilde{w}_j = \frac{1}{\sqrt{j(j-1)}}\mathbf{1}_{j-1} - \sqrt{\frac{j-1}{j}}\delta_j \tag{3.50}$$

where δ_j is the Dirac function at j and $\mathbf{1}_j = \delta_1 + \delta_2 + \cdots + \delta_j$, for $j = 2, 3, \ldots, n$.

It is easy to see (exercise) that $\{\widetilde{w}_j : j = 2, 3, \ldots, n\}$ is an orthonormal basis for V_1.

We now show that the action of an adjacent transposition satisfies (3.46), (3.47) and (3.48). Indeed,

$$\begin{aligned}\frac{1}{j}\widetilde{w}_j + \sqrt{1-\frac{1}{j^2}}\widetilde{w}_{j+1} &= \frac{1}{j\sqrt{j(j-1)}}\mathbf{1}_{j-1} - \frac{1}{j}\sqrt{\frac{j-1}{j}}\delta_j \\ &\quad + \frac{1}{j}\sqrt{\frac{j-1}{j}}\mathbf{1}_j - \sqrt{\frac{j-1}{j}}\delta_{j+1} \\ &= \frac{1}{\sqrt{j(j-1)}}\mathbf{1}_{j-1} - \sqrt{\frac{j-1}{j}}\delta_{j+1} \\ &= s_j\widetilde{w}_j.\end{aligned}$$

Similarly one proves (3.47), while (3.48) is obvious.

Exercise 3.4.8 Show that $\widetilde{w}_j$ in (3.50) corresponds to the path

$$\begin{aligned}&(n-1,1) \to (n-2,1) \to \cdots \to (j,1) \to (j-1,1) \to (j-1) \to (j-2) \\ &\quad \to \cdots \to (2) \to (1)\end{aligned}$$

in the Young diagram by examining the action of $\mathfrak{S}_n \geq \mathfrak{S}_{n-1} \geq \cdots \geq \mathfrak{S}_2 \geq \mathfrak{S}_1$. This again identifies $\widetilde{w}_j$ with w_j.

Exercise 3.4.9 Let $\widetilde{w}_j$ be as in (3.50). Show, by direct computation, that

$$X_i\widetilde{w}_j = \begin{cases} (i-1)\widetilde{w}_j & \text{for } i < j \\ -\widetilde{w}_j & \text{for } i = j \\ (i-2)\widetilde{w}_j & \text{for } i > j \end{cases}$$

(so that $\alpha(T_j) = (0, 1, 2, \ldots, j-2, -1, j-1, j, \ldots, n-2) \in \text{Spec}(n)$, compare with (3.45)).

3.4.3 The Murnaghan–Nakayama rule for a cycle

Let $\lambda, \mu \vdash n$. We denote by χ^λ_μ the value of the character of S^λ evaluated at a permutation π belonging to the conjugacy class of type μ (i.e. π has cycle structure μ). In Section 3.5.5 we shall give a recursive formula for χ^λ_μ. Now we limit ourselves to a particular case that will be used to prove the formula in the general case.

The diagram of λ is a *hook* if $\lambda = (n-k, 1^k) \equiv (n-k, 1, 1, \ldots, 1)$ (with k-many 1's) for some $0 \le k \le n-1$. See Figure 3.20. The integer k is called the *height* of the hook. We shall also call the partition λ a hook.

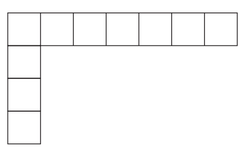


Figure 3.20 The hook of height 3 corresponding to $\lambda = (7, 1, 1, 1) \vdash 10$.

Note that a diagram is a hook if and only if it does not contain the box of coordinates (2, 2). We now give a formula for χ^λ_μ when $\mu = (n)$.

Theorem 3.4.10 *For $\lambda \vdash n$ we have*

$$\chi^\lambda_{(n)} = \begin{cases} (-1)^k & \text{if } \lambda = (n-k, 1^k) \\ 0 & \text{if } \lambda \text{ is not a hook.} \end{cases} \tag{3.51}$$

Proof First note that

$$X_2 X_3 \cdots X_n = \text{ sum of all } n \text{ cycles of } \mathfrak{S}_n. \tag{3.52}$$

This may be easily proved by induction, using the identity:

$$(\alpha_1 \to \alpha_2 \to \cdots \to \alpha_{k-1} \to \alpha_k \to \alpha_1)(\alpha_k \to k+1 \to \alpha_k)$$
$$= (\alpha_1 \to \alpha_2 \to \cdots \to \alpha_{k-1} \to \alpha_k \to k+1 \to \alpha_1) \tag{3.53}$$

for $\alpha_1, \alpha_2, \dots, \alpha_k$, $k+1$ distinct. Indeed, for $n = 2$ (3.52) is trivial. Now, if we fix an $n-1$ cycle, say, $(\alpha_1 \to \alpha_2 \to \cdots \to \alpha_{n-2} \to \alpha_{n-1} \to \alpha_1) \in \mathfrak{S}_{n-1}$ and we multiply it by X_n, then the product splits as the sum $\sum_{j=1}^{n-1} (\alpha_1 \to \alpha_2 \to \cdots \to \alpha_{n-2} \to \alpha_{n-1} \to \alpha_1)(j, n)$. For each summand we can express the product as $(\alpha_1' \to \alpha_2' \to \cdots \to \alpha_{n-2}' \to j \to \alpha_1')(j, n)$ (where $\alpha_1', \alpha_2', \dots, \alpha_{n-2}', j$ is a permutation of $\alpha_1, \alpha_2, \dots, \alpha_{n-1}$) which, by virtue of (3.53), becomes $(\alpha_1' \to \alpha_2' \to \cdots \to \alpha_{n-2}' \to j \to n \to \alpha_1')$ and this is an n cycle. It is clear that all n cycles appear (exactly one time) in the global sum.

Moreover, note that if λ is not a hook, $T \in \mathrm{Tab}(\lambda)$, j is the number in the box $(2, 2)$ of the diagram of λ, and $C(T) = (a_1, a_2, \dots, a_n)$, then $a_j = 0$ (Figure 3.21).

0	1	$\cdots$
-1	0	$\cdots$
$\vdots$	$\vdots$	

Figure 3.21

Consequently, $X_j w_T = 0 w_T = 0$ and

$$X_2 X_3 \cdots X_n w_T = 0 \quad \text{for all } T \in \mathrm{Tab}(\lambda). \tag{3.54}$$

If λ is a hook, say $\lambda = (n-k, 1^k)$, we clearly have $C(T^\lambda) = (0, 1, 2, \dots, n-k-1, -1, -2, \dots, -k)$, so that, more generally, for $T \in \mathrm{Tab}(\lambda)$, $C(T) = (0, a_2, a_3, \dots, a_n)$ with $\{a_2, a_3, \dots, a_n\} \equiv \{1, 2, \dots, n-k-1, -1, -2, \dots, -k\}$ (Figure 3.22).

0	1	$\cdots$	$\cdots$	$\cdots$	$n-k-1$
-1					
$\vdots$					
$\vdots$					
$-k$					

Figure 3.22

Therefore, for $T \in \mathrm{Tab}(\lambda)$, recalling the definition of $\alpha(T) \in \mathrm{Spec}(n)$ (see (3.26)) and Theorem 3.3.7, we have

$$X_2 X_3 \cdots X_n w_T = a_2 a_3 \cdots a_n w_T = (-1)^k k!(n-k-1)! w_T \tag{3.55}$$

Note also that, from Proposition 3.3.9 we get

$$\dim S^\lambda = \binom{n-1}{k}. \tag{3.56}$$

Indeed, a standard tableau of shape λ is determined by the k numbers placed in the boxes $(2, 1), (3, 1), \ldots, (k+1, 1)$.

Finally, since $(n-1)!$ is the number of all n cycles in $\mathfrak{S}_n$, from (3.52) and Lemma 1.3.13 we get, for any $\lambda \vdash n$

$$X_2 X_3 \cdots X_n w = \frac{(n-1)!}{\dim S^\lambda} \chi^\lambda_{(n)} w \tag{3.57}$$

for any $w \in S^\lambda$. Recall that, in our notation, $X_2 X_3 \cdots X_n w$ is the Fourier transform of $X_2 X_3 \cdots X_n$ at the representation S^λ applied to w.

Then, (3.51) follows from (3.54), (3.55), (3.56) and (3.57). □

3.4.4 The Young seminormal units

In this section, we give an expression, in terms of the YJM elements, of the primitive idempotents in $L(\mathfrak{S}_n)$ associated with the Gelfand–Tsetlin bases for the irreducible representations (see Proposition 1.5.17).

For each $T \in \mathrm{Tab}(n)$, the primitive idempotent in $L(\mathfrak{S}_n)$ associated with the Gelfand–Tsetlin vector w_T is the normalized matrix coefficient

$$e_T(\pi) = \frac{d_\lambda}{n!} \langle \pi w_T, w_T \rangle_{S^\lambda} \tag{3.58}$$

for all $\pi \in \mathfrak{S}_n$ where λ is the shape of T and $d_\lambda = \dim S^\lambda$. We recall that, following the notation in this chapter, πw_T indicates the action of $\pi \in \mathfrak{S}_n$ on the vector w_T of the representation S^λ. Moreover, if $S \in \mathrm{Tab}(n)$ we denote by $e_T w_S = \sum_{\pi \mathfrak{S}_n} e_T(\pi) \pi w_S$ the action of the Fourier transform of e_T on w_S and by $e_T e_S$ the convolution of e_T and e_S. We can now state (see Proposition 1.5.17) the main properties of the basis $\{e_T : T \in \mathrm{Tab}(n)\}$ of the Gelfand–Tsetlin algebra $GZ(n)$:

(i) $e_T e_S = \delta_{T,S} e_T$
(ii) $e_T w_S = \delta_{T,S} w_T$

for all $T, S \in \mathrm{Tab}(n)$.

Property (ii) clearly characterizes e_T among all the elements of the group algebra $L(\mathfrak{S}_n)$. The elements e_T, $T \in \text{Tab}(n)$, are also called *Young seminormal units*.

In [96, 97], Murphy gave a remarkable expression of e_T in terms of the YJM elements. This is also nicely exposed in Garsia's lecture notes [44].

However, our approach is quite different from those sources, because we have already established the basic properties of the Gelfand–Tsetlin bases, from which such an expression quite easily follows.

We introduce some notation (from [44]). For $T \in \text{Tab}(n)$, we denote by $\overline{T}$ the tableau in $\text{Tab}(n-1)$ obtained by removing from T the box containing n. Moreover, we denote by $a_T(j)$ the jth component in $C(T)$, that is, $C(T) = (a_T(1), a_T(2), \ldots, a_T(n))$. With this notation, the spectral analysis of the YJM elements may be summarized by the formula $X_k w_T = a_T(k) w_T$ for all $T \in \text{Tab}(n)$, $k = 1, 2, \ldots, n$ (see (3.26) and Theorem 3.3.7).

Theorem 3.4.11 *We have the following recursive expression for* e_T*,* $T \in \text{Tab}(n)$

$$e_T = e_{\overline{T}} \cdot \prod_{\substack{S \in \text{Tab}(n): \\ \overline{S} = \overline{T}, S \neq T}} \frac{X_n - a_S(n)}{a_T(n) - a_S(n)} \tag{3.59}$$

and $e_T = 1$ *for the unique* $T \in \text{Tab}(1)$.

Proof We show that indeed, denoting by $\widetilde{e}_T$ the group algebra element (recursively defined) on the right-hand side of (3.59), then $\widetilde{e}_T w_S = \delta_{T,S} w_S$ for all $S \in \text{Tab}(n)$. By the characterizing property (ii) above, we deduce that $\widetilde{e}_T = e_T$.

The proof is by induction on n. Suppose that $\overline{S} \neq \overline{T}$. This means that either $\overline{S}$ and $\overline{T}$ have different shapes, or that they have the same shape but a different placement of the numbers $1, 2, \ldots, n-1$. In both cases, we have:

$$\widetilde{e}_{\overline{T}} w_S = \widetilde{e}_{\overline{T}} w_{\overline{S}} = 0. \tag{3.60}$$

Indeed, the first equality follows from the fact that $\widetilde{e}_{\overline{T}} \in L(\mathfrak{S}_{n-1})$, and therefore, to get the action of $\widetilde{e}_{\overline{T}}$ on w_S we have to restrict to $\mathfrak{S}_{n-1}$ the irreducible representation containing w_S (thus obtaining the vector $w_{\overline{S}}$). Moreover, the second equality follows from the inductive hypothesis.

On the other hand, if $\overline{S} = \overline{T}$ but $S \neq T$, then $X_n w_S = a_S(n) w_S$, so that $\widetilde{e}_T w_S = 0$, because the factor $X_n - a_S(n)$ appears in $\widetilde{e}_T$ (the second factor in the right-hand side of (3.59)).

Finally, we have $X_n w_T = a_T(n) w_T$ and therefore $\frac{X_n - a_S(n)}{a_T(n) - a_S(n)} w_T = w_T$ for all $S \in \mathrm{Tab}(n)$ such that $\overline{S} = \overline{T}$ and $S \neq T$, so that

$$\begin{aligned}\widetilde{e}_T w_T &= \widetilde{e}_{\overline{T}} \cdot \prod_{\substack{S \in \mathrm{Tab}(n):\\ \overline{S} = \overline{T}, S \neq T}} \frac{X_n - a_S(n)}{a_T(n) - a_S(n)} w_T \\ &= \widetilde{e}_{\overline{T}} w_T \\ &= w_T.\end{aligned}$$

Note that the third equality may be derived by restricting to $\mathfrak{S}_{n-1}$, along the lines of the first equality in (3.60). □

On the other hand, the Fourier inversion formula in Proposition 1.5.17 immediately yields the following expression of X_k in terms of the Young seminormal units:

Proposition 3.4.12 *For $k = 1, 2, \dots, n$ we have*

$$X_k = \sum_{T \in \mathrm{Tab}(n)} a_T(k) e_T.$$

Exercise 3.4.13 (1) Let T be the unique standard tableau of shape (n) (see Example 3.4.5). Show that

$$e_T = \frac{1}{n!} \prod_{j=1}^{n} (1 + X_j).$$

Prove also that $e_T = \frac{1}{n!} \sum_{\pi \in \mathfrak{S}_n} \pi$ in two ways: (i) by means of the representation theory of $\mathfrak{S}_n$ and (ii) as an algebraic identity in $\mathfrak{S}_n$.

Hint. If $\pi \in \mathfrak{S}_n$, then there exists a unique $\sigma \in \mathfrak{S}_{n-1}$ and $j \in \{1, 2, \dots, n-1\}$ such that $\pi = \sigma(j \to n \to j)$.

(2) Let T be the unique standard tableau of shape (1^n) (see Example 3.4.6). Show that

$$e_T = \frac{1}{n!} \prod_{j=1}^{n} (1 - X_j).$$

As in (1), give two proofs of the fact that $e_T = \frac{1}{n!} \sum_{\pi \in \mathfrak{S}_n} \varepsilon(\pi) \pi$.

(3) Let T_j be the standard tableau of Example 3.4.7. Show that

$$e_{T_j} = -\frac{(j-2)!}{n!(n-2)!} \left[\prod_{i=1}^{j-1} (X_i + 1) \right] \cdot (X_j - j + 1) \cdot \left[\prod_{i=j+1}^{n} X_i (X_i + 2) \right].$$

Exercise 3.4.14 Set $S_T^\lambda := L(\mathfrak{S}_n)e_T$ (the set of all convolutions fe_T, with $f \in L(\mathfrak{S}_n)$). Show that $L(\mathfrak{S}_n) = \bigoplus_{\lambda \vdash n} \bigoplus_{T \in \mathrm{Tab}(\lambda)} S_T^\lambda$ is an orthogonal decomposition of the left regular representation of $\mathfrak{S}_n$ into irreducible representations and that S_T^λ is an eigenspace of the operator $f \mapsto fX_k$ (right convolution by X_k), with corresponding eigenvalue equal to $a_T(k)$.

Hint: look at Exercise 1.5.18 and Exercise 2.2.4.

3.5 Skew representations and the Murnhagan–Nakayama rule

In this section, we define the skew representations of the symmetric group. Then, we prove the Murnhagan–Nakayama rule which enables us to recursively compute the characters of $\mathfrak{S}_n$.

3.5.1 Skew shapes

Let $0 < k < n$. Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r) \vdash n$ and $\mu = (\mu_1, \mu_2, \dots, \mu_s) \vdash (n - k)$ with $s \le r$ and $\mu \preceq \lambda$ (recall that the latter means that $\mu_j \le \lambda_j$ for all $j = 1, 2, \dots, s$). Then, the *skew shape* λ/μ is obtained by removing from the shape of λ all the boxes belonging to μ. For instance, if $\lambda = (8, 4, 3)$ and $\mu = (3, 2)$, then λ/μ is shown in Figure 3.23.

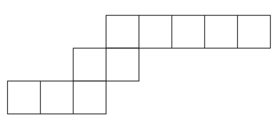


Figure 3.23 The skew shape (8, 4, 3)/(3, 2).

We also denote by $|\lambda/\mu| = k$ the number of boxes of the skew shape λ/μ.

Let $1 \le i \le s - 1$. If $\mu_i < \lambda_{i+1}$ we say that the rows i and $i + 1$ of λ/μ are *connected*. On the contrary, if $\mu_i \ge \lambda_{i+1}$ we say that they are *disconnected*: see Figures 3.24(a)(b).

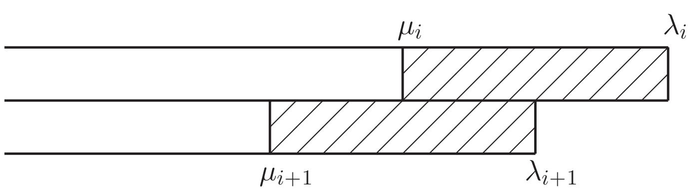


Figure 3.24(a) $\mu_i < \lambda_{i+1}$: the rows i and $i + 1$ are connected.

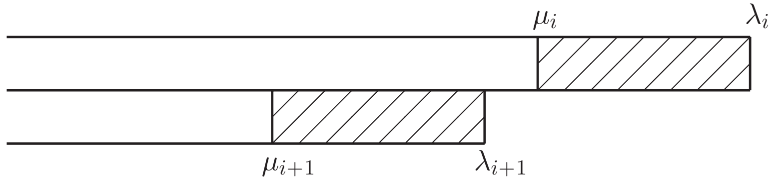


Figure 3.24(b) $\mu_i \geq \lambda_{i+1}$: the rows i and $i+1$ are disconnected.

We say that λ/μ is *connected* if all pairs of consecutive rows in λ/μ are connected. Otherwise, we shay that λ/μ is *disconnected* . If all pairs of consecutive rows are disconnected we say that λ/μ is *totally disconnected* (in Macdonald [83], a totally disconnected skew diagram is called a *horizontal m-strip*). This last condition is equivalent to having: $s \leq r \leq s+1$ and

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots \geq \mu_{s-1} \geq \lambda_s \geq \mu_s \geq \lambda_{s+1}.$$

Example 3.5.1

(i) Let $\lambda = (8, 4, 3)$ and $\mu = (3, 2)$, then λ/μ is connected (see Figure 3.23).

(ii) Let $\lambda = (8, 3, 3)$ and $\mu = (3, 2)$, then λ/μ is disconnected (Figure 3.25):

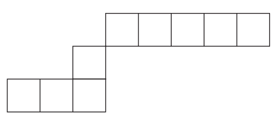


Figure 3.25 (8,3,3)/(3,2) is disconnected.

(iii) Let $\lambda = (8, 3, 1)$ and $\mu = (3, 2)$, then λ/μ is totally disconnected (Figure 3.26):

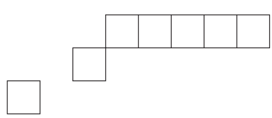


Figure 3.26 (8,3,1)/(3,2) is totally disconnected.

Note that if the diagonals of λ are numbered by the quantity (jth coordinate $-$ ith coordinate) (see Section 3.1.5), then λ/μ is connected if and only if the numbers of the diagonals of λ/μ form a *segment* in $\mathbb{Z}$.

A *standard tableau* of shape λ/μ is a bijective filling of the boxes of λ/μ with $1, 2, \ldots, k$ such that the numbers are in increasing order from left to right along the rows, and from up to down along the columns.

If T is a standard tableau of shape λ/μ, the *content* of T is (as in (3.5))

$$C(T) = (j(1) - i(1), j(2) - i(2), \dots, j(k) - i(k)),$$

where $(i(t), j(t))$ are the (i, j)-coordinates of the box containing $t = 1, 2, \dots, k$. For instance, the tableau T in Figure 3.27 is standard

			1	4	5	9	10
		2	3				
6	7	8					

Figure 3.27 A standard tableau of shape (8,4,3)/(3,2).

and its content is

$$C(T) = (3, 1, 2, 4, 5, -2, -1, 0, 6, 7).$$

Denote by $\mathrm{Tab}(\lambda/\mu)$ the set of all standard tableaux of shape λ/μ. If $\lambda = \lambda^{(k)} \to \lambda^{(k-1)} \to \cdots \to \lambda^{(2)} \to \lambda^{(1)} = \mu$ is a path from λ to μ in the Young poset $\mathbb{Y}$, then we may construct a standard tableau $T \in \mathrm{Tab}(\lambda/\mu)$ by placing the number j in the box $\lambda^{(j)}/\lambda^{(j-1)}$, $j = k, k-1, \dots, 2$. This way, we get a bijection

$$\{\text{Path in } \mathbb{Y} \text{ from } \lambda \text{ to } \mu\} \to \mathrm{Tab}(\lambda/\mu) \tag{3.61}$$

that generalizes (3.8).

Let $T \in \mathrm{Tab}(\lambda/\mu)$. An adjacent transposition s_j is *admissible* for T if s_jT is still standard. The following theorem is a straightforward generalization of Theorem 3.1.5 and its corollary.

Theorem 3.5.2 *Let $T, R \in \mathrm{Tab}(\lambda/\mu)$, $\pi \in \mathfrak{S}_k$ and suppose that $\pi T = R$. Then R may be obtained from T by a sequence of $\ell(\pi)$ admissible transpositions.*

Proof Consider the box that contains the number k in R and let j denote the corresponding number in T. Then s_h is admissible for $s_{h-1}s_{h-2}\cdots s_jT$, $h = j, j+1, \dots, k-1$ and k is placed in the same box of $s_{k-1}s_{k-2}\cdots s_jT$ and R. Continuing this way, with $k-1, k-2, \dots, 2$, one proves the theorem. □

3.5.2 Skew representations of the symmetric group

Let $1 < k < n$ and denote by $\mathfrak{S}_{n-k}$ and $\mathfrak{S}_k$ the groups of all permutations of $\{1, 2, \dots, n-k\}$ and $\{n-k+1, n-k+2, \dots, n\}$, respectively. Viewed as subgroups of $\mathfrak{S}_n$, they commute: if $\sigma \in \mathfrak{S}_{n-k}$ and $\pi \in \mathfrak{S}_k$, then $\sigma\pi = \pi\sigma$.

Let $\lambda \vdash n$ and $\mu \vdash n-k$.

Recall that S^μ is contained in $\mathrm{Res}^{\mathfrak{S}_n}_{\mathfrak{S}_{n-k}} S^\lambda$ if and only if there exists a path in $\mathbb{Y}$ from λ to μ, and this is in turn equivalent to $\lambda \succeq \mu$.

There is a natural representation of $\mathfrak{S}_k$ on $\mathrm{Hom}_{\mathfrak{S}_{n-k}}(S^\mu, S^\lambda)$: for $\Phi \in \mathrm{Hom}_{\mathfrak{S}_{n-k}}(S^\mu, S^\lambda)$ and $\pi \in \mathfrak{S}_k$, then $\pi\Phi$ is well defined. Indeed, π acts on S^λ (recall that we have identified π with the Fourier transform $\lambda(\delta_\pi) \equiv \lambda(\pi)$). Moreover, $\pi\Phi \in \mathrm{Hom}_{\mathfrak{S}_{n-k}}(S^\mu, S^\lambda)$, because π commutes with $\mathfrak{S}_{n-k}$. This representation is called the *skew representation* of $\mathfrak{S}_k$ associated with λ/μ. It is denoted by $S^{\lambda/\mu}$.

Note that $\dim S^{\lambda/\mu} = |\mathrm{Tab}(\lambda/\mu)|$. Indeed, $\dim S^{\lambda/\mu}$ equals the number of paths in $\mathbb{Y}$ from λ to μ and this, in turn, equals $|\mathrm{Tab}(\lambda/\mu)|$ (see (3.61)).

For the moment, we assume that the tableaux in $\mathrm{Tab}(\lambda/\mu)$ are filled (in a standard way) with the numbers $n-k+1, n-k+2, \dots, n$. Let $R \in \mathrm{Tab}(\mu)$ (filled with $1, 2, \dots, n-k$) and $T \in \mathrm{Tab}(\lambda/\mu)$. We can define a standard tableau $R \sqcup T$ simply by filling the boxes of λ according to R (for the numbers $1, 2, \dots, n-k$) and to T (for $n-k+1, n-k+2, \dots, n$).

For instance, if R is as shown in Figure 3.28(a),

1	2	4
3	5	

Figure 3.28(a)

and T is as shown in Figure 3.28(b),

			7	9	11	13
		6	8			
10	12	14				

Figure 3.28(b)

then $R \sqcup T$ is equal to the tableau shown in Figure 3.28(c)

1	2	4	7	9	11	13
3	5	6	8			
10	12	14				

Figure 3.28(c)

For every $T \in \mathrm{Tab}(\lambda/\mu)$ we define a linear operator

$$\Phi_T : S^\mu \to S^\lambda$$

by setting

$$\Phi_T w_R = w_{R\sqcup T}$$

where w_R (resp. $w_{R\sqcup T}$) is the vector of Young's orthogonal form (see Theorem 3.4.4) associated with R (resp. $R \sqcup T$).

Theorem 3.5.3 *The set* $\{\Phi_T : T \in \mathrm{Tab}(\lambda/\mu)\}$ *is a basis for* $S^{\lambda/\mu}$. *Moreover, for* $T \in \mathrm{Tab}(\lambda/\mu)$ *and* $j = n-k+1, n-k+2, \dots, n-1$, *we have*

$$s_j \Phi_T = \frac{1}{r}\Phi_T + \sqrt{1 - \frac{1}{r^2}}\Phi_{s_j T}$$

where, for $C(T) = (a_{n-k+1}, a_{n-k+2}, \dots, a_n)$, $r = a_{j+1} - a_j$ *is the axial distance from* $j+1$ *to* j *in* T.

Proof This is an immediate consequence of Theorem 3.4.4 and of the following obvious identities. For $R \in \mathrm{Tab}(\mu)$ and $T \in \mathrm{Tab}(\lambda/\mu)$ then

$$s_j(R \sqcup T) = (s_j R) \sqcup T \quad \text{for } j = 1, 2, \dots, n-k-1, \tag{3.62}$$

$$s_j(R \sqcup T) = R \sqcup (s_j T) \quad \text{for } j = n-k+1, n-k+2, \dots, n-1. \tag{3.63}$$

Suppose that $C(R \sqcup T) = (a_1, a_2, \dots, a_n)$. Then, from (3.62) and Theorem 3.4.4 we get, for $j = 1, 2, \dots, n-k-1$,

$$\begin{aligned} s_j \Phi_T w_R &= s_j w_{R\sqcup T} \\ &= \frac{1}{r} w_{R\sqcup T} + \sqrt{1 - \frac{1}{r^2}} w_{(s_j R)\sqcup T} \\ &= \Phi_T s_j w_R \end{aligned}$$

where $r = a_{j+1} - a_j$ is the axial distance from $j+1$ to j in R (and therefore, also in $R \sqcup T$). This shows that $\Phi_T \in S^{\lambda/\mu}$. Moreover, $\Phi_{T_1} S^{\lambda/\mu} \perp \Phi_{T_2} S^{\lambda/\mu}$ if $T_1 \neq T_2$ and since $\dim S^{\lambda/\mu} = |\mathrm{Tab}(\lambda/\mu)|$, then $\{\Phi_T : T \in \mathrm{Tab}(\lambda/\mu)\}$ is a basis for $S^{\lambda/\mu}$.

Finally, for $j = n-k+1, n-k+2, \dots, n-1$, from (3.63) we get

$$\begin{aligned} s_j \Phi_T w_R &= s_j w_{R\sqcup T} \\ &= \frac{1}{r} w_{R\sqcup T} + \sqrt{1 - \frac{1}{r^2}} w_{R\sqcup (s_j T)} \\ &= \left(\frac{1}{r}\Phi_T + \sqrt{1 - \frac{1}{r^2}}\Phi_{s_j T} \right) w_R \end{aligned}$$

where $r = a_{j+1} - a_j$ is the axial distance from $j+1$ to j in T (and therefore, also in $R \sqcup T$). □

Let t be a nonnegative integer and denote by $\mathfrak{S}_k$ the symmetric group acting on the segment $\{t+1, t+2, \ldots, t+k\}$. Then we can construct $S^{\lambda/\mu}$ just using the explicit formulas in Theorem 3.5.3. We shall use these formulas in a slightly different notation: we say that in $S^{\lambda/\mu}$ there exists an orthonormal basis $\{w_T : T \in \mathrm{Tab}(\lambda/\mu)\}$ (T filled with $t+1, t+2, \ldots, t+k$), called the *Young orthogonal basis*, such that, for $j = t+1, t+2, \ldots, t+k-1$, we have

$$s_j w_T = \frac{1}{r} w_T + \sqrt{1 - \frac{1}{r^2}} w_{s_j T} \tag{3.64}$$

where r is the axial distance from $j+1$ to j in T.

Remark 3.5.4 The representation $S^{\lambda/\mu}$ depends only on the shape of the skew tableau λ/μ and it is invariant with respect to translations. This means that if $t > 0$, $\lambda^{(1)} = (\lambda_1 + t, \lambda_2 + t, \ldots, \lambda_r + t)$ and $\mu^{(1)} = (\mu_1 + t, \mu_2 + t, \ldots, \mu_s + t, \underbrace{t, t, \ldots, t}_{r-s})$, then $S^{\lambda^{(1)}/\mu^{(1)}} \cong S^{\lambda/\mu}$.

Similarly, if $\lambda^{(2)} = (\alpha_1, \alpha_2, \ldots, \alpha_t, \lambda_1, \lambda_2, \ldots, \lambda_r)$ and $\mu^{(2)} = (\alpha_1, \alpha_2, \ldots, \alpha_t, \mu_1, \mu_2, \ldots, \mu_s)$, with $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_t \geq \lambda_1$, then, again, $S^{\lambda^{(2)}/\mu^{(2)}} \cong S^{\lambda/\mu}$.

Moreover, if λ/μ is connected, $\lambda_1 > \mu_1$ and $r > s$, then we say that λ and μ give the *minimal realization* of λ/μ.

3.5.3 Basic properties of the skew representations and Pieri's rule

In this section, we collect some basic properties of the skew representations that will be used to prove the Murnaghan–Nakayama formula and the Young rule (Section 3.7).

We start by introducing a notion which is a slight generalization of that of a partition.

Let k be a positive integer. A sequence $b = (b_1, b_2, \ldots, b_\ell)$ of positive integers is an *ℓ-parts composition* of k if $b_1 + b_2 + \cdots + b_\ell = k$ (therefore, a partition is a composition with the property that $b_1 \geq b_2 \geq \cdots \geq b_\ell$). Let t be a nonnegative integer and denote by $\mathfrak{S}_k$ the symmetric group acting on the segment $\{t+1, t+2, \ldots, t+k\}$. The *Young subgroup* associated with b is given by

$$\mathfrak{S}_b = \mathfrak{S}_{b_1} \times \mathfrak{S}_{b_2} \times \cdots \times \mathfrak{S}_{b_\ell}$$

where $\mathfrak{S}_{b_j}$ acts on $\{t + b_1 + b_2 + \cdots + b_{j-1} + 1, t + b_1 + b_2 + \cdots + b_{j-1} + 2, \ldots, t + b_1 + b_2 + \cdots + b_j\}$.

Proposition 3.5.5 *Let $b = (b_1, b_2, \ldots, b_\ell)$ be a composition of k. Then*

$$\mathrm{Res}^{\mathfrak{S}_k}_{\mathfrak{S}_b} S^{\lambda/\mu} = \bigoplus \left(S^{\lambda^{(1)}/\mu} \boxtimes S^{\lambda^{(2)}/\lambda^{(1)}} \boxtimes \cdots \boxtimes S^{\lambda/\lambda^{(\ell-1)}} \right)$$

where the sum runs over all sequences $\lambda^{(0)} = \mu \preceq \lambda^{(1)} \preceq \cdots \preceq \lambda^{(\ell-1)} \preceq \lambda^{(\ell)} = \lambda$ such that $|\lambda^{(j)}/\lambda^{(j-1)}| = b_j$, for $j = 1, 2, \ldots, \ell$.

Proof A closer look at the proof of Theorem 3.5.3 reveals that

$$\mathrm{Res}^{\mathfrak{S}_n}_{\mathfrak{S}_{n-k} \times \mathfrak{S}_k} S^{\lambda} = \bigoplus_{\substack{\mu \vdash n-k: \\ \mu \preceq \lambda}} \left(S^{\mu} \boxtimes S^{\lambda/\mu} \right). \tag{3.65}$$

It suffices to note the following fact: if $T \in \mathrm{Tab}(\lambda)$, then the boxes occupied by the numbers $1, 2, \ldots, n-k$ in T form the diagram of a partition μ of $n-k$. The present proposition is an obvious generalization of (3.65). □

In general, $S^{\lambda/\mu}$ is not irreducible. However, the following holds.

Proposition 3.5.6 *For every $T \in \mathrm{Tab}(\lambda/\mu)$, w_T is a cyclic vector in $S^{\lambda/\mu}$.*

Proof Fix $T \in \mathrm{Tab}(\lambda/\mu)$ and let V be the linear span $< \pi w_T : \pi \in \mathfrak{S}_k >$. For every $R \in \mathrm{Tab}(\lambda/\mu)$, let $\pi_R \in \mathfrak{S}_k$ denote the permutation such that $\pi_R T = R$. We shall prove, by induction on $\ell(\pi_R)$, that w_R is in V. If $\ell(\pi_R) = 1$, the statement follows from (3.64). If $\ell(\pi_R) > 1$, let $R_1 \in \mathrm{Tab}(\lambda/\mu)$ be such that $\ell(\pi_{R_1}) = \ell(\pi_R) - 1$ and $\pi_R = s_j \pi_{R_1}$ for some adjacent transposition s_j (recall Theorem 3.5.2). Therefore $R = s_j R_1$, and if the proposition holds true for R_1, that is, $w_{R_1} \in V$, then, applying again (3.64), we get $w_R \in V$. □

Let now $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r) \vdash n$ and $\mu = (\mu_1, \mu_2, \ldots, \mu_s) \vdash n-k$ with $\mu \preceq \lambda$.

Proposition 3.5.7 *Suppose that the rows h and $h+1$ are disconnected in λ/μ and denote by θ/ν and η/τ the skew diagrams formed with the rows $1, 2, \ldots, h$ and $h+1, h+2, \ldots, r$ of λ/μ, respectively. Suppose that $|\theta/\nu| = m$ (so that $|\eta/\tau| = k-m$). Then*

$$S^{\lambda/\mu} \cong \mathrm{Ind}^{\mathfrak{S}_k}_{\mathfrak{S}_m \times \mathfrak{S}_{k-m}} \left(S^{\theta/\nu} \boxtimes S^{\eta/\tau} \right).$$

Proof Consider the subspace W of $S^{\lambda/\mu}$ spanned by all the w_T, with $T \in \mathrm{Tab}(\lambda/\mu)$, such that: the boxes in θ/ν contain the numbers $1, 2, \ldots, m$, and those in η/τ contain the numbers $m+1, m+2, \ldots, k$. Clearly, $W \cong S^{\theta/\nu} \boxtimes S^{\eta/\tau}$ as a $\mathfrak{S}_m \times \mathfrak{S}_{k-m}$ representation (apply the Young orthogonal form). Let $\mathcal{S}$ be a system of representatives for the cosets of $\mathfrak{S}_m \times \mathfrak{S}_{k-m}$ in $\mathfrak{S}_k$. We must show that $S^{\lambda/\mu} = \oplus_{\pi\in\mathcal{S}}\pi W$ (with direct sum). First of all, $S^{\lambda/\mu}$ is spanned by the subspaces πW because every Young basis vector is cyclic (by Proposition 3.5.6).

On the other hand, we have

$$\dim S^{\lambda/\mu} = |\mathrm{Tab}(\lambda/\mu)| = \binom{k}{m}\dim S^{\theta/\nu}\cdot\dim S^{\eta/\tau} = |\mathcal{S}|\dim W.$$

Indeed, to form a standard λ/μ tableau T we must choose an m-subset A in $\{1, 2, \ldots, k\}$ and then form a θ/ν tableau with the numbers in A and an η/τ tableau with numbers in A^c (θ/ν and η/τ are disconnected). □

Corollary 3.5.8 *Suppose that λ/μ is totally disconnected and that $b_1, b_2, \ldots, b_t$ are the length of the (non-trivial) rows of λ/μ. Then*

$$S^{\lambda/\mu} \cong \mathrm{Ind}_{\mathfrak{S}_b}^{\mathfrak{S}_k}\left(S^{(b_1)} \boxtimes S^{(b_2)} \boxtimes \cdots \boxtimes S^{(b_t)}\right)$$

where $b = (b_1, b_2, \ldots, b_t)$, that is, $S^{\lambda/\mu}$ is isomorphic to the permutation representation of $\mathfrak{S}_k$ over $\mathfrak{S}_k/\mathfrak{S}_b$.

Example 3.5.9 Take $\lambda = (k, 1)$ and $\mu = (1)$. Let w_j be the Young basis vector associated with the standard tableau

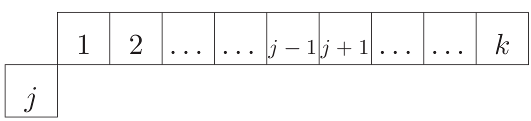


Figure 3.29

for $j = 1, 2, \ldots, k$ (Figure 3.29). Then the Young formula gives

$$s_j w_j = \frac{1}{j+1}w_j + \sqrt{1 - \frac{1}{(j+1)^2}}\,w_{j+1} \tag{3.66}$$

$$s_{j-1}w_j = -\frac{1}{j}w_j + \sqrt{1 - \frac{1}{j^2}}\,w_{j-1} \tag{3.67}$$

and

$$s_i w_j = w_j, \qquad \text{for } i \neq j-1, j. \tag{3.68}$$

On the other hand, by Corollary 3.5.8 we have that $S^{\lambda/\mu}$ is isomorphic to the permutation module $L(X)$, where $X = \mathfrak{S}_k/(\mathfrak{S}_{k-1}\times\mathfrak{S}_1) \equiv \{1, 2, \ldots, k\}$.

Indeed, an explicit formula for $w_1, w_2, \dots, w_k$ as vectors in $L(X)$ is given by

$$w_j = \frac{1}{\sqrt{j(j+1)}}\mathbf{1}_{j-1} - \sqrt{\frac{j}{j+1}}\delta_j + \frac{1-\sqrt{k+1}}{k\sqrt{j(j+1)}}\mathbf{1}_k$$

where $\mathbf{1}_j = \delta_1 + \delta_2 + \cdots + \delta_j$.

Exercise 3.5.10 Prove that $w_1, w_2, \dots, w_k$ is an orthonormal system in $L(X)$ and that (3.66), (3.67) and (3.68) are satisfied.

Exercise 3.5.11 (a) Let $\mathfrak{S}_{k+1}$ act on $Y = \{0, 1, \dots, k\}$. For $j = 1, 2, \dots, k$, set

$$u_j = \delta_j + \frac{1-\sqrt{k+1}}{k\sqrt{k(k+1)}}\mathbf{1}_k - \frac{1}{\sqrt{k+1}}\delta_0,$$

where $\mathbf{1}_k = \delta_1 + \delta_2 + \cdots + \delta_k$. Prove that $u_1, u_2, \dots, u_k$ is an orthonormal basis in $S^{k,1} \subseteq L(Y)$ (see Example 3.4.7) and that the map

$$L(X) \ni \delta_j \mapsto u_j \in \mathrm{Res}^{\mathfrak{S}_{k+1}}_{\mathfrak{S}_k} S^{k,1}$$

is a unitary isomorphism of $\mathfrak{S}_k$-representations.

(b) Use (a) and Example 3.4.7 to give another solution of Exercise 3.5.10 (note also that $S^{(k,1)/(1)} = \mathrm{Hom}_{\mathfrak{S}_k}(S^{(1)}, S^{k,1}) \cong \mathrm{Res}^{\mathfrak{S}_{k+1}}_{\mathfrak{S}_k} S^{k,1}$).

(c) Show that for $k > i > j$, the vector w_j is not an eigenvector of the YJM operator X_i.

Proposition 3.5.12 *The multiplicity of the trivial representation $S^{(k)}$ in $S^{\lambda/\mu}$ is equal to* 1 *if λ/μ is totally disconnected,* 0 *otherwise.*

Proof If λ/μ is totally disconnected, then $S^{\lambda/\mu}$ is a permutation module (with respect to a transitive action) and therefore it contains $S^{(k)}$ with multiplicity one.

Suppose now that λ/μ is not totally disconnected, that is, there are two connected rows. Then, there exists a standard λ/μ-tableau T with two consecutive numbers ℓ and $\ell+1$ one the same column. This is clear if λ/μ has only two rows (Figure 3.30):

			ℓ	$\ell+2$			$\ell+2t-1$	$\ell+2t+1$		$\ell+2t+r$
1		$\ell-1$	$\ell+1$	$\ell+3$		$\ell+2t$				

Figure 3.30

and this trick may be easily adapted to the general case. Then $s_\ell w_T = -w_T$ and therefore $S^{\lambda/\mu}$ cannot contain the trivial representation. See Corollary 1.1.11 and Proposition 3.5.6. $\square$

The following corollary is a generalization of the branching rule.

Corollary 3.5.13 *Let $\lambda \vdash n$ and $\mu \vdash n-k$ be two partitions. Then the multiplicity of $S^\mu \boxtimes S^{(k)}$ in $\mathrm{Res}^{\mathfrak{S}_n}_{\mathfrak{S}_{n-k}\times\mathfrak{S}_k} S^\lambda$ is equal to* 1 *if $\mu \preceq \lambda$ and λ/μ is totally disconnected, and* 0 *otherwise.*

Proof This is an immediate consequence of Proposition 3.5.5 (more precisely of (3.65)) and Proposition 3.5.12. $\square$

Applying the Frobenius reciprocity (cf. Theorem 1.6.11) we immediately get the following.

Corollary 3.5.14 (Pieri's rule) *For $\mu \vdash n-k$ we have*

$$\mathrm{Ind}^{\mathfrak{S}_n}_{\mathfrak{S}_{n-k}\times\mathfrak{S}_k}\left(S^\mu \boxtimes S^{(k)}\right) = \bigoplus_\lambda S^\lambda$$

where the sum runs over all partitions $\lambda \vdash n$ such that $\mu \preceq \lambda$ and λ/μ is totally disconnected.

3.5.4 Skew hooks

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r) \vdash n$ and $\mu \vdash n-k$ be two partitions. We say that λ/μ is a *skew hook* if the corresponding diagram is connected and does not have two boxes on the same diagonal. This is equivalent to the following condition: the content of λ/μ is a segment of integers of length $|\lambda/\mu|$. Moreover, if λ and μ form a minimal realization of λ/μ (see Remark 3.5.4), then $\lambda_j = \mu_{j-1} + 1$ for $j = 2, 3, \ldots, r$.

The *height* of a skew hook λ/μ, denoted by $\langle\lambda/\mu\rangle$, is equal to the number of rows -1; if λ and μ are minimal, it is equal to $r-1$ and $\lambda_1 = k - r + 1 \equiv k - \langle\lambda/\mu\rangle$.

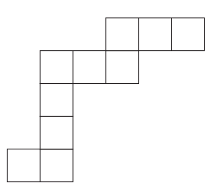


Figure 3.31 The skew hook (6, 4, 2, 2, 2)/(3, 1, 1, 1).

In the example in Figure 3.31, one has $\langle(6, 4, 2, 2, 2)/(3, 1, 1, 1)\rangle = 4$.

Theorem 3.5.15 *Suppose that λ/μ is connected and that $\gamma \vdash k$ is the hook of height h, that is, $\gamma = (k-h, 1^h)$. Then, the multiplicity of S^γ in $S^{\lambda/\mu}$ is equal to* 1 *if λ/μ is a skew hook of height h,* 0 *in all other cases.*

Proof We split the proof into three steps.

Step 1 First suppose that λ/μ is not a skew hook. Then it contains a subdiagram of the form $\boxplus$. Moreover, there exists $T_0 \in \text{Tab}(\lambda/\mu)$ such that $\boxplus$ contains four consecutive numbers, say $t, t+1, t+2, t+3$. Indeed, if we divide λ/μ as in Figure 3.32,

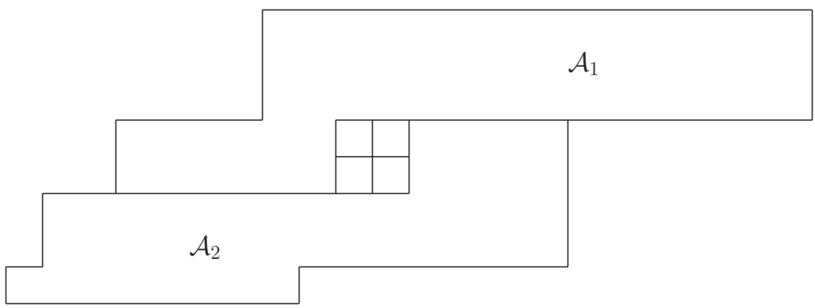


Figure 3.32

we can certainly take $t =$ (number of boxes in $\mathcal{A}_1$) + 1.

Let V be the subspace spanned by all tableaux w_T with $T \in \text{Tab}(\lambda/\mu)$ that have $t, t+1, t+2, t+3$ in $\boxplus$ and coincide with T_0 in $\mathcal{A}_1$ and $\mathcal{A}_2$, and let $\mathfrak{S}_4$ be the symmetric group on $\{t, t+1, t+2, t+3\}$. Then, from the Young orthogonal form we deduce that V is $\mathfrak{S}_4$-invariant and isomorphic to $S^{2,2}$ (note also that $\mathfrak{S}_4$ is conjugate to the symmetric group on $\{1, 2, 3, 4\}$).

Since $(2, 2)$ is not contained in γ (this is a hook), then $S^{2,2}$ is not contained in $\text{Res}^{\mathfrak{S}_k}_{\mathfrak{S}_4} S^\gamma$ and therefore, by Frobenius reciprocity (Theorem 1.6.11), we deduce that S^γ is not contained in $\text{Ind}^{\mathfrak{S}_k}_{\mathfrak{S}_4} S^{2,2}$.

Arguing as in Proposition 3.5.7 (that is, using Proposition 3.5.6; see also Exercise 1.6.3) we can construct a surjective intertwining map

$$\text{Ind}^{\mathfrak{S}_k}_{\mathfrak{S}_4} S^{2,2} \cong \bigoplus_{\pi \in \mathcal{S}} \pi S^{2,2} \to S^{\lambda/\mu}$$

(where $\mathcal{S}$ is a system of representatives for the $\mathfrak{S}_4$ cosets in $\mathfrak{S}_k$. Clearly, in the present case, the map above fails to be injective).

In particular, $S^{\lambda/\mu}$ also cannot contain S^γ.

Step 2 Suppose now that λ/μ is a skew hook with $\langle\lambda/\mu\rangle \neq h$. Assume also that λ and μ yield a minimal realization of λ/μ. Then $\langle\lambda/\mu\rangle = r-1$ and $\lambda_1 + r - 1 = k$ and we have $h > r-1$ (if $\langle\lambda/\mu\rangle < h$), or $\lambda_1 < k-h$ (if $\langle\lambda/\mu\rangle > h$). In both cases, $\gamma \not\preceq \lambda$ and therefore $\mathrm{Res}^{\mathfrak{S}_n}_{\mathfrak{S}_k} S^\lambda$ does not contain S^γ; since $S^{\lambda/\mu} \preceq \mathrm{Res}^{\mathfrak{S}_n}_{\mathfrak{S}_k} S^\lambda$ (see (3.65)), then $S^{\lambda/\mu}$ does not contain S^γ either.

Step 3 Finally, suppose that λ/μ is a hook with $\langle\lambda/\mu\rangle = h$ and that λ and μ yield a minimal realization of λ/μ. In this case, λ/γ has shape μ (Figure 3.33):

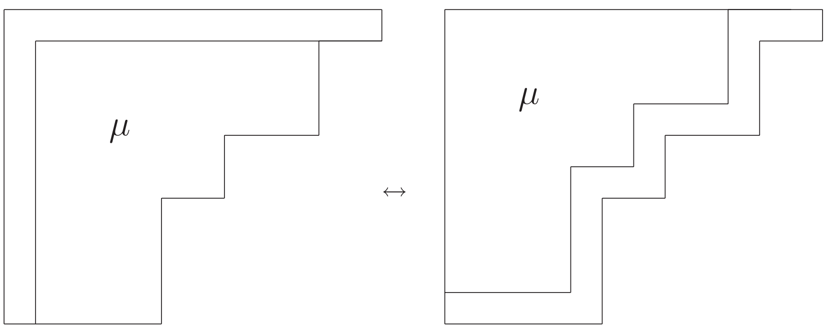


Figure 3.33

Therefore, $S^{\lambda/\gamma} \cong S^\mu$ as $\mathfrak{S}_{n-k}$-representations. Then, an application of (3.65) ensures that the multiplicity of $S^\gamma \boxtimes S^\mu \cong S^\gamma \boxtimes S^{\lambda/\gamma}$ in $\mathrm{Res}^{\mathfrak{S}_n}_{\mathfrak{S}_k \times \mathfrak{S}_{n-k}} S^\lambda$ is equal to 1.

Therefore, we end the proof if we show that the multiplicity of S^γ in $S^{\lambda/\mu}$ is equal to the multiplicity of $S^\mu \boxtimes S^\gamma$ in $\mathrm{Res}^{\mathfrak{S}_n}_{\mathfrak{S}_{n-k} \times \mathfrak{S}_k} S^\lambda$. But this is, again, a consequence of (3.65) and the following observation: if $\theta \vdash k$ and $\theta \preceq \lambda$, then λ/θ is a skew hook if and only if $\theta = \mu$ (since λ, μ are minimal, every skew hook of the form λ/θ is contained in λ/μ). Therefore, from Step 1, if $S^{\lambda/\theta}$ contains S^γ, then $\theta = \mu$. □

3.5.5 The Murnaghan–Nakayama rule

Let λ/μ be a skew shape with $|\lambda/\mu| = k$.

Theorem 3.5.16 *The character of $S^{\lambda/\mu}$ on a cycle of length k is given by the following rule:*

$$\chi^{\lambda/\mu}(12\cdots k) = \begin{cases} (-1)^{\langle\lambda/\mu\rangle} & \text{if } \lambda/\mu \text{ is a skew hook} \\ 0 & \text{otherwise.} \end{cases}$$

Proof First of all, suppose that λ/μ is not connected. Then, from Proposition 3.5.7 we know that

$$S^{\lambda/\mu} \cong \mathrm{Ind}_{\mathfrak{S}_m \times \mathfrak{S}_{k-m}}^{\mathfrak{S}_k} (S^{\theta/\nu} \boxtimes S^{\eta/\tau})$$

for suitable m, θ, ν, η and τ.

But a cycle of length k is not conjugate to any element in $\mathfrak{S}_m \times \mathfrak{S}_{k-m}$, and therefore Frobenius character formula for an induced representation (cf. Theorem 1.6.7) ensures that $\chi^{\lambda/\mu}(12\cdots k) = 0$.

Now suppose that λ/μ is connected but is not a skew hook. Then no irreducible representation of the form S^γ, with $\gamma \vdash k$ a hook, is contained in $S^{\lambda/\mu}$ (cf. Theorem 3.5.15) and therefore, by Theorem 3.4.10, we again have $\chi^{\lambda/\mu}(12\cdots k) = 0$.

Finally, suppose that λ/μ is a skew hook and set $h = \langle \lambda/\mu \rangle$. Again, from Theorem 3.5.15 we deduce that $S^{k-h,1^h}$ is the unique irreducible representation of $\mathfrak{S}_k$ associated with a hook contained in $S^{\lambda/\mu}$, and its multiplicity is equal to 1. Another application of Theorem 3.4.10 ends the proof. □

Theorem 3.5.17 (The Murnaghan–Nakayama rule) *Let $\sigma \vdash k$. Then*

$$\chi_\sigma^{\lambda/\mu} = \sum_S (-1)^{\langle S \rangle}$$

where, if $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_\ell)$, the sum runs over all sequences $S = (\lambda^{(0)} = \mu \preceq \lambda^{(1)} \preceq \cdots \preceq \lambda^{(\ell-1)} \preceq \lambda^{(\ell)} = \lambda)$ such that $|\lambda^{(j)}/\lambda^{(j-1)}| = \sigma_j$ and $\lambda^{(j)}/\lambda^{(j-1)}$ is a skew hook, $j = 1, 2, \ldots, \ell$, and $\langle S \rangle := \sum_{j=1}^{\ell} \langle \lambda^{(j)}/\lambda^{(j-1)} \rangle$.

Proof Set $\mathfrak{S}_\sigma = \mathfrak{S}_{\sigma_1} \times \mathfrak{S}_{\sigma_2} \times \cdots \times \mathfrak{S}_{\sigma_\ell}$, where $\mathfrak{S}_{\sigma_j}$ acts on $\{\sigma_1 + \cdots + \sigma_{j-1} + 1, \ldots, \sigma_1 + \cdots + \sigma_j\}$. Set $\pi = (1, 2, \ldots, \sigma_1)(\sigma_1 + 1, \sigma_1 + 2, \ldots, \sigma_1 + \sigma_2) \cdots (\sigma_1 + \cdots + \sigma_{\ell-1} + 1, \ldots, \sigma_1 + \cdots + \sigma_\ell)$. Note that it belongs to $\mathfrak{S}_\sigma$ and has cycle type σ. Then, from Proposition 3.5.5 we get

$$\chi^{\lambda/\mu}(\pi) = \sum_S \prod_{j=1}^{\ell} \chi_{\sigma_j}^{\lambda^{(j)}/\lambda^{(j-1)}}$$

where the sum runs over all sequences $\lambda^{(0)} = \mu \preceq \lambda^{(1)} \preceq \cdots \preceq \lambda^{(\ell-1)} \preceq \lambda^{(\ell)} = \lambda$ such that $|\lambda^{(j)}/\lambda^{(j-1)}| = \sigma_j$, for $j = 1, 2, \ldots, \ell$, and an application of Theorem 3.5.16 ends the proof. □

Remark 3.5.18 In Theorem 3.5.17, the condition $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_\ell$ is not necessary. Indeed, let $b = (b_1, b_2, \ldots, b_\ell)$ be a composition of k. A *rim hook tableau* of *shape* λ/μ and *content* b is a skew tableau T (of shape λ/μ), filled with the numbers $1, 2, \dots, \ell$ (with repetitions) (see Figure 3.34) in such a way that

(i) b_j is the number of j's in T, $j = 1, 2, \dots, \ell$;
(ii) the numbers are weakly increasing along the rows and the columns;
(iii) for $j = 1, 2, \dots, \ell$ the boxes occupied by j form a skew hook T_j.

The *height* $\langle T \rangle$ of T is the sum of the heights of the T_j's:

$$\langle T \rangle = \sum_{j=1}^{\ell} \langle T_j \rangle.$$

		1	1	3	3	4	4
	1	1	2	3	5	5	
2	2	2	2	5	5		

Figure 3.34 A rim hook tableau T of shape $(8, 7, 6)/(2, 1)$ and content $(4, 5, 3, 2, 4)$ with $\langle T \rangle = 1 + 1 + 1 + 0 + 1 = 4$.

Then the Murghanan–Nakayama rule may be also expressed in the following way: if π is a permutation of cycle type $(b_1, b_2, \dots, b_\ell)$ then

$$\chi^\lambda(\pi) = \sum_T (-1)^{\langle T \rangle}$$

where the sum is over all rim hook tableaux of shape λ/μ and content $(b_1, b_2, \dots, b_\ell)$.

Example 3.5.19 Take $\lambda = (5, 4, 3)$ and $\mu = \emptyset$ and suppose that $\pi \in \mathfrak{S}_{12}$ has cycle type $(1, 2, 3, 6)$. There are only two rim hook tableaux of shape λ and content $(1, 2, 3, 6)$, namely those in Figure 3.35.

$T_1 =$

1	3	3	4	4
2	3	4	4	
2	4	4		

$T_2 =$

1	2	2	4	4
3	3	4	4	
3	4	4		

Figure 3.35 The two rim hook tableaux of shape $(5, 4, 3)$ and content $(1, 2, 3, 6)$.

Since $\langle T_1 \rangle = 4$ and $\langle T_2 \rangle = 3$, we have $\chi^\lambda(\pi) = (-1)^4 + (-1)^3 = 1 - 1 = 0$.

3.6 The Frobenius–Young correspondence

In this section, we introduce a class of permutation representations of $\mathfrak{S}_n$ naturally parameterized by the partitions of n. Our exposition is inspired by the monographs by James and Kerber [66], Macdonald [83] and Sternberg [115]. In particular, Sections 3.6.2 and 3.6.3 are based on Vershik's paper [120].

3.6.1 The dominance and the lexicographic orders for partitions

Definition 3.6.1 Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_h)$ and $\mu = (\mu_1, \mu_2, \ldots, \mu_k)$ be two partitions of n.

We say that μ *dominates* λ and we write

$$\lambda \trianglelefteq \mu$$

if $k \leq h$ and

$$\sum_{i=1}^{j} \lambda_i \leq \sum_{i=1}^{j} \mu_i$$

for $j = 1, 2, \ldots, k$.

For instance one has $(5, 3, 1) \trianglerighteq (4, 3, 2)$. The relation $\trianglelefteq$ is a partial order and the partitions $(1, 1, \ldots, 1)$ and (n) are the minimal and maximal elements, respectively. The order is not total: for instance, the partitions $\lambda = (3, 3)$ and $\mu = (4, 1, 1)$ are not comparable.

We say that μ is obtained from λ by a *single-box up-move* if there exist positive integers i and j with $i < j$ such that $\mu_\ell = \lambda_\ell$ for all $\ell \neq i, j$ and $\mu_i = \lambda_i + 1$ and $\lambda_j = \mu_j - 1$. Clearly, if μ is obtained from λ by a single-box up-move, then $\lambda \trianglelefteq \mu$.

Proposition 3.6.2 *Let λ and μ be two partitions of n. Then $\mu \trianglerighteq \lambda$ if and only if there exists a chain*

$$\lambda^0 \trianglelefteq \lambda^1 \trianglelefteq \cdots \trianglelefteq \lambda^{s-1} \trianglelefteq \lambda^s$$

where $\lambda^0 = \lambda$, $\lambda^s = \mu$ and λ^{i+1} is obtained from λ^i by a single-box up-move, $i = 0, 1, \ldots, s-1$.

Proof The "if" part is obvious. Suppose $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_h)$ is dominated by $\mu = (\mu_1, \mu_2, \ldots, \mu_k) \neq \lambda$ and set $\mu_j = 0$ for $j = k+1, k+2, \ldots, h$. Let s be the smallest positive integer such that $\sum_{i=1}^{s} \lambda_i < \sum_{i=1}^{s} \mu_i$, denote by t the smallest positive integer such that $\mu_t < \lambda_t$ and by v the largest integer such that $\lambda_v = \lambda_t$. Then we have

(P1) $\mu_1 = \lambda_1,\ \mu_2 = \lambda_2, \ldots, \mu_{s-1} = \lambda_{s-1}$ and $\mu_s > \lambda_s$;
(P2) $\mu_i \geq \lambda_i \quad i = s+1, s+2, \ldots, t-1$;
(P3) $\lambda_i = \lambda_t > \mu_t \geq \mu_i \quad i = t+1, t+2, \ldots, v$.

It particular, from (P1) it follows that $t > s$. Moreover, from (P1), (P2) and (P3) it follows that the the quantity $\sum_{i=1}^{z} \mu_i - \sum_{i=1}^{z} \lambda_i$ is positive for $z = s$ and $z = v-1$, not decreasing for $s \leq z \leq t-1$, decreasing for $t \leq z \leq v-1$. Then we have

(P4) $\sum_{i=1}^{z} \mu_i > \sum_{i=1}^{z} \lambda_i \quad$ for $s \leq z \leq v-1$.

Define a partition $\overline{\lambda} \vdash n$ by setting

$$\overline{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_{s-1}, \lambda_s + 1, \lambda_{s+1}, \ldots, \lambda_{v-1}, \lambda_v - 1, \lambda_{v+1}, \ldots, \lambda_h).$$

Note that $\overline{\lambda}$ is a partition because (P1) implies that $\lambda_{s-1} \geq \lambda_s + 1$ while $\lambda_v > \lambda_{v+1}$ implies $\lambda_v - 1 \geq \lambda_{v+1}$. Moreover, $\overline{\lambda}$ is obtained from λ by a single-box up-move.

Now, (P4) implies that $\overline{\lambda} \trianglelefteq \mu$. Thus, setting $\lambda^1 = \overline{\lambda}$, $\lambda^2 = \overline{\lambda^1}$ and so on, then, after a finite number of steps, we eventually obtain $\lambda^s = \mu$. □

For instance, if $\lambda = (4, 4, 4)$ and $\mu = (8, 3, 1)$ we obtain the following chain of partitions:

$$(4, 4, 4) \triangleleft (5, 4, 3) \triangleleft (6, 3, 3) \triangleleft (7, 3, 2) \triangleleft (8, 3, 1).$$

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \vdash n$. Note that in the diagram of λ there are $t := \lambda_1$ columns and the jth column contains exactly $\lambda'_j = |\{i : \lambda_i \geq j\}|$ boxes. The *conjugate* partition λ' of λ is defined by $\lambda' = (\lambda'_1, \lambda'_2, \ldots, \lambda'_t)$. Clearly its associated Young diagram is obtained by transposing (that is, reflecting with respect to the main diagonal) the Young diagram of λ (Figure 3.36).

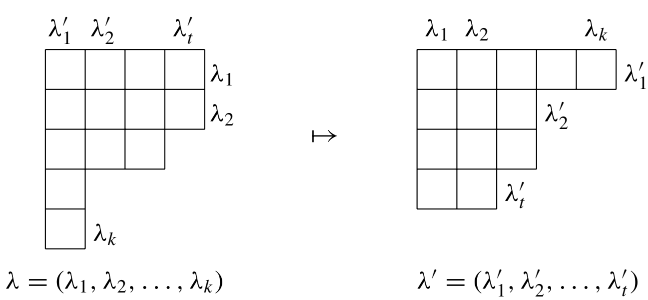


Figure 3.36

Note that $(\lambda')' = \lambda$ so that $\lambda_j = |\{i : \lambda'_i \geq j\}|$. The following proposition connects the partial order $\trianglelefteq$ with the conjugation.

Proposition 3.6.3 *We have $\lambda \trianglelefteq \mu$ if and only if $\mu' \trianglelefteq \lambda'$.*

Proof In view of Proposition 3.6.2 it suffices to prove this fact when μ is obtained from λ by a single-box up-move. But this is obvious. □

Suppose now that $\lambda \vdash n$ and $\gamma \vdash n-1$. We recall (see Section 3.1.6) that λ covers γ if the Young frame of γ can be obtained by removing a single box from the Young frame of λ.

Lemma 3.6.4 *Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_h)$ and $\mu = (\mu_1, \mu_2, \dots, \mu_k)$ be two partitions of n. Then the following two conditions are equivalent:*

(a) $\lambda \trianglelefteq \mu$
(b) *For every ρ covered by μ there exists γ covered by λ with $\gamma \trianglelefteq \rho$.*

Proof We first show the implication (a)⇒(b). Suppose that $\lambda \trianglelefteq \mu$ and that ρ is obtained from μ by removing a single box from row i. Let $j \geq 1$ be the largest integer such that $\lambda_1 = \lambda_j$. Note that if $j > i$ then $\mu_1 > \lambda_1$ (because $\mu_i > \mu_{i+1}$ and $\lambda_1 = \lambda_2 = \cdots = \lambda_{i+1}$). Therefore, defining ℓ as the smallest positive integer such that $\lambda_1 + \lambda_2 + \cdots + \lambda_\ell = \mu_1 + \mu_2 + \cdots + \mu_\ell$, then, necessarily, $\ell \geq j$: otherwise, we would have $\mu_{\ell+1} \leq \mu_\ell < \lambda_\ell = \lambda_{\ell+1}$ and therefore $\mu_1 + \mu_2 + \cdots + \mu_{\ell+1} < \lambda_1 + \lambda_2 + \cdots + \lambda_{\ell+1}$, contradicting $\lambda \trianglelefteq \mu$. This shows that removing a box from the jth row of λ one gets a partition γ satisfying $\gamma \trianglelefteq \rho$ (if $j \leq i$ this is obvious).

Now assume that λ and μ satisfy condition (b). Let j be the largest positive integer such that $\mu_1 = \mu_j$, so that the rightmost box in the jth row of μ is removable (it is the highest removable box). Let ρ be the partition obtained by removing that box. Then there exists a partition $\gamma \vdash n-1$ covered by λ such that $\gamma \trianglelefteq \rho$. Suppose that γ is obtained by removing a box from the ith row. If $j \leq i$ we immediately conclude that also $\lambda \trianglelefteq \mu$. Suppose that $j > i$. Since $\rho_1 = \rho_2 = \cdots = \rho_{j-1} = \mu_1$, $\rho_1 + \rho_2 + \cdots + \rho_{i-1} \geq \gamma_1 + \gamma_2 + \cdots + \gamma_{i-1}$ and $\mu_1 \geq \gamma_{i-1} = \lambda_{i-1} \geq \lambda_i > \gamma_i \geq \gamma_{i+1} \geq \cdots \geq \gamma_{j-1}$, then

$$\rho_1 + \rho_2 + \cdots + \rho_\ell > \gamma_1 + \gamma_2 + \cdots + \gamma_\ell \qquad \ell = i, i+1, \dots, j-1$$

and this shows that also in this case we have $\lambda \trianglelefteq \mu$. □

Now suppose that $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ and $\mu = (\mu_1, \mu_2, \dots, \mu_s)$ are partitions of n. We write $\lambda < \mu$ if there exists an index $j \in \{1, 2, \dots, \min\{k, s\}\}$ such that $\lambda_i = \mu_i$ for $i = 1, 2, \dots, j-1$ and $\lambda_j < \mu_j$. This is a *total order* on the set of all partitions on n, that is, it is an order relation and we always have $\lambda < \mu$, $\lambda = \mu$ or $\lambda > \mu$; it is called the *lexicographic order*. Moreover, it is a refinement of the dominance order (cf. Definition 3.6.1), as is shown in the following proposition.

Proposition 3.6.5 *If* $\lambda \trianglelefteq \mu$ *then* $\lambda \leq \mu$.

Proof Suppose that $\lambda \neq \mu$ and let j be the largest index such that $\lambda_i = \mu_i$ for $i = 1, 2, \ldots, j-1$. If $\lambda \trianglelefteq \mu$ then necessarily $\lambda_j < \mu_j$. □

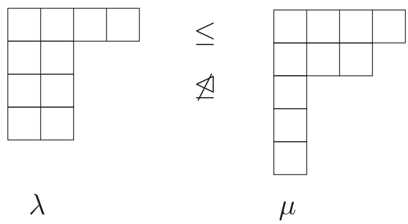


Figure 3.37 $\lambda, \mu \vdash 10$ with $\lambda \leq \mu$ and $\lambda \ntrianglelefteq \mu$

Let $M = (M(\lambda, \mu))_{\lambda,\mu \vdash n}$ be a matrix indexed by the partitions of n. We say that M is *upper unitriangular* with respect to $\trianglerighteq$ if $M(\lambda, \lambda) = 1$ for all $\lambda \vdash n$ and $M(\lambda, \mu) = 0$ unless $\lambda \trianglerighteq \mu$.

Lemma 3.6.6 *Let* $M = (M(\lambda, \mu))_{\lambda,\mu \vdash n}$ *be an integer valued matrix upper unitriangular with respect to* $\trianglerighteq$. *Then* M *is invertible and its inverse is still integer valued and upper unitriangular.*

Proof By Proposition 3.6.5, M is also upper unitriangular with respect to $\geq$ which is a total order. Therefore, the statement follows from Cramer's rule. □

3.6.2 The Young modules

Let m be a positive integer. Given two h-parts compositions of m, $a = (a_1, a_2, \ldots, a_h)$ and $b = (b_1, b_2, \ldots, b_h)$, we say that b is obtained from a by a permutation if there exists $\pi \in \mathfrak{S}_h$ such that $(b_1, b_2, \ldots, b_h) = (a_{\pi(1)}, a_{\pi(2)}, \ldots, a_{\pi(h)})$; if this is the case, we then write $b = \pi a$.

In particular, there is a unique partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_h) \vdash n$ that may be obtained from a by a permutation; we say that λ is the *partition associated* with a.

For instance, if $a = (5, 7, 2, 3, 4)$ then $\lambda = (7, 5, 4, 3, 2)$.

Let $a = (a_1, a_2, \ldots, a_h)$ be a composition of n. A *composition* of $\{1, 2, \ldots, n\}$ of *type* a is an ordered sequence $A = (A_1, A_2, \ldots, A_h)$ of subsets of $\{1, 2, \ldots, n\}$ such that $|A_j| = a_j$ for $j = 1, 2 \ldots, h$ and $A_i \cap A_j = \emptyset$ for $i \neq j$. Note that these facts imply that $A_1 \coprod A_2 \coprod \cdots \coprod A_h = \{1, 2, \ldots, n\}$; in other words, $(A_1, A_2, \ldots, A_h)$ is an *ordered partition* of $\{1, 2, \ldots, n\}$. Moreover, A_h is always determined by $A_1, A_2, \ldots, A_{h-1}$.

Denote by Ω_a the set of all compositions of $\{1, 2, \ldots, n\}$ of type a. The symmetric group $\mathfrak{S}_n$ acts on Ω_a in the natural way: if $\sigma \in \mathfrak{S}_n$ and $A = (A_1, A_2, \ldots, A_h) \in \Omega_a$ then $\sigma A = (\sigma A_1, \sigma A_2, \ldots, \sigma A_h)$. This action is transitive. Moreover, if we fix $\overline{A} = (\overline{A_1}, \overline{A_2}, \ldots, \overline{A_h}) \in \Omega_a$ then the stabilizer of $\overline{A}$ is isomorphic to the Young subgroup $\mathfrak{S}_{a_1} \times \mathfrak{S}_{a_2} \times \cdots \times \mathfrak{S}_{a_h}$ (see Section 3.5.3). Indeed, it consists of all $\sigma \in \mathfrak{S}_n$ such that $\sigma \overline{A_i} = \overline{A_i}$ for all $i = 1, 2, \ldots, h$, so that $\sigma = \sigma_1 \sigma_2 \cdots \sigma_h$ with $\sigma_i \in \mathfrak{S}_{a_i}$, the symmetric group on $\overline{A_i}$, $i = 1, 2, \ldots, h$. Then we can write

$$\Omega_a \cong \mathfrak{S}_n / (\mathfrak{S}_{a_1} \times \mathfrak{S}_{a_2} \times \cdots \times \mathfrak{S}_{a_h}).$$

Note that if the composition b is obtained from a by a permutation, then $\Omega_a \cong \Omega_b$ as homogeneous spaces, so that $\Omega_a \cong \Omega_\lambda$ where λ is the partition associated with a.

The *Young module of type* a is the $\mathfrak{S}_n$-permutation module $L(\Omega_a)$. It is usually denoted by M^a.

Again, if b and λ are as above, then $M^a \cong M^b \cong M^\lambda$, so that we might limit ourselves to partitions. Note also that when $\lambda = (n-k, k)$ has two parts, then $M^{n-k,k}$ is exactly the permutation module introduced in Example 1.4.10.

In terms of induced representations we can write

$$M^a = \mathrm{Ind}_{\mathfrak{S}_{a_1} \times \mathfrak{S}_{a_2} \times \cdots \times \mathfrak{S}_{a_h}}^{\mathfrak{S}_n} \iota_{\mathfrak{S}_{a_1} \times \mathfrak{S}_{a_2} \times \cdots \times \mathfrak{S}_{a_h}}$$

and obviously

$$\iota_{\mathfrak{S}_{a_1} \times \mathfrak{S}_{a_2} \times \cdots \times \mathfrak{S}_{a_h}} \sim \iota_{\mathfrak{S}_{a_1}} \otimes \iota_{\mathfrak{S}_{a_2}} \otimes \cdots \otimes \iota_{\mathfrak{S}_{a_h}}.$$

Suppose that $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_h)$ is a partition of n and γ is a partition of $n-1$ covered by λ (see Section 3.1.6). Then there exists $1 \le j \le k$ such that $\gamma_j = \lambda_j - 1$ and $\gamma_t = \lambda_t$ for $t \neq j$. We define $C(\lambda, \gamma)$ as the number of $1 \le i \le h$ such that $\lambda_i = \lambda_j$.

Lemma 3.6.7

$$\mathrm{Res}_{\mathfrak{S}_{n-1}}^{\mathfrak{S}_n} M^\lambda = \bigoplus_{\substack{\gamma \vdash n-1: \\ \gamma \preceq \lambda}} C(\lambda, \gamma) M^\gamma. \tag{3.69}$$

Proof This is an application of (1.64). Set $\Omega_j = \{(B_1, B_2, \ldots, B_h) \in \Omega_\lambda : n \in B_j\}$. Then $\coprod_{j=1}^h \Omega_j = \Omega_\lambda$ is the decomposition of Ω_λ into $\mathfrak{S}_{n-1}$-orbits. Moreover, if $\lambda_{i-1} > \lambda_i = \lambda_{i+1} = \cdots = \lambda_j > \lambda_{j+1}$ and $\gamma \vdash n-1$ is obtained from λ by removing a box from the jth row, then $\Omega_i \cong \Omega_{i+1} \cong \cdots \cong \Omega_j \cong \Omega_\gamma$ as $\mathfrak{S}_{n-1}$-homogeneous spaces. Therefore, (3.69) holds. □

For a partition $\lambda \vdash n$, denote by ε_λ the alternating representation of the Young subgroup $\mathfrak{S}_\lambda = \mathfrak{S}_{\lambda_1} \times \mathfrak{S}_{\lambda_2} \times \cdots \times \mathfrak{S}_{\lambda_h}$. This means that if $\sigma \in \mathfrak{S}_\lambda$

then $\varepsilon_\lambda(\sigma) = 1$ if σ is even, while $\varepsilon_\lambda(\sigma) = -1$ if σ is odd. We may also write $\varepsilon_\lambda = \mathrm{Res}^{\mathfrak{S}_n}_{\mathfrak{S}_\lambda} \varepsilon_{\mathfrak{S}_n}$, or $\varepsilon_\lambda \sim \varepsilon_{\mathfrak{S}_{\lambda_1}} \otimes \varepsilon_{\mathfrak{S}_{\lambda_2}} \otimes \cdots \otimes \varepsilon_{\mathfrak{S}_{\lambda_h}}$, where $\varepsilon_{\mathfrak{S}_{\lambda m}}$ is the alternating representation of $\mathfrak{S}_m$. Also set $\widetilde{M}^\lambda := \mathrm{Ind}^{\mathfrak{S}_n}_{\mathfrak{S}_\lambda} \varepsilon_\lambda$. Similarly, we denote by ι_λ the trivial representation of $\mathfrak{S}_\lambda$, so that $M^\lambda \cong \mathrm{Ind}^{\mathfrak{S}_n}_{\mathfrak{S}_\lambda} \iota_\lambda$.

Proposition 3.6.8 *For every $\lambda \vdash n$ we have*

$$\dim \mathrm{Hom}_{\mathfrak{S}_n}(\widetilde{M}^{\lambda'}, M^\lambda) = 1,$$

where λ' is the conjugate partition of λ.

Proof We apply Mackey's intertwining number theorem (Theorem 1.6.15). Let T be a system of representatives for the cosets $\mathfrak{S}_{\lambda'} \backslash \mathfrak{S}_n / \mathfrak{S}_\lambda$ and, for all $t \in T$, set $G_t = t\mathfrak{S}_\lambda t^{-1} \cap \mathfrak{S}_{\lambda'}$. Then we have:

$$\dim \mathrm{Hom}_{\mathfrak{S}_n}(\mathrm{Ind}^{\mathfrak{S}_n}_{\mathfrak{S}_{\lambda'}} \varepsilon_{\lambda'}, \mathrm{Ind}^{\mathfrak{S}_n}_{\mathfrak{S}_\lambda} \iota_\lambda) = \sum_{t \in T} \dim \mathrm{Hom}_{G_t}(\varepsilon_{G_t}, \iota_{G_t}).$$

Indeed, $\mathrm{Res}^{\mathfrak{S}_{\lambda'}}_{G_t} \varepsilon_{\lambda'} = \varepsilon_{G_t}$ and $\iota_\lambda(t^{-1}gt) = 1$ for all $g \in G$. But ε_{G_t} and ι_{G_t} are clearly irreducible (they are one-dimensional) and they are also inequivalent if G_t is nontrivial. Therefore we are left to show that there exists exactly one trivial G_t. Suppose that $\mathfrak{S}_{\lambda'}$ is the stabilizer of $(B_1, B_2, \ldots, B_{\lambda_1}) \in \Omega_{\lambda'}$. First note that $(A_1, A_2, \ldots, A_h), (A'_1, A'_2, \ldots, A'_h) \in \Omega_\lambda$ belong to the same $\mathfrak{S}_{\lambda'}$-orbit if and only if

$$|A_i \cap B_j| = |A'_i \cap B_j| \qquad \forall i, j. \tag{3.70}$$

Moreover, the stabilizer of $(A_1, A_2, \ldots, A_h)$ in $\mathfrak{S}_{\lambda'}$ is the group of all permutations $\pi \in \mathfrak{S}_{\lambda'}$ such that $\pi(A_i \cap B_j) = A_i \cap B_j$ for all i, j; in particular it is trivial if and only if

$$|A_i \cap B_j| = 1 \qquad \forall i, j. \tag{3.71}$$

Indeed, if $|A_i \cap B_j| \geq 2$ then there exists $\sigma \in \mathfrak{S}_{\lambda'}$ nontrivial such that $\sigma(A_i \cap B_j) = A_i \cap B_j$ while $|A_i \cap B_j| \leq 1$ for all i, j gives (3.71). From the conditions (3.70) and (3.71) it follows that there exists just one orbit of $\mathfrak{S}_{\lambda'}$ on Ω_λ such that the stabilizer of an element is trivial, namely the set of all $(A_1, A_2, \ldots, A_h)$ satisfying (3.71). This shows that there exists exactly one trivial G_t. □

3.6.3 The Frobenius–Young correspondence

Theorem 3.6.9 *Let λ, μ be two partitions of n. Then S^μ is contained in M^λ if and only if $\lambda \trianglelefteq \mu$.*

Proof First we prove, by induction on n, the "only if" part. Suppose that S^μ is contained in M^λ and that the "only if" part is true for $n-1$. By (3.69) and the branching rule, we can say that for every ρ covered by μ there exists γ covered by λ such that $S^\rho \preceq M^\gamma$ and therefore, by the inductive hypothesis, $\gamma \trianglelefteq \rho$. Then Lemma 3.6.4 ensures that $\lambda \trianglelefteq \mu$. Note also that we can take $n=3$ as the basis of induction, since $M^{2,1} \cong S^{(3)} \oplus S^{2,1}$ (see Example 1.4.5) and $M^{1,1,1} \cong S^{(3)} \oplus 2S^{2,1} \oplus S^{1,1,1}$.

In particular (see Example 1.4.10) we know that $M^{n-k,k}$ decomposes, without multiplicity, into the sum of $k+1$ irreducible representations. Since $(n-k,k) \trianglelefteq \mu$ if and only if $\mu = (n-j, j)$ with $0 \le j \le k$, we conclude that

$$M^{n-k,k} \cong \bigoplus_{j=0}^{k} S^{n-j,j} \tag{3.72}$$

(see also Exercise 1.4.11). Now we observe that

$$S^\lambda \preceq M^\lambda \quad \forall \lambda \vdash n. \tag{3.73}$$

Indeed, consider the tableau T^λ (presented in Figure 3.6) and the associated Young basis vector w_{T^λ}. From the Young formula (Theorem 3.4.4) it follows that w_{T^λ} is $\mathfrak{S}_\lambda$-invariant (if $\mathfrak{S}_\lambda$ stabilizes the rows of T^λ). Then, by Theorem 1.4.12, it follows that S^λ is contained in M^λ.

Then, to get the reverse implication, it suffices to prove the following claim.

Claim $\lambda \trianglelefteq \mu$ if and only if there exists an invariant subspace $V_{\lambda,\mu}$ in M^λ such that $M^\lambda \cong M^\mu \oplus V_{\lambda,\mu}$.

Note that, by virtue of Proposition 3.6.2, we can limit ourselves to prove the claim in the case μ is obtained from λ by a single-box up-move, that is, when there exist $i<j$ such that $\mu = (\lambda_1, \lambda_2, \ldots, \lambda_{i-1}, \lambda_i + 1, \ldots, \lambda_j - 1, \ldots, \lambda_h)$. Set $m = \lambda_1 + \lambda_2 + \cdots + \lambda_{i-1} + \lambda_{i+1} + \cdots + \lambda_{j-1} + \lambda_{j+1} + \cdots + \lambda_h \equiv n - (\lambda_i + \lambda_j)$ and $\nu = (\lambda_1, \lambda_2, \ldots, \lambda_{i-1}, \lambda_{i+1}, \ldots, \lambda_{j-1}, \lambda_{j+1}, \ldots, \lambda_h)$, so that $\nu \vdash m$. We have

$$\begin{aligned}
M^\lambda &= \mathrm{Ind}_{\mathfrak{S}_\lambda}^{\mathfrak{S}_n} \left[S^{(\lambda_1)} \otimes S^{(\lambda_2)} \otimes \cdots \otimes S^{(\lambda_h)}\right] \\
&\cong \mathrm{Ind}_{\mathfrak{S}_\nu \times \mathfrak{S}_{\lambda_i} \times \mathfrak{S}_{\lambda_j}}^{\mathfrak{S}_n} \left[\iota_{\mathfrak{S}_\nu} \otimes S^{(\lambda_i)} \otimes S^{(\lambda_j)}\right] \\
\text{(by Proposition 1.6.6)} \quad &\cong \mathrm{Ind}_{\mathfrak{S}_m \times \mathfrak{S}_{\lambda_i+\lambda_j}}^{\mathfrak{S}_n} \left[M^\nu \otimes M^{\lambda_i,\lambda_j}\right] \\
\text{(by (3.72))} \quad &\cong \mathrm{Ind}_{\mathfrak{S}_m \times \mathfrak{S}_{\lambda_i+\lambda_j}}^{\mathfrak{S}_n} \left\{M^\nu \otimes \left[M^{\lambda_i+1,\lambda_j-1} \oplus S^{\lambda_i,\lambda_j}\right]\right\} \\
\text{(by Proposition 1.6.6)} \quad &\cong M^\mu \bigoplus \mathrm{Ind}_{\mathfrak{S}_m \times \mathfrak{S}_{\lambda_i+\lambda_j}}^{\mathfrak{S}_n} \left[M^\nu \otimes S^{\lambda_i,\lambda_j}\right]
\end{aligned}$$

and the claim follows.

This ends the proof of the theorem. □

Recalling (see Example 3.4.6) that $S^{1,1,\dots,1}$ coincides with the alternating representation of $\mathfrak{S}_n$, we have:

Lemma 3.6.10 *For every partition $\lambda \vdash n$, we have*

$$S^{\lambda'} \cong S^{\lambda} \otimes S^{1,1,\dots,1}. \tag{3.74}$$

Proof For every $T \in \text{Tab}(\lambda)$, let T' be the transposed tableau. Then $T' \in \text{Tab}(\lambda')$ and

$$\begin{array}{ccc} \text{Tab}(\lambda) & \to & \text{Tab}(\lambda') \\ T & \mapsto & T' \end{array}$$

is clearly a bijection. Moreover, if $C(T) = (a_1, a_2, \dots, a_n)$ then $C(T') = (-a_1, -a_2, \dots, -a_n)$, and the lemma follows from the Young orthogonal form (Theorem 3.4.4), considering the base $(-1)^{\ell(\pi_T)} w_{T'}$ in $S^{\lambda'}$, where π_T is as in Proposition 3.4.1. □

Theorem 3.6.11 *Denote by $K(\mu, \lambda)$ the multiplicity of S^μ in M^λ. Then*

$$K(\mu, \lambda) \begin{cases} = 0 & \text{if } \lambda \not\trianglelefteq \mu \\ = 1 & \text{if } \lambda = \mu \\ \geq 1 & \text{if } \lambda \triangleleft \mu \end{cases} \tag{3.75}$$

so that

$$M^\lambda \cong \bigoplus_{\lambda \trianglelefteq \mu} K(\mu, \lambda) S^\mu \equiv S^\lambda \oplus \bigoplus_{\lambda \triangleleft \mu} K(\mu, \lambda) S^\mu$$

and all the indicated multiplicities are positive.

Moreover, we also have

$$\widetilde{M}^{\lambda'} \cong \bigoplus_{\mu \trianglelefteq \lambda} K(\mu', \lambda') S^\mu.$$

Proof The cases $\lambda \not\trianglelefteq \mu$ and $\lambda \triangleleft \mu$ in (3.75) follow from Theorem 3.6.9. Note that from Corollary 1.6.10 we get

$$\widetilde{M}^\lambda \cong \text{Ind}_{\mathfrak{S}_\lambda}^{\mathfrak{S}_n} \text{Res}_{\mathfrak{S}_\lambda}^{\mathfrak{S}_n} S^{1,1,\dots,1} \cong S^{1,1,\dots,1} \otimes M^\lambda. \tag{3.76}$$

Denote by ψ^λ and $\widetilde{\psi}^{\lambda'}$ the characters of M^λ and $\widetilde{M}^{\lambda'}$, respectively. From (3.76), Lemma 3.6.10 and Proposition 1.3.4.(vii), we get

$$\begin{aligned} \langle \chi^\mu, \widetilde{\psi}^{\lambda'} \rangle &= \langle \chi^\mu, \chi^{1,1,\dots,1} \psi^{\lambda'} \rangle \\ &= \langle \chi^{1,1,\dots,1} \chi^\mu, \psi^{\lambda'} \rangle \\ \text{(by (3.74))} \quad &= \langle \chi^{\mu'}, \psi^{\lambda'} \rangle \\ &= K(\mu', \lambda'), \end{aligned}$$

that is, the multiplicity of S^μ in $\widetilde{M}^{\lambda'}$ is equal to $K(\mu', \lambda')$. In particular, S^μ is contained in $\widetilde{M}^{\lambda'}$ if and only if $\lambda' \trianglelefteq \mu'$, and this is equivalent to $\mu \trianglelefteq \lambda$ (Proposition 3.6.3).

This implies that S^λ is the unique common irreducible representation between M^λ and $\widetilde{M}^{\lambda'}$. Since $\dim\mathrm{Hom}_{\mathfrak{S}_n}(\widetilde{M}^{\lambda'}, M^\lambda) = 1$ (Proposition 3.6.8), we conclude that the multiplicity of S^λ is one in both spaces and this ends the proof. □

Corollary 3.6.12 (Frobenius–Young correspondence) *For every $\lambda \vdash n$, S^λ may be characterized as the unique irreducible representation which is common to both M^λ and $\widetilde{M}^{\lambda'}$.*

The coefficients $K(\mu, \lambda)$ are called the *Kostka numbers*. We shall prove a combinatorial expression for these numbers in Section 3.7 (see, in particular, Corollary 3.7.11). Now we give and interesting linear relation connecting the Kostka numbers for $\mathfrak{S}_n$ with those for $\mathfrak{S}_{n-1}$.

Theorem 3.6.13 (Vershik's relations for the Kostka numbers) *For any $\lambda \vdash n$ and $\rho \vdash n-1$, we have*

$$\sum_{\substack{\mu \vdash n: \\ \mu \succeq \rho}} K(\mu, \lambda) = \sum_{\substack{\gamma \vdash n-1: \\ \gamma \preceq \lambda}} C(\lambda, \gamma) K(\rho, \gamma).$$

Proof From Lemma 3.6.7 and Theorem 3.6.11 we get

$$\begin{aligned}
\mathrm{Res}^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}} M^\lambda &\cong \bigoplus_{\substack{\gamma \vdash n-1: \\ \gamma \preceq \lambda}} C(\lambda, \gamma) M^\gamma \\
&\cong \bigoplus_{\substack{\gamma \vdash n-1: \\ \gamma \preceq \lambda}} \left(C(\lambda, \gamma) \bigoplus_{\substack{\rho \vdash n-1: \\ \rho \trianglerighteq \gamma}} K(\rho, \gamma) S^\rho \right) \\
&\cong \bigoplus_{\rho \vdash n-1} \left[\sum_{\substack{\gamma \vdash n-1: \\ \rho \trianglerighteq \gamma, \\ \lambda \succeq \gamma}} C(\lambda, \gamma) K(\rho, \gamma) \right] S^\rho.
\end{aligned}$$

Moreover, another application of Theorem 3.6.11 together with the branching rule yields

$$\begin{aligned}\mathrm{Res}^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}} M^\lambda &\cong \bigoplus_{\substack{\mu\vdash n:\\ \mu\trianglerighteq\lambda}} K(\mu,\lambda)\mathrm{Res}^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}} S^\mu\\ &\cong \bigoplus_{\rho\vdash n-1}\left[\sum_{\substack{\mu\vdash n:\\ \mu\trianglerighteq\lambda,\\ \mu\succeq\rho}} K(\mu,\lambda)\right] S^\rho.\end{aligned}$$

Therefore we get

$$\sum_{\substack{\mu\vdash n:\\ \mu\trianglerighteq\lambda,\\ \mu\succeq\rho}} K(\mu,\lambda) = \sum_{\substack{\gamma\vdash n-1:\\ \rho\trianglerighteq\gamma,\\ \lambda\succeq\gamma}} C(\lambda,\gamma)K(\rho,\gamma).$$

The formula in the statement keeps into account the fact that $K(\mu,\lambda)=0$ when $\mu \not\trianglerighteq \lambda$. □

3.6.4 Radon transforms between Young's modules

The results of this section will be used only in Section 6.2.

Let $a=(a_1,a_2,\dots,a_h)$ and $b=(b_1,b_2,\dots,b_k)$ be two compositions of n. We want to describe the orbits of $\mathfrak{S}_n$ on $\Omega_a\times\Omega_b$.

For $A\in\Omega_a$ and $B\in\Omega_b$ denote by $C(A,B)$ the composition of $\{1,2,\dots,n\}$ whose elements are the non-empty $A_i\cap B_j$'s ordered lexicographically (i.e. $(i,j)\le(i',j')$ if $i<i'$, or $i=i'$ and $j\le j'$). We call $C(A,B)$ the *intersection* of A and B; be aware that, however, $C(A,B)\neq C(B,A)$, in general. We also denote by c the corresponding type of $C(A,B)$. The following fact is obvious:

Lemma 3.6.14 *Let $A\in\Omega_a$ and $B\in\Omega_b$ and $C(A,B)\in\Omega_c$ and consider the actions of $\mathfrak{S}_n$ on $\Omega_a\times\Omega_b$ and on Ω_c. Then $\sigma\in\mathfrak{S}_n$ fixes $(A,B)\in\Omega_a\times\Omega_b$ if and only if it fixes $C(A,B)$. In other words $\mathfrak{S}_A\cap\mathfrak{S}_B=\mathfrak{S}_{C(A,B)}$.*

We replace the notion of intersection of two partitions with the following notion: denote by $\mathfrak{M}_{a,b}$ the set of all matrices $(m_{i,j})\in M_{h\times k}(\mathbb{N})$ (with non-negative integer entries) such that $\sum_{i=1}^h m_{i,j}=b_j$ for all $j=1,2,\dots,k$ and $\sum_{j=1}^k m_{i,j}=a_i$ for all $i=1,2,\dots,h$. We may also say that the column sums (resp. the row sums) of the matrix $(m_{i,j})$ equal the b_j's (resp. a_i's).

With $A\in\Omega_a$ and $B\in\Omega_b$, we associate the matrix $m=m(A,B)\in\mathfrak{M}_{a,b}$ by setting $m_{ij}=|A_i\cap B_j|$ for all $j=1,2,\dots,k$ and $i=1,2,\dots,h$. We then have:

Lemma 3.6.15 *(A, B) and $(A', B') \in \Omega_a \times \Omega_b$ belong to the same $\mathfrak{S}_n$-orbit if and only if $m(A, B) = m(A', B')$. In particular the set of $\mathfrak{S}_n$ orbits on $\Omega_a \times \Omega_b$ is in one-to-one correspondence with $\mathfrak{M}_{a,b}$.*

Proof The "only if" part is obvious ($|A_i \cap B_j| = |\sigma(A_i \cap B_j)| = |\sigma A_i \cap \sigma B_j|$ for all $\sigma \in \mathfrak{S}_n$). Conversely, suppose that $m(A, B) = m(A', B')$. Then the intersection compositions $C(A, B)$ and $C(A', B')$ are of the same type, say c. As $\mathfrak{S}_n$ acts transitively on Ω_c, there exists $\sigma \in \mathfrak{S}_n$ such that $\sigma(C(A, B)) = C(A', B')$ and thus $\sigma(A, B) = (A', B')$ (compare with (3.70)). □

From Exercise 1.4.3 and Lemma 3.6.15 we deduce:

Corollary 3.6.16

$$\dim\mathrm{Hom}_{\mathfrak{S}_n}(M^a, M^b) = |\mathfrak{M}_{a,b}|.$$

With each $m_0 \in \mathfrak{M}_{a,b}$, we associate the *Radon transform* $R_{m_0} : M^a \to M^b$ (see Section 1.4.1) defined by setting

$$R_{m_0} f(B) = \sum_{\substack{A \in \Omega_a: \\ m(A,B)=m_0}} f(A)$$

for all $f \in M^a$ and $B \in \Omega_b$.

Exercise 3.6.17 Show that $\{R_m : m \in \mathfrak{M}_{a,b}\}$ is a basis for $\mathrm{Hom}_{\mathfrak{S}_n}(M^a, M^b)$.

3.7 The Young rule

In this section, we give the rule for decomposing the Young modules into irreducible representations. In particular, we construct a multiplicity-free chain and study the associated Gelfand–Tsetlin decomposition along the lines of Section 2.2.3. We construct a poset that generalizes the Young poset leading to the GZ-basis for $L(\mathfrak{S}_n) \cong M^{(1^n)}$.

3.7.1 Semistandard Young tableaux

Let $\mu = (\mu_1, \mu_2, \dots, \mu_k)$ and $a = (a_1, a_2, \dots, a_h)$ be a partition and a composition of n, respectively. A *semistandard tableau* of shape μ and weight a is a filling of the Young frame of μ with the numbers $1, 2, \dots, h$ in such a way that

(i) i occupies a_i boxes, $i = 1, 2, \dots, h$;
(ii) the numbers are strictly increasing down the columns and weakly increasing along the rows.

For example, the tableau in Figure 3.38 is semistandard of shape $\mu = (6, 4, 2, 2, 1)$ and weight $a = (4, 3, 3, 1, 2, 2)$

1	1	1	1	2	2
2	3	3	4		
3	5				
5	6				
6					

Figure 3.38

We denote by $\mathrm{STab}(\mu, a)$ the set of all semistandard tableaux of shape μ and weight a. Now we give two elementary results on semistandard tableaux that may be obtained as a consequence of representation theoretic results, but whose elementary proof deserves consideration.

Lemma 3.7.1 *If the composition b is obtained from a by a permutation then* $|\mathrm{STab}(\mu, a)| = |\mathrm{STab}(\mu, b)|$.

Proof We may suppose that b is obtained from a by an adjacent transposition s_i, that is,

$$b = (a_1, a_2, \ldots, a_{i-1}, a_{i+1}a_i, a_{i+2}, \ldots, a_h) \text{ (and } a_i \neq a_{i+1}).$$

We now construct an explicit bijection

$$\begin{array}{ccc} \mathrm{STab}(\mu, a) & \to & \mathrm{STab}(\mu, b) \\ T & \mapsto & S. \end{array}$$

We say that a box containing i in a tableau $T \in \mathrm{STab}(\mu, a)$ is *i-good* if the box below does not contain $i + 1$; we say that a box containing $i + 1$ is *i-good* if the box above does not contain i. We replace each i inside an i-good box by $i + 1$ and each $i + 1$ in an i-good box by i, respecting the following rule: if a row of T contains α i-good boxes containing i and β i-good boxes containing $(i + 1)$, then we change only the last $(\alpha - \beta)$-many i's if $\alpha > \beta$, and only the first $\beta - \alpha$ $(i + 1)$'s if $\beta > \alpha$ (if $\alpha = \beta$ we do not change anything in that row; note also that the i-good boxes containing i and the i-good boxes containing $i + 1$ occupy a set of consecutive boxes in the row).

This way, we obtain a tableau $S \in \mathrm{STab}(\mu, b)$ and the map $T \mapsto S$ is the desired bijection (the inverse map is defined in the same way). □

Example 3.7.2 In Figure 3.39, the algorithm presented in the proof of Lemma 3.7.1 is used to transform a semistandard tableau of shape (7, 3, 5) and weight (5, 3, 4, 2) into one with the same shape and weight (5, 4, 3, 2) (so that $i = 2$).

1	1	1	1	1	2
2	2	3	3	3	
3	4	4			

$\mapsto$

1	1	1	1	1	3
2	2	2	2	3	
3	4	4			

Figure 3.39

Lemma 3.7.3 $\mathrm{STab}(\mu, a)$ *is nonempty if and only if* $a \trianglelefteq \mu$. *Moreover,* $|\mathrm{STab}(\mu, \mu)| = 1$.

Proof By virtue of the preceding lemma, we may suppose that $a = \lambda$ is a partition. Suppose that T is a semistandard tableau of shape μ and weight λ. Then all the numbers $1, 2, \dots, j$ must be placed in the first j rows of T: otherwise the columns could not be strictly increasing. Thus $\sum_{i=1}^{j} \lambda_i \leq \sum_{i=1}^{j} \mu_i$, $j = 1, 2, \dots, k$. In other words, if $|\mathrm{STab}(\mu, \lambda)| > 0$ then $\lambda \trianglelefteq \mu$. It is also clear that $\mathrm{STab}(\mu, \mu)$ only contains the (unique) tableau T with all the j's in the jth row , for $j = 1, 2, \dots K$.

Now we prove the reverse implication, by induction on n. Suppose that $\lambda \trianglelefteq \mu$. We say that a box of coordinates (i, μ_i) is *removable for the pair* (μ, λ) if, setting $\widetilde{\mu} = (\mu_1, \mu_2, \dots, \mu_{i-1}, \mu_i - 1, \mu_{i+1}, \dots, \mu_k)$ and $\widetilde{\lambda} = (\lambda_1, \lambda_2, \dots, \lambda_{h-1}, \lambda_h - 1)$, $\widetilde{\mu}$ is still a partition (that is, $\mu_i > \mu_{i+1}$) and we still have $\widetilde{\lambda} \trianglelefteq \widetilde{\mu}$ (that is, $\sum_{j=1}^{t} \mu_j > \sum_{j=1}^{t} \lambda_j$, for $t = i, i+1, \dots, k-1$). Clearly, in a tableau $T \in \mathrm{STab}(\mu, \lambda)$ if the row i contains the number h then the box (i, μ_i) is removable for (μ, λ).

Let (i, μ_i) be the *highest* removable box for the pair (μ, λ) and let $\widetilde{\mu}$ and $\widetilde{\lambda}$ as above. By the inductive hypothesis, we may construct a semistandard tableau $\widetilde{T}$ of shape $\widetilde{\mu}$ and weight $\widetilde{\lambda}$. Moreover, $\widetilde{T}$ does not contain h in a row higher than the one containing the box (i, μ_i) (for the choice of (i, μ_i)). Then if we add to $\widetilde{T}$ a box with h in position (i, μ_i), we get a tableau $T \in \mathrm{STab}(\mu, \lambda)$. □

Example 3.7.4 The tableau in Figure 3.40 of shape $\mu = (8, 5, 4, 3, 3, 2)$ and weight $(7, 4, 3, 3, 2, 2, 2, 1, 1)$ has been constructed by a repeated application of the algorithm used in the last part of the preceding proof.

1	1	1	1	1	1	1	9
2	2	2	2	8			
3	3	3	7				
4	4	4					
5	5	7					
6	6						

Figure 3.40

Remark 3.7.5 The "ingenuous" algorithm: from the first to the last row, in each row, from the left to the right, place $\underbrace{1, 1, \ldots, 1}_{\lambda_1}, \underbrace{2, 2, \ldots, 2}_{\lambda_2}, \ldots, \underbrace{h, h, \ldots, h}_{\lambda_h}$ does not produce a semistandard tableau (if $\lambda_1 = \lambda_2 = \cdots = \lambda_h = 1$ then it clearly produces a standard tableau). The tableau in Figure 3.41(a) of shape $\mu = (8, 5, 5)$ and weight $\lambda = (6, 6, 6)$ is constructed using this algorithm and it is not semistandard:

1	1	1	1	1	1	2	2
2	2	2	2	3			
3	3	3	3	3			

Figure 3.41(a)

Our algorithm produces the semistandard tableau shown in Figure 3.41(b).

1	1	1	1	1	1	2	3
2	2	2	2	2			
3	3	3	3	3			

Figure 3.41(b)

3.7.2 The reduced Young poset

Now we give an alternative description of the semistandard tableaux. Fix a composition $a = (a_1, a_2, \ldots, a_h)$ of n. We define a poset $\mathbb{Y}_a$ that we call the *reduced Young poset* associated with a. It is formed by all partitions $\nu \vdash (a_1 + a_2 + \cdots + a_i)$ such that $\nu \trianglerighteq (a_1, a_2, \ldots, a_i)$, for $i = 1, 2, \ldots, h$. For a fixed i, they form the i-th level of the poset. Given θ and ν we say that ν is *totally contained* in θ, and we write $\nu \Subset \theta$, when

- θ belongs to an higher level then ν, that is, $\nu \vdash (a_1 + a_2 + \cdots + a_i)$ and $\theta \vdash (a_1 + a_2 + \cdots + a_j)$, with $i < j$;
- there exists a sequence $\nu = \nu^{(0)} \preceq \nu^{(1)} \preceq \cdots \preceq \nu^{(j-i)} = \theta$ of partitions in $\mathbb{Y}_a$ such that $\nu^{(t)}$ belongs to the level $i + t$ and $\nu^{(t+1)}/\nu^{(t)}$ is totally disconnected, $t = 0, 1, \ldots, j - i - 1$.

If $j = i + 1$ we write $\theta \Rightarrow \nu$ so that the above conditions may be written in the form $\nu^{(j-i)} \Rightarrow \nu^{(j-i-1)} \Rightarrow \cdots \Rightarrow \nu^{(1)} \Rightarrow \nu^{(0)}$. Clearly, $\theta \Rightarrow \nu$ if and only if θ covers ν in $\mathbb{Y}_a$ (in particular, if $\theta \Rightarrow \nu$ then θ/ν is totally disconnected). In other words, we have taken the levels $a_1, a_1 + a_2, \ldots, a_1 + a_2 + \cdots + a_h$ of the Young poset $\mathbb{Y}$, eliminated those partitions that do not satisfy

$\nu \trianglerighteq (a_1, a_2, \ldots, a_i)$ and replaced the order $\preceq$ with the more restrictive condition $\Subset$. Note also that the last level is always formed just by the trivial partition (a_1).

Example 3.7.6 Figure 3.42 is the Hasse diagram of the poset $\mathbb{Y}_{(2,2,2)}$.

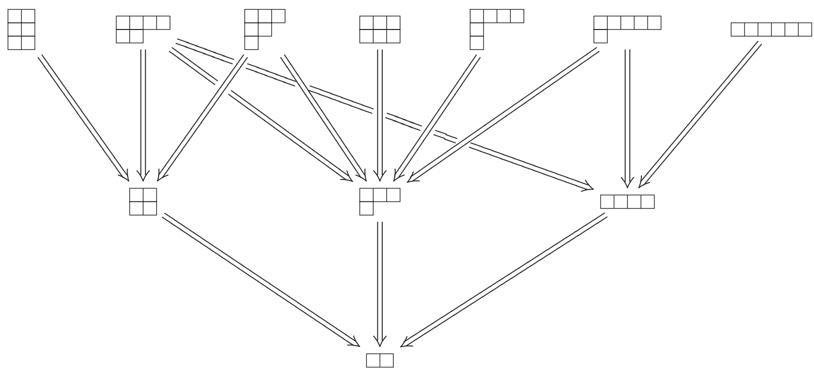


Figure 3.42 The Hasse diagram of $\mathbb{Y}_{(2,2,2)}$.

A *path* in $\mathbb{Y}_a$ is a sequence $\nu^{(h)} \Rightarrow \nu^{(h-1)} \Rightarrow \cdots \Rightarrow \nu^{(1)}$, where $\nu^{(j)}$ belongs to the level j, so that $\nu^{(1)} = (a_1)$. With each path in $\mathbb{Y}_a$ we can associate a semistandard tableau of weight a. We can place h in the boxes of $\nu^{(h)}/\nu^{(h-1)}$, $h-1$ in the boxes of $\nu^{(h-1)}/\nu^{(h-2)}$ and so on. For instance, the semistandard tableau in Figure 3.40 (cf. Example 3.7.4) is associated with the path shown in Figure 3.43.

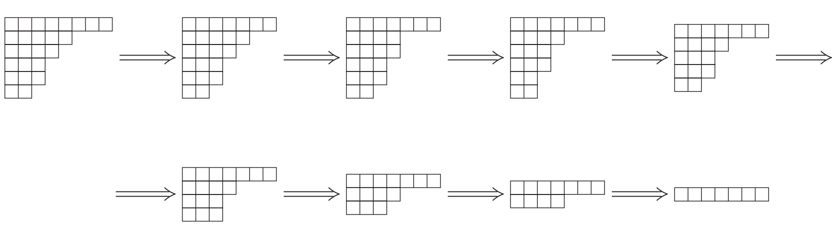


Figure 3.43

Define

$$\mathrm{STab}(a) = \coprod_{\substack{\mu \vdash n \\ \mu \trianglerighteq a}} \mathrm{STab}(\mu, a).$$

Then as in (3.9) we can say that the map described above is a bijection

$$\{\text{paths in } \mathbb{Y}_a\} \to \mathrm{STab}(a)$$

that restricted to the paths starting at μ yields a bijection

$$\{\text{paths in } \mathbb{Y}_a \text{ starting at } \mu\} \to \mathrm{STab}(\mu, a).$$

Remark 3.7.7 Suppose that $\mu \vdash n$ and there exists a chain $\nu^{(1)} \preceq \nu^{(2)} \preceq \cdots \preceq \nu^{(h)} = \mu$ such that $\nu^{(j)} \vdash (a_1 + a_2 + \cdots + a_j)$ and $\nu^{(j+1)}/\nu^{(j)}$ is totally disconnected, $j = 1, 2, \ldots, h$. Then, by the procedure described above, we can construct a semistandard Young tableau of shape μ and weight a, so that, by virtue of Lemma 3.7.3, we can deduce that $\mu \trianglerighteq a$ and $\nu^{(j)} \trianglerighteq (a_1, a_2, \ldots, a_j)$, for $j = 1, 2, \ldots, h-1$. This means that, in the definition of $\mathbb{Y}_a$, we may replace the condition $\nu^{(j)} \trianglerighteq (a_1, a_2, \ldots, a_j)$ with the existence, for every $\nu \in \mathbb{Y}_a$ such that $\nu \vdash (a_1 + a_2 + \cdots + a_j)$, of a path $\nu \Rightarrow \nu^{(j-1)} \Rightarrow \cdots \Rightarrow \nu^{(1)} = (a_1)$.

3.7.3 The Young rule

In this section, we give the general rule to decompose the permutation module M^a for a composition $a = (a_1, a_2, \ldots, a_h)$ of n. We will use the results in Section 2.2.3 and Section 3.5.

Theorem 3.7.8
(i) *The chain*

$$\begin{aligned}\mathfrak{S}_n &\geq \mathfrak{S}_{a_1+a_2+\cdots+a_{h-1}} \times \mathfrak{S}_{a_h} \\ &\geq \mathfrak{S}_{a_1+a_2+\cdots+a_{h-2}} \times \mathfrak{S}_{a_{h-1}} \times \mathfrak{S}_{a_h} \\ &\geq \cdots \\ &\geq \mathfrak{S}_{a_1} \times \mathfrak{S}_{a_2} \times \cdots \times \mathfrak{S}_{a_h}\end{aligned}$$

is multiplicity-free for the pair $(\mathfrak{S}_n, \mathfrak{S}_{a_1} \times \mathfrak{S}_{a_2} \times \cdots \times \mathfrak{S}_{a_h})$.

(ii) *The associated Bratteli diagram coincides with* $\mathbb{Y}_a$ *(if every partition* μ *is replaced with the corresponding irreducible representation* S^μ*).*

Proof (i) is a consequence of Corollary 3.5.13. If $\mu \vdash n$ that corollary ensures that

$$\mathrm{Res}^{\mathfrak{S}_n}_{\mathfrak{S}_{a_1+a_2+\cdots+a_{h-1}} \times \mathfrak{S}_{a_h}} S^\mu \cong \left\{ \bigoplus_\nu [S^\nu \boxtimes S^{(a_h)}] \right\} \oplus W \tag{3.77}$$

where the sum is over all $\nu \vdash a_1 + a_2 + \cdots + a_{h-1}$ such that μ/ν is totally disconnected and W contains all the sub-representations of the form $S^\nu \boxtimes S^\theta$, with $\theta \vdash a_h$, $\theta \neq (a_h)$. On the other hand, a representation $S^\nu \boxtimes S^\theta$ of

$\mathfrak{S}_{a_1+a_2+\cdots+a_{h-1}} \times \mathfrak{S}_{a_h}$ contains nontrivial $(\mathfrak{S}_{a_1} \times \mathfrak{S}_{a_2} \times \cdots \times \mathfrak{S}_{a_h})$-invariant vectors only if it is of the form $S^\nu \boxtimes S^{(a_h)}$: it suffices to consider its restriction to $\mathfrak{S}_{a_1} \times \mathfrak{S}_{a_2} \times \cdots \times \mathfrak{S}_{a_h}$. Therefore (3.77) ensures that $\mathfrak{S}_{a_1+a_2+\cdots+a_{h-1}} \times \mathfrak{S}_{a_h}$ is multiplicity–free for the pair $(\mathfrak{S}_n, \mathfrak{S}_{a_1} \times \mathfrak{S}_{a_2} \times \cdots \times \mathfrak{S}_{a_h})$. Iterating this argument, one ends the proof.

(ii) For what we have just proved, S^μ contains a nontrivial $(\mathfrak{S}_{a_1} \times \mathfrak{S}_{a_2} \times \cdots \times \mathfrak{S}_{a_h})$-invariant vector if and only if there exists a chain $\nu^{(1)} = (a_1) \preceq \nu^{(2)} \preceq \cdots \preceq \nu^{(h)} = \mu$ such that $\nu^{(j)} \vdash a_1 + a_2 + \cdots + a_j$ and $\nu^{(j+1)}/\nu^{(j)}$ is totally disconnected, $j = 1, 2, \ldots, h-1$ (for the "only if" part, just note that if w is a nontrivial $(\mathfrak{S}_{a_1} \times \mathfrak{S}_{a_2} \times \cdots \times \mathfrak{S}_{a_h})$-invariant vector then the projection on some $S^\nu \boxtimes S^{(a_h)}$ in (3.77) must be nontrivial and that one can iterate this procedure). In particular, we must have $\mu \trianglerighteq a$ and $\nu^{(j)} \trianglerighteq (a_1, a_2, \ldots, a_j)$, for all $j = 1, 2, \ldots, h-1$ (see also Remark 3.7.7). Then it is clear that the corresponding Bratteli diagram is exactly $\mathbb{Y}_a$. □

In Section 3.7.4 we shall give another proof of (i) in the last theorem, along the lines of Corollary 3.2.2.

We can now construct a Gelfand–Tsetlin basis for the $(\mathfrak{S}_{a_1} \times \mathfrak{S}_{a_2} \times \cdots \times \mathfrak{S}_{a_h})$-invariant vectors in S^μ following the procedure in Section 2.2.3. Take $T \in \mathrm{STab}(\mu, a)$ and let

$$\mu = \nu^{(h)} \Rightarrow \nu^{(h-1)} \Rightarrow \cdots \Rightarrow \nu^{(2)} \Rightarrow \nu^{(1)} = (a_1)$$

be the associated path in $\mathbb{Y}_a$. Thus we have a chain of sub-representations

$$\begin{aligned} S^\mu &\succeq S^{\nu^{(h-1)}} \boxtimes S^{(a_h)} \\ &\succeq S^{\nu^{(h-2)}} \boxtimes S^{(a_{h-1})} \boxtimes S^{(a_h)} \\ &\succeq \cdots \\ &\succeq S^{(a_1)} \boxtimes S^{(a_2)} \boxtimes \cdots \boxtimes S^{(a_h)} \end{aligned}$$

corresponding to (2.30) and this determines, up to a multiplicative constant of modulus one, a unitary K-invariant vector in S^μ that we denote by w_T. Then we have:

Proposition 3.7.9 *The set* $\{w_T : T \in \mathrm{STab}(\mu, a)\}$ *is an orthonormal basis for the* $(\mathfrak{S}_{a_1} \times \mathfrak{S}_{a_2} \times \cdots \times \mathfrak{S}_{a_h})$*-invariant vectors in* S^μ.

Now we apply Theorem 2.2.9. Denote by $\mathcal{S}_T : S^\mu \to M^a$ the intertwining operator associated with w_T. Moreover, for every $T \in \mathrm{STab}(\mu, a)$ with associated path $\mu = \nu^{(h)} \Rightarrow \cdots \Rightarrow \nu^{(2)} \Rightarrow \nu^{(1)} = (a_1)$, we can use the procedure that led to (2.29). We take the subspace $S^{\nu^{(2)}} \boxtimes S^{(a_3)} \boxtimes \cdots \boxtimes S^{(a_h)}$ into

$$\mathrm{Ind}_{\mathfrak{S}_{a_1} \times \mathfrak{S}_{a_2} \times \cdots \times \mathfrak{S}_{a_h}}^{\mathfrak{S}_{a_1+a_2} \times \mathfrak{S}_{a_3} \times \cdots \times \mathfrak{S}_{a_h}} \left[S^{(a_1)} \boxtimes S^{(a_2)} \boxtimes \cdots \boxtimes S^{(a_h)} \right]$$

and iterating we take the subspace $S^{\nu^{(j+1)}} \boxtimes S^{(a_{j+2})} \boxtimes \cdots \boxtimes S^{(a_h)}$ into

$$\mathrm{Ind}_{\mathfrak{S}_{a_1+a_2+\cdots+a_j} \times \mathfrak{S}_{a_{j+1}} \times \cdots \times \mathfrak{S}_{a_h}}^{\mathfrak{S}_{a_1+a_2+\cdots+a_{j+1}} \times \mathfrak{S}_{a_{j+2}} \times \cdots \times \mathfrak{S}_{a_h}} \left[S^{\nu^{(j)}} \boxtimes S^{(a_{j+1})} \boxtimes \cdots \boxtimes S^{(a_h)} \right]. \tag{3.78}$$

We denote by S_T^{μ} the subspace isomorphic to S^{μ} inside M^a obtained in the finale stage of this procedure.

Theorem 3.7.10 (Orthogonal Young decomposition)

$$\bigoplus_{T \in \mathrm{STab}(\mu,a)} S_T^{\mu}$$

is an orthogonal decomposition of the S^{μ}-isotypic component in M^a. Moreover, $\mathcal{S}_T$ intertwines S^{μ} with S_T^{μ}.

Corollary 3.7.11 (Young's rule) *The multiplicity $K(\mu, a)$ of S^{μ} in the permutation module M^a is equal to the number of semistandard tableaux of shape μ and weight a. In formulae*

$$K(\mu, a) = |\mathrm{STab}(\mu, a)|.$$

Exercise 3.7.12 Use Corollary 3.7.11 to derive Vershik's relations in Theorem 3.6.13.

Corollary 3.7.13 *The integer valued matrix $(K(\mu, \nu))_{\mu,\nu \vdash n}$ is upper unitriangular, with respect to the partial order $\trianglelefteq$, so that it is invertible and its inverse $(H(\mu, \nu))_{\mu,\nu \vdash n}$ is integer valued and upper unitriangular.*

Proof This follows from Lemma 3.7.3 (alternatively, from Theorem 3.6.11) and Lemma 3.6.6. □

Denote, as before, by ψ^{ν} the character of M^{ν}.

Corollary 3.7.14 *For any $\lambda \vdash n$ we have*

$$\chi^{\lambda} = \sum_{\substack{\nu \vdash n: \\ \nu \trianglerighteq \lambda}} H(\nu, \lambda) \psi^{\nu}.$$

Proof In terms of characters, the Young rule is just $\psi^{\nu} = \sum_{\substack{\lambda \vdash n: \\ \lambda \trianglerighteq \nu}} K(\lambda, \nu) \chi^{\lambda}$. □

For instance, $\chi^{n-1,1} = \psi^{n-1,1} - \psi^{(n)}$ and $\chi^{n-2,1,1} = \psi^{n-2,1,1} - \psi^{n-2,2} - \psi^{n-1,1} + \psi^{(n)}$.

3.7.4 A Greenhalgebra with the symmetric group

In this section, which is based on A. S. Greenhalgh's thesis [53] and on Section 9.8 of our monograph [20], we prove that if V is an irreducible representation of $\mathfrak{S}_n$ and W is an irreducible representation of $\mathfrak{S}_{n-k}$, then the multiplicity of $W \otimes S^{(k)}$ in $\mathrm{Res}^{\mathfrak{S}_n}_{\mathfrak{S}_{n-k}\times\mathfrak{S}_k} V$ is ≤ 1. This will be achieved without using the representation theory of $\mathfrak{S}_n$, but simply generalizing the arguments that led to Corollary 3.2.2. Here, the algebras of conjugacy invariant functions will be replaced by suitable Greenhalgebras (see Section 2.1.3).

Let Θ_k be the set of all injective functions $\theta\colon \{1, 2, \dots, k\} \to \{1, 2, \dots, n\}$. Then Θ_k may be also viewed as the set of all *ordered* sequences of k *distinct* elements in $\{1, 2, \dots, n\}$ via the identifying map $\theta \mapsto (\theta(1), \theta(2), \dots, \theta(k))$. The group $\mathfrak{S}_n$ naturally acts on Θ_k: $(\pi\theta)(j) = \pi(\theta(j))$ for all $\pi \in \mathfrak{S}_n, \theta \in \Theta_k$ and $j \in \{1, 2, \dots, k\}$. Note that the action is transitive. Denote by $\theta_0 \in \Theta_k$ the map such that $\theta_0(j) = j$ for all $j \in \{1, 2, \dots, k\}$. Then the stabilizer of θ_0 in $\mathfrak{S}_n$ is the subgroup $\mathfrak{S}_{n-k}$ that fixes each $j \in \{1, 2, \dots, k\}$ and permutes the remaining elements $k + 1, k + 2, \dots, n$. In other words, $\Theta_k \equiv \mathfrak{S}_n/\mathfrak{S}_{n-k}$ as homogeneous spaces.

Let now $\mathfrak{S}_k$ be the subgroup of $\mathfrak{S}_n$ that fixes each $j \in \{k + 1, k + 2, \dots, n\}$ and permutes $\{1, 2, \dots, k\}$. The group $\mathfrak{S}_n \times (\mathfrak{S}_{n-k} \times \mathfrak{S}_k)$ acts on Θ_k in the following way: for $\theta \in \Theta_k, \pi \in \mathfrak{S}_n, \alpha \in \mathfrak{S}_{n-k}$ and $\beta \in \mathfrak{S}_k$, then

$$[(\pi, \alpha\beta)\theta](j) = \pi[\theta(\beta^{-1}(j))], \qquad \text{for all } j = 1, 2, \dots, k.$$

That is, we compose the maps π, θ and β^{-1}:

$$\{1, 2, \dots, k\} \stackrel{\beta^{-1}}{\to} \{1, 2, \dots, k\} \stackrel{\theta}{\to} \{1, 2, \dots, n\} \stackrel{\pi}{\to} \{1, 2, \dots, n\}.$$

In other words, if $(\theta(1), \theta(2), \dots, \theta(k)) = (i_1, i_2, \dots, i_k)$ then the $(\pi, \alpha\beta)$-image of $(i_1, i_2, \dots, i_k)$ is $(\pi(i_{\beta^{-1}(1)}), \pi(i_{\beta^{-1}(2)}), \dots, \pi(i_{\beta^{-1}(k)}))$: π changes the value and β permutes the coordinates.

From still another point of view, we can say that the groups $\mathfrak{S}_n$, $\mathfrak{S}_{n-k}$ and $\mathfrak{S}_k$ act on Θ_k (the group $\mathfrak{S}_{n-k}$ trivially) and their actions commute, so that their Cartesian product acts on Θ_k too.

Lemma 3.7.15 *The stabilizer of θ_0 in $\mathfrak{S}_n \times (\mathfrak{S}_{n-k} \times \mathfrak{S}_k)$ is the subgroup*

$$B = \{(\eta\sigma, \sigma) \equiv (\sigma\eta, \sigma) : \eta \in \mathfrak{S}_{n-k}, \sigma \in \mathfrak{S}_{n-k} \times \mathfrak{S}_k\}.$$

In particular, $\Theta_k \cong [\mathfrak{S}_n \times (\mathfrak{S}_{n-k} \times \mathfrak{S}_k)]/B$.

Proof We have, for $\pi \in \mathfrak{S}_n$, $\alpha \in \mathfrak{S}_{n-k}$ and $\beta \in \mathfrak{S}_k$,

$$\begin{aligned}(\pi, \alpha\beta)\theta_0 = \theta_0 &\Leftrightarrow \pi[\theta_0(\beta^{-1}(j))] = j \text{ for all } j = 1, 2, \ldots, k\\ &\Leftrightarrow \beta^{-1}(j) = \pi^{-1}(j) \text{ for all } j = 1, 2, \ldots, k\\ &\Leftrightarrow \pi = \beta\gamma \text{ with } \gamma \in \mathfrak{S}_{n-k}\\ &\Leftrightarrow (\pi, \alpha\beta) = (\beta\gamma, \alpha\beta) = (\beta\alpha(\alpha^{-1}\gamma), \alpha\beta)\\ &\quad= (\alpha\beta\eta, \alpha\beta) \text{ with } \eta = \alpha^{-1}\gamma \in \mathfrak{S}_{n-k}.\end{aligned}$$ □

In the terminology of Section 2.1.3, we can say that the algebra of bi-B-invariant functions on $\mathfrak{S}_n \times (\mathfrak{S}_{n-k} \times \mathfrak{S}_k)$ is isomorphic to the Greenhalgebra $\mathcal{G}(\mathfrak{S}_n, \mathfrak{S}_{n-k} \times \mathfrak{S}_k, \mathfrak{S}_{n-k})$, that is, the algebra of all functions on $\mathfrak{S}_n$ that are $(\mathfrak{S}_{n-k} \times \mathfrak{S}_k)$-conjugacy invariant and bi-$\mathfrak{S}_{n-k}$-invariant. The main task of this section is to prove that these algebras are commutative.

To this end, we introduce a parameterization of the orbits of $\mathfrak{S}_n \times (\mathfrak{S}_{n-k} \times \mathfrak{S}_k)$ on $\Theta_k \times \Theta_k$. For any ordered pair $(\theta, \vartheta) \in \Theta_k \times \Theta_k$, we construct an *oriented graph* $\mathcal{G}(\theta, \vartheta)$ as follows. The vertex set is $V = \{\theta(1), \theta(2), \ldots, \theta(k)\} \cup \{\vartheta(1), \vartheta(2), \ldots, \vartheta(k)\}$ and the edge set is $E = \{(\theta(1), \vartheta(1)), (\theta(2), \vartheta(2)), \ldots, (\theta(k), \vartheta(k))\}$.

Figure 3.44 is an example with $n = 15$ and $k = 8$.

$$\begin{aligned}(\theta(1), \theta(2), \ldots, \theta(8)) &= (1, 7, 11, 6, 3, 14, 9, 10)\\ (\vartheta(1), \vartheta(2), \ldots, \vartheta(8)) &= (3, 13, 9, 6, 14, 10, 15, 1)\end{aligned}$$

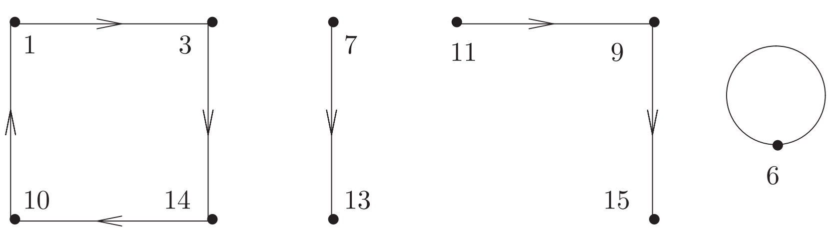


Figure 3.44

Note that every vertex of $\mathcal{G}(\theta, \vartheta)$ has degree ≤ 2. More precisely,

- if $\theta(j) = \vartheta(j)$ then the vertex $\theta(j)$ has a loop and it is not connected to any other vertex;
- if $i \neq j$ in $\{1, 2, \ldots, k\}$ are such that $\theta(i) = \vartheta(j)$ then $\theta(i)$ has degree two: it is the terminal vertex of the edge $(\theta(j), \vartheta(j))$ and the initial vertex of the edge $(\theta(i), \vartheta(i))$;

- if $v \in \{\theta(1), \theta(2), \dots, \theta(k)\} \triangle \{\vartheta(1), \vartheta(2), \dots, \vartheta(k)\}$ then v has degree 1 and it is an initial vertex when $v \in \{\theta(1), \theta(2), \dots, \theta(k)\}$ and it is a terminal edge when $v \in \{\vartheta(1), \vartheta(2), \dots, \vartheta(k)\}$.

Lemma 3.7.16 *Two ordered pairs (θ, ϑ), $(\theta', \vartheta') \in \Theta_k \times \Theta_k$ belong to the same orbit of $\mathfrak{S}_n \times (\mathfrak{S}_{n-k} \times \mathfrak{S}_k)$ if and only if the two oriented graphs $\mathcal{G}(\theta, \vartheta)$ and $\mathcal{G}(\theta', \vartheta')$ are isomorphic.*

Proof If there exists $(\pi, \gamma) \in \mathfrak{S}_n \times (\mathfrak{S}_{n-k} \times \mathfrak{S}_k)$ such that $(\pi, \gamma)(\theta, \vartheta) = (\theta', \vartheta')$ then, for all $j = 1, 2, \dots, k$,

$$\theta'(j) = \pi[\theta(\gamma^{-1}(j))], \qquad \vartheta'(j) = \pi[\vartheta(\gamma^{-1}(j))],$$

and the map $\phi : \mathcal{G}(\theta, \vartheta) \to \mathcal{G}(\theta', \vartheta')$ given by

$$\phi(v) = \pi(v) \quad v \in \{\theta(1), \theta(2), \dots, \theta(k)\} \cup \{\vartheta(1), \vartheta(2), \dots, \vartheta(k)\},$$

is an isomorphism. Indeed, γ permutes the edges, π permutes the vertices, but neither changes the structure of the graph.

Now suppose that $\phi : \mathcal{G}(\theta, \vartheta) \to \mathcal{G}(\theta', \vartheta')$ is an isomorphism. We can take $\gamma \in \mathfrak{S}_{n-k} \times \mathfrak{S}_k$ such that

$$\phi(\theta(\gamma^{-1}(j)), \vartheta(\gamma^{-1}(j))) = (\theta'(j), \vartheta'(j)) \quad j = 1, 2, \dots, k.$$

Setting, for all $j = 1, 2, \dots, k$,

$$\pi[\theta(\gamma^{-1}(j))] = \theta'(j), \qquad \pi[\vartheta(\gamma^{-1}(j))] = \vartheta'(j)$$

(if $\theta(\gamma^{-1}(j)) = \vartheta(\gamma^{-1}(j))$ then we have a loop at $\theta(\gamma^{-1}(j))$ but also at $\theta'(j) = \vartheta'(j)$, because ϕ is an isomorphism) and extending π to a permutation of the whole set $\{1, 2, \dots, n\}$, we have $(\pi, \gamma)(\theta, \vartheta) = (\theta', \vartheta')$. □

Corollary 3.7.17 $(\mathfrak{S}_n \times (\mathfrak{S}_{n-k} \times \mathfrak{S}_k), B)$ *is a symmetric Gelfand pair.*

Proof The graphs $\mathcal{G}(\theta, \vartheta)$ and $\mathcal{G}(\vartheta, \theta)$ are always isomorphic. □

Corollary 3.7.18 *The Greenhalgebra $\mathcal{G}(\mathfrak{S}_n, \mathfrak{S}_{n-k} \times \mathfrak{S}_k, \mathfrak{S}_{n-k})$ is commutative.*

Recalling that $S^{(n-k)}$ denotes the trivial representation of $\mathfrak{S}_{n-k}$, we have:

Corollary 3.7.19 *For any irreducible representation V of $\mathfrak{S}_n$ and any irreducible representation W of $\mathfrak{S}_k$, we have:*

(i) *the multiplicity of $W \boxtimes S^{(n-k)}$ in $\mathrm{Res}^{\mathfrak{S}_n}_{\mathfrak{S}_k \times \mathfrak{S}_{n-k}} V$ is ≤ 1;*
(ii) *the multiplicity of V in $\mathrm{Ind}^{\mathfrak{S}_n}_{\mathfrak{S}_k \times \mathfrak{S}_{n-k}} [W \boxtimes S^{(n-k)}]$ is ≤ 1.*

4

Symmetric functions

4.1 Symmetric polynomials

In this section, we investigate the fundamentals of the theory of symmetric polynomials, which are homogeneous polynomials invariant under permutations of the variables. The basic reference is Macdonalds' monograph [83] (see also [84]). Other important references are: Stanley's book [113], which is the best reference for the combinatorial aspects, and Sagan's [108], where the connections with the representation theory of the symmetric group are treated in great detail. We have also benefited greatly from the expositions in Fulton's and Fulton and Harris monographs [42, 43] (we adopt their point of view focusing on symmetric polynomials in a finite number of variables), and Simon's book [111], especially for the proof of the Frobenius character formula, that indeed will be given in the next subsection. We also mention the recent book by Procesi [103] and [46] which is a very nice historical survey, from Cauchy's formula to Macdonald polynomials. For particular aspects and applications of symmetric functions, we also refer to the books by Lascoux [77], Manivel [85] and Bressoud [12].

4.1.1 More notation and results on partitions

In this section, we collect some notation and results on partitions that will be frequently used in this chapter. Let n be a positive integer and $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k) \vdash n$ be a partition. We represent it also in the following way: $\lambda = (1^{u_1}, 2^{u_2}, \dots, n^{u_n})$, where u_j is the number of $i \in \{1, 2, \dots, k\}$ such that $\lambda_i = j$. Clearly, $1u_1 + 2u_2 + \cdots + nu_n = n$. We omit the terms of the form i^{u_i} with $u_i = 0$ and simply write i when $u_i = 1$. For instance, the partition $\lambda = (7, 5, 5, 5, 3, 3, 1)$ of 29 may be represented in the form $\lambda = (1, 3^2, 5^3, 7)$. We give an elementary application of this notation.

Proposition 4.1.1 *For $\lambda = (1^{u_1}, 2^{u_2}, \dots, n^{u_n}) \vdash n$, denote by $\mathcal{C}_\lambda$ the corresponding conjugacy class of $\mathfrak{S}_n$ and by z_λ the cardinality of the centralizer of an element $\pi \in \mathcal{C}_\lambda$. Then we have:*

$$z_\lambda = u_1!1^{u_1}u_2!2^{u_2}\cdots u_n!n^{u_n} \tag{4.1}$$

and

$$|\mathcal{C}_\lambda| = \frac{n!}{u_1!1^{u_1}u_2!2^{u_2}\cdots u_n!n^{u_n}}. \tag{4.2}$$

Proof Consider the action of $\mathfrak{S}_n$ on $\mathcal{C}_\lambda$ by conjugation: $\pi \mapsto \sigma\pi\sigma^{-1}$, for $\pi \in \mathcal{C}_\lambda$ and $\sigma \in \mathfrak{S}_n$. The stabilizer of π is its centralizer: $Z_\pi = \{\sigma \in \mathfrak{S}_n : \sigma\pi\sigma^{-1} = \pi\}$. From Proposition 3.1.3, it follows that $\sigma \in Z_\pi$ if and only if (a) for $j = 1, 2, \dots, n, \sigma$ permutes among themselves the cycles of length j and/or (b) σ cyclically permutes each cycle of π. But for each j, (a) may be performed in $u_j!$ different ways while, for each cycle of lenght j of π, (b) may be performed in j different ways. Therefore we get $z_\lambda := |Z_\lambda| = u_1!1^{u_1}u_2!2^{u_2}\cdots u_n!n^{u_n}$ and $\mathcal{C}_\lambda = \frac{|\mathfrak{S}_n|}{z_\lambda} = \frac{n!}{u_1!1^{u_1}u_2!2^{u_2}\cdots u_n!n^{u_n}}$. □

In particular, the orthogonality relations for the characters of $\mathfrak{S}_n$ may be written in the form

$$\sum_{\lambda \vdash n} \frac{1}{z_\lambda}\chi_\lambda^\mu \chi_\lambda^\nu = \delta_{\mu,\nu}. \tag{4.3}$$

For a partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$, with $\lambda_k > 0$, we set $|\lambda| = \lambda_1 + \lambda_2 + \cdots + \lambda_k$ and denote by $\ell(\lambda) = k$ its *length*. For instance, if $\lambda = (6, 4, 4, 3, 3, 3, 2, 1, 1)$, we have $|\lambda| = 27$ and $\ell(\lambda) = 9$. If λ' is the conjugate partition, then $\ell(\lambda') = \lambda_1$ and $\ell(\lambda) = \lambda'_1$. Moreover, $|\lambda| = n$ is just an equivalent way to write $\lambda \vdash n$ and if $\lambda = (1^{u_1}, 2^{u_2}, \dots, n^{u_n})$ then $\ell(\lambda) = u_1 + u_2 + \cdots + u_n$. We also denote by $m_k(\lambda)$ the number of parts of λ that are equal to k, that is, if $\lambda = (1^{u_1}, 2^{u_2}, \dots, n^{u_n})$, then $m_k(\lambda) = u_k$ for all $k = 1, 2, \dots, n$.

4.1.2 Monomial symmetric polynomials

Let $\mathcal{X}_n^r$ be the set of all monomials of degree r in the variables $x_1, x_2, \dots, x_n$ and W_r^n the space of all linear combinations with complex coefficients of the elements of $\mathcal{X}_n^r$. In other words, W_n^r is the space of all homogeneous polynomials of degree r in the indeterminates $x_1, x_2, \dots, x_n$. We can define a natural action of $\mathfrak{S}_n$ on $\mathcal{X}_n^r$ by setting $\pi(x_1^{v_1}x_2^{v_2}\cdots x_n^{v_n}) = x_{\pi(1)}^{v_1}x_{\pi(2)}^{v_2}\cdots x_{\pi(n)}^{v_n}$, for $\pi \in \mathfrak{S}_n$ and $x_1^{v_1}x_2^{v_2}\cdots x_n^{v_n} \in \mathcal{X}_n^r$. The corresponding permutation representation on W_n^r is given by: $(\pi p)(x_1, x_2, \dots, x_n) = p(x_{\pi(1)}, x_{\pi(2)}, \dots, x_{\pi(n)})$, for all $\pi \in \mathfrak{S}_n$ and $p \in W_n^r$. We want to identify the orbits of $\mathfrak{S}_n$ on $\mathcal{X}_n^r$ and consequently, the decomposition of W_n^r into transitive permutation representations (that turn out to be Young modules). We need the following particular notation: if $\lambda = (1^{u_1}, 2^{u_2}, \ldots, n^{u_n})$, we denote by $u(\lambda)$ the composition $(u_{i_1}, u_{i_2}, \ldots, u_{i_k}, n - \ell(\lambda))$, where $u_{i_1}, u_{i_2}, \ldots, u_{i_k}$ are the nonzero integers in $u_1, u_2, \ldots, u_n$.

Proposition 4.1.2 *The decomposition of W_n^r as a direct sum of transitive permutation representations is given by:*

$$W_n^r \cong \bigoplus_{\lambda \vdash r : \ell(\lambda) \leq n} M^{u(\lambda)},$$

where $M^{u(\lambda)}$ is the Young module associated with $u(\lambda)$.

Proof Let $\lambda \vdash r$, $\ell(\lambda) \leq n$ and $u(\lambda) = (u_{i_1}, u_{i_2}, \ldots, u_{i_k}, n - \ell(\lambda))$, as above. In the notation of Section 3.6.2, consider the map

$$\begin{array}{cccc} \Phi_\lambda : & \Omega_{u(\lambda)} & \longrightarrow & \mathcal{X}_n^r \\ & (A_1, A_2, \ldots, A_k, A_{k+1}) & \longmapsto & \prod_{j=1}^k \left(\prod_{v \in A_j} x_v \right)^{i_j}. \end{array}$$

Clearly, Φ_λ commutes with the action of $\mathfrak{S}_n$ and it is injective. Moreover, $\mathcal{X}_n^r = \coprod_{\lambda \vdash r : \ell(\lambda) \leq n} \Phi_\lambda(\Omega_{u(\lambda)})$ is the decomposition of $\mathcal{X}_n^r$ into $\mathfrak{S}_n$-orbits. Indeed, $\Phi_\lambda(\Omega_{u(\lambda)})$ is the set of all monomials $x_{j_1}^{\lambda_1} x_{j_2}^{\lambda_2} \cdots x_{j_{\ell(\lambda)}}^{\lambda_{\ell(\lambda)}}$ with $\{j_1, j_2, \ldots, j_{\ell(\lambda)}\}$ an $\ell(\lambda)$-subset of $\{1, 2, \ldots, n\}$, and therefore it is precisely the orbit containing $x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_{\ell(\lambda)}^{\lambda_{\ell(\lambda)}}$. Since $M^{u(\lambda)} = L(\Omega_{u(\lambda)})$, the proposition follows. □

The vector space of all *homogeneous symmetric polynomials* of degree r in the variables $x_1, x_2, \ldots, x_n$, denoted by Λ_n^r, is just the trivial isotypic component in W_n^r, that is the set of all $p \in W_n^r$ such that $\pi p = p$ for all $\pi \in \mathfrak{S}_n$. From Proposition 4.1.2, we can get easily a basis for Λ_n^r. For $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \vdash r$, with $\ell(\lambda) \equiv k \leq n$, the *monomial symmetric polynomial* associated with λ is given by:

$$m_\lambda(x) = \sum_{i_1, i_2, \ldots, i_k} x_{i_1}^{\lambda_1} x_{i_2}^{\lambda_2} \cdots x_{i_k}^{\lambda_k}$$

where the sum is over all distinct k-subsets $\{i_1, i_2, \ldots, i_k\}$ of $\{1, 2, \ldots, n\}$. In other words, m_λ is the sum of all monomials with exponents $\lambda_1, \lambda_2, \ldots, \lambda_k$. Here and in the sequel, we write $p(x)$ to denote $p(x_1, x_2, \ldots, x_n)$, if the variables $x_1, x_2, \ldots, x_n$ are fixed.

Corollary 4.1.3 *The set* $\{m_\lambda : \lambda \vdash r, \ell(\lambda) \le n\}$ *is a basis for* Λ_n^r.

Note that if $p = \sum_{\substack{\lambda \vdash r: \\ \ell(\lambda) \le n}} a_\lambda m_\lambda \in \Lambda_n^r$, then a_λ (with $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \vdash r$) is the coefficient of $x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_k^{\lambda_k}$ in p.

Example 4.1.4 For instance, $W_n^1 \cong M^{n-1,1}$, while $W_n^2 \cong M^{n-1,1} \oplus M^{n-2,2}$; the monomial symmetric polynomials in Λ_n^2 are: $m_{(2)} = x_1^2 + x_2^2 + \cdots + x_n^2$ and $m_{1,1} = \sum_{1 \le i < j \le n} x_i x_j$. We also have $m_{(3)} = x_1^3 + x_2^3 + \cdots + x_n^3$, $m_{(2,1)} = \sum_{i \ne j, i,j=1}^n x_i^2 x_j$ and $m_{(1,1,1)} = \sum_{1 \le i < j < k \le n} x_i x_j x_k$. Moreover, $W_n^3 \cong M^{n-1,1} \oplus M^{n-2,1,1} \oplus M^{n-3,3}$.

Exercise 4.1.5 Show that, for $n \ge 4$, $W_n^4 \cong M^{n-1,1} \oplus M^{n-2,1,1} \oplus M^{n-3,2,1} \oplus M^{n-2,2} \oplus M^{n-4,4}$, while $W_2^r \cong \frac{r}{2} M^{1,1} \oplus M^{(2)}$ if r is even, $W_2^r \cong \frac{r+1}{2} M^{1,1}$ if r is odd.

4.1.3 Elementary, complete and power sums symmetric polynomials

In this section, we introduce three other families of symmetric polynomials. We set $e_0 = 1$ and, for $j = 1, 2, \ldots, n$,

$$e_j(x_1, x_2, \ldots, x_n) = \sum_{1 \le i_1 < \cdots < i_j \le n} x_{i_1} x_{i_2} \cdots x_{i_j}.$$

In other words, $e_j(x)$ is the sum of all square-free (monic) monomials of degree j. The functions $e_1, e_2, \ldots, e_n$ are usually introduced to express the cofficients of a polynomial when we know its roots: if $P(z) = (z - \alpha_1)(z - \alpha_2) \cdots (z - \alpha_n)$, then $P(z) = z^n - e_1(\alpha_1, \ldots, \alpha_n) z^{n-1} + e_2(\alpha_1, \ldots, \alpha_n) z^{n-2} + \cdots + (-1)^n e_n(\alpha_1, \ldots, \alpha_n)$; see [76]. For every $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \vdash r$, with $\ell(\lambda') \equiv \lambda_1 \le n$, we set

$$e_\lambda(x) = e_{\lambda_1}(x) e_{\lambda_2}(x) \cdots e_{\lambda_k}(x).$$

Then $e_\lambda(x)$ is the *elementary symmetric polynomial* associated with λ.

We set $h_0 = 1$ and, for $j = 1, 2, \ldots$, we denote by $h_j(x)$ the sum of all (monic) monomials of degree j:

$$h_j(x) = \sum_{i_1, i_2, \ldots, i_j = 1}^n x_{i_1} x_{i_2} \cdots x_{i_j}.$$

For instance, for $n = 2$, $h_2(x) = x_1^2 + x_1 x_2 + x_2^2$ and $h_3(x) = x_1^3 + x_1^2 x_2 + x_1 x_2^2 + x_2^3$. If $\lambda \vdash r$, the *complete symmetric polynomial* associated with $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ is given by

$$h_\lambda(x) = h_{\lambda_1}(x) h_{\lambda_2}(x) \cdots h_{\lambda_k}(x).$$

Finally, we set $p_0 = 1$ and $p_j(x) = x_1^j + x_2^j + \cdots + x_n^j$ for $j = 1, 2, \ldots$. If $\lambda \vdash r$, the *power sum symmetric polynomial* associated with $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ is given by

$$p_\lambda(x) = p_{\lambda_1}(x) p_{\lambda_2}(x) \cdots p_{\lambda_k}(x).$$

Note that the condition $\ell(\lambda') \leq n$ for the existence of e_λ is dual to the condition $\ell(\lambda) \leq n$ for m_λ. Moreover, both conditions become trivial when $n \geq r$. On the other hand, the polynomials h_λ and p_λ are defined for *all* partitions of r.

If $a_0, a_1, \ldots, a_n, \ldots$ is a sequence of numbers (or functions), its *generating function* is the power series $\sum_{k=0}^{\infty} a_k t^k$. It may be a formal power series, or it may be a power series with a positive radius of convergence. We shall encounter convergent generating functions, whose sum is computable; but we leave to the reader the easy task of verifying the convergence.

The following exercise contains some useful criteria in order to easily develop products and compositions of functions (see the books by Ahlfors [1] and Weinberger [123]).

Exercise 4.1.6 (1) Show that if $f, g : \mathbb{C} \to \mathbb{C}$ are two functions which are analytic in z_0 with power series expansions $f(z) = \sum_{n=0}^{+\infty} a_n (z - z_0)^n$ and $g(z) = \sum_{n=0}^{+\infty} b_n (z - z_0)^n$, then the power series expansion of their product $f \cdot g$ is $f(z)g(z) = \sum_{n=0}^{+\infty} \left[\sum_{k=0}^{n} a_k b_{n-k}\right] (z - z_0)^n$ (Cauchy product).

(2) Suppose that f is analytic in an open set U containing z_0, that g is analytic in an open set containing $w_0 = f(z_0)$ and that $f(U) \subset W$. Let $f = \sum_{n=0}^{+\infty} a_n (z - z_0)^n$ and $g(w) = \sum_{n=0}^{+\infty} b_n (w - w_0)^n$ be the corresponding power series expansions. Show that in the power series expansion of $F(z) := g[f(z)] = \sum_{n=0}^{+\infty} c_n (z - z_0)^n$ of the composition of g and f, the coefficients c_n depends only on $a_0, a_1, \ldots, a_n, b_0, b_1, \ldots, b_n$. In particular, the Taylor polynomial $\sum_{n=0}^{N} c_n (z - z_0)^n$ may be obtained by replacing f with $\sum_{n=0}^{N} a_n (z - z_0)^n$ and g with $\sum_{n=0}^{N} b_n (w - w_0)^n$ (thus, omitting the powers of $(z - z_0)$ of degree $\geq N + 1$).

Lemma 4.1.7 *The sequences of symmetric polynomials $e_0, e_1, \ldots, e_n$, $h_0, h_1, h_2, \ldots$ and $p_1, p_2, p_3, \ldots$ have the following generating functions:*

(i) $\displaystyle\sum_{j=0}^{n} e_j(x) t^j = \prod_{i=1}^{n} (1 + x_i t) =: E(t)$;

(ii) $\displaystyle\sum_{j=0}^{\infty} h_j(x) t^j = \prod_{i=1}^{n} \frac{1}{1 - x_i t} =: H(t)$;

(iii) $\displaystyle\sum_{j=1}^{\infty} p_j(x) t^{j-1} = \sum_{i=1}^{n} \frac{x_i}{1 - x_i t} =: P(t)$.

Proof We use the well-known formal geometric series expansions ($s \neq 1$):

$$\sum_{j=0}^{\infty} s^j = \frac{1}{1-s} \tag{4.4}$$

and

$$\sum_{j=1}^{\infty} s^j = \frac{s}{1-s}. \tag{4.5}$$

(i) Set $E_n(t) = \prod_{i=1}^{n}(1 + x_i t)$ and $e_j^{(n)}(x) = e_j(x_1, x_2, \ldots, x_n)$. We also set $e_j^{(n)}(x) = 0$ if $j > n$. We proceed by induction on n. For $n = 1$ we have $e_0(x) + e_1(x)t = e_0^{(1)}(x) + e_1^{(1)}(x)t = 1 + x_1 t = E_1(t)$. Moreover, we note that for $j \geq 1$ one has

$$e_j^{(n)}(x) = e_j^{(n-1)}(x) + e_{j-1}^{(n-1)}(x)x_n. \tag{4.6}$$

Then,

$$\begin{aligned}
\sum_{j=0}^{n} e_j^{(n)}(x)t^j &= 1 + \sum_{j=1}^{n}[e_j^{(n-1)}(x) + e_{j-1}^{(n-1)}(x)x_n]t^j \quad \text{(by (4.6))} \\
&= 1 + \sum_{j=1}^{n} e_j^{(n-1)}(x)t^j + \sum_{j=1}^{n} e_{j-1}^{(n-1)}(x)x_n t^j \\
&= \sum_{j=0}^{n-1} e_j^{(n-1)}(x)t^j + (\sum_{j=0}^{n-1} e_j^{(n-1)}(x)t^j)x_n t \\
&= E_{n-1}(t)(1 + x_n t) \\
\text{(by induction)} \quad &= \left(\prod_{i=1}^{n-1}(1 + x_i t)\right)(1 + x_n t) \\
&= \prod_{i=1}^{n}(1 + x_i t) \\
&= E_n(t).
\end{aligned}$$

(ii) Set $H_n(t) = \prod_{i=1}^{n} \frac{1}{1-x_i t}$ and $h_j^{(n)}(x) = h_j(x_1, x_2, \ldots, x_n)$. We also set $h_j^{(n)} = 0$ if $j < 0$. We proceed by induction on n. For $n = 1$ we have $h_j(x) = h_j^{(1)}(x) = x_1^j$ and $\sum_{j=0}^{\infty} h_j^{(1)}(x)t^j = \sum_{j=0}^{\infty} x_1^j t^j = \frac{1}{1-x_1 t} = H_1(t)$. Now, for $j \geq 1$ one has

$$\begin{aligned}
h_j^{(n)}(x) &= h_j^{(n-1)}(x) + h_{j-1}^{(n-1)}(x)x_n + \cdots + h_1^{(n-1)}(x)x_n^{j-1} + h_0^{(n-1)}(x)x_n^j \\
&= \sum_{k=0}^{\infty} h_{j-k}^{(n-1)}(x)x_n^k.
\end{aligned} \tag{4.7}$$

Then

$$\begin{aligned}
\sum_{j=0}^{n} h_j^{(n)}(x)t^j &= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} h_{j-k}^{(n-1)}(x)x_n^k t^j \quad \text{(by (4.7))}\\
&= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} h_{j-k}^{(n-1)}(x)t^{j-k}(x_n t)^k\\
\text{(by setting } \ell = j-k) \quad &= \sum_{\ell=0}^{\infty} h_{\ell}^{(n-1)}(x)t^{\ell}\sum_{k=0}^{\infty}(x_n t)^k\\
\text{(by induction and (4.4))} \quad &= \left(\prod_{i=1}^{n-1}\frac{1}{1-x_i t}\right)\frac{1}{1-x_n t}\\
&= \prod_{i=1}^{n}\frac{1}{1-x_i t}\\
&= H_n(t).
\end{aligned}$$

(iii) Set $P_n(t) = \sum_{i=1}^{n}\frac{x_i}{1-x_i t}$ and $p_j^{(n)}(x) = p_j(x_1, x_2, \ldots, x_n)$. We proceed by induction on n. For $n = 1$ we have $p_j(x) = p_j^{(1)}(x) = x_1^j$ and $\sum_{j=1}^{\infty} p_j^{(1)}(x)t^{j-1} = \sum_{j=1}^{\infty} x_1^{j-1}t^{j-1}x_1 = \sum_{j=0}^{\infty} x_1^j t^j x_1 = \frac{x_1}{1-x_1 t} = P_1(t)$. Now, for $j \geq 1$ one has

$$p_j^{(n)}(x) = p_j^{(n-1)}(x) + x_n^j. \tag{4.8}$$

Then

$$\begin{aligned}
\sum_{j=1}^{n} p_j^{(n)}(x)t^j &= \sum_{j=1}^{\infty}\left(p_j^{(n-1)}(x)t^j + x_n^j t^j\right) \quad \text{(by (4.8))}\\
&= \sum_{j=1}^{\infty} p_j^{(n-1)}(x)t^j + \sum_{j=1}^{\infty} x_n^j t^j\\
\text{(by induction and (4.5))} \quad &= \left(\sum_{i=1}^{n-1}\frac{x_i}{1-x_i t}\right) + \frac{x_n}{1-x_n t}\\
&= \sum_{i=1}^{n}\frac{x_i}{1-x_i t}\\
&= P_n(t).
\end{aligned}$$

□

Exercise 4.1.8 Prove that $\sum_{j=1}^{\infty} p_j(x)\frac{t^j}{j} = \log\prod_{i=1}^{n}\frac{1}{1-x_i t}$. Hint: use the Taylor expansion $\log\frac{1}{1-s} = \sum_{j=1}^{\infty}\frac{s^j}{j}$. See also the proof of Lemma 4.1.10.

Lemma 4.1.9 *We have the following relations between the symmetric polynomials e_j, h_j and p_j:*

(i) $\sum_{j=0}^{\min\{k,n\}} (-1)^{k-j} e_j h_{k-j} = 0, \quad$ *for* $k = 1, 2, 3, \ldots$;

(ii) $kh_k = \sum_{j=1}^{k} p_j h_{k-j}, \quad$ *for* $k = 1, 2, 3, \ldots$;

(iii) $ke_k = \sum_{j=1}^{k} (-1)^{j-1} p_j e_{k-j}, \quad$ *for* $k = 1, 2, \ldots, n$ *(Newton's formulae).*

Proof (i) In the notation of Lemma 4.1.7, we have $E(t)H(-t) = 1$. This implies that

$$1 = \left(\sum_{j=0}^{n} e_j t^j\right)\left(\sum_{j=0}^{\infty} h_j(-t)^j\right) = \sum_{k=0}^{\infty}\left(\sum_{j=0}^{\min\{k,n\}} (-1)^{k-j} e_j h_{k-j}\right) t^k.$$

(ii) We have

$$\begin{aligned}\frac{dH}{dt} &= \sum_{i=1}^{n} \frac{x_i}{(1-x_i t)^2} \prod_{\substack{j=1\\ j\neq i}}^{n} \frac{1}{1-x_j t}\\ &= \sum_{i=1}^{n} \frac{x_i}{1-x_i t}\left(\prod_{j=1}^{n} \frac{1}{1-x_j t}\right)\\ &= P(t)H(t).\end{aligned}$$

On the other hand, $dH/dt = \sum_{k=0}^{\infty} kh_k t^{k-1}$.

(iii) We have

$$\begin{aligned}\frac{dE}{dt} &= \sum_{i=1}^{n} x_i \prod_{\substack{j=1\\ j\neq i}}^{n} (1 + x_j t)\\ &= \sum_{i=1}^{n} \frac{x_i}{1+x_i t} \prod_{j=1}^{n} (1 + x_j t)\\ &= P(-t)E(t).\end{aligned}$$

On the other hand, $dE/dt = \sum_{k=1}^{n} ke_k t^{k-1}$. □

In the following Lemma, we express h_r in terms of the p_λ's.

Lemma 4.1.10 *For $r = 1, 2, \ldots,$ we have:*

$$h_r(x) = \sum_{\lambda \vdash r} \frac{1}{z_\lambda} p_\lambda(x),$$

where z_λ is as in (4.1).

Proof We have:

$$\frac{d}{dt} \log H(t) = \sum_{i=1}^{n} \frac{x_i}{1 - x_i t} = P(t),$$

and therefore

$$\begin{aligned} H(t) = \exp\Big(\int_0^t P(\tau) d\tau\Big) &= \exp\Big(\sum_{k=1}^{\infty} \frac{p_k(x) t^k}{k}\Big) \\ &= \prod_{k=1}^{\infty} \exp\left(\frac{p_k(x) t^k}{k}\right) \\ &= \prod_{k=1}^{\infty} \sum_{u_k=0}^{\infty} \frac{1}{u_k!} \Big(\frac{p_k(x) t^k}{k}\Big)^{u_k} \\ &= \sum_{r=0}^{\infty} \Big(\sum_{\lambda \vdash r} z_\lambda^{-1} p_\lambda(x)\Big) t^r, \end{aligned}$$

if $\lambda = (1^{u_1}, 2^{u_2}, \ldots, r^{u_r})$. □

Now suppose that $y_1, y_2, \ldots, y_n$ is another set of variables. We denote by $h_\lambda(xy)$, (resp. $p_\lambda(xy)$), the complete (resp. power sums) polynomial constructed using the n^2 variables

$$x_1 y_1, \ldots, x_n y_1, x_1 y_2, \ldots, x_n y_2, \ldots, x_1 y_n, \ldots x_n y_n.$$

Note that the power sums have the following property: $p_k(xy) = p_k(x) p_k(y)$, and therefore $p_\lambda(xy) = p_\lambda(x) p_\lambda(y)$.

Lemma 4.1.11 *We have the following expansions for the product* $\prod_{i,j=1}^{n} \frac{1}{1 - x_i y_j}$:

$$\prod_{i,j=1}^{n} \frac{1}{1 - x_i y_j} = \sum_{r=0}^{\infty} \sum_{\lambda \vdash r} \frac{1}{z_\lambda} p_\lambda(x) p_\lambda(y); \tag{4.9}$$

and

$$\prod_{i,j=1}^{n} \frac{1}{1 - x_i y_j} = \sum_{r=0}^{\infty} \sum_{\substack{\lambda \vdash r: \\ \ell(\lambda) \leq n}} h_\lambda(x) m_\lambda(y). \tag{4.10}$$

Proof The first expansion is an immediate consequence of Lemma 4.1.10:

$$\begin{aligned}\prod_{i,j=1}^{n} \frac{1}{1 - x_i y_j} &= \sum_{r=0}^{\infty} h_r(xy) \\ &= \sum_{r=0}^{\infty} \sum_{\lambda \vdash r} \frac{1}{z_\lambda} p_\lambda(xy) \\ &= \sum_{r=0}^{\infty} \sum_{\lambda \vdash r} \frac{1}{z_\lambda} p_\lambda(x) p_\lambda(y).\end{aligned}$$

The second expansion is a consequence of the following fact: since

$$\prod_{i,j=1}^{n} \frac{1}{1 - x_i y_j} = \prod_{j=1}^{n} H(y_j) = \prod_{j=1}^{n} \sum_{\alpha_j=0}^{\infty} h_{\alpha_j}(x) y_j^{\alpha_j},$$

for each $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \vdash r$ with $\ell(\lambda) \leq n$, the coefficient of $y_1^{\lambda_1} y_2^{\lambda_2} \cdots y_k^{\lambda_k}$ in the product $\prod_{i,j=1}^{n} \frac{1}{1 - x_i y_j}$ is equal to $h_\lambda(x)$. □

4.1.4 The fundamental theorem on symmetric polynomials

Now we have all the necessary tools to prove the main theorem of this section. Actually, the term "fundamental theorem" is usually restricted to case (i) of such a theorem.

Theorem 4.1.12 *Each of the following sets is a basis for the space* Λ_n^r*:*

(i) $\{e_\lambda : \lambda \vdash r$ *and* $\ell(\lambda') \leq n\}$*;*
(ii) $\{h_\lambda : \lambda \vdash r$ *and* $\ell(\lambda') \leq n\}$*;*
(iii) $\{h_\lambda : \lambda \vdash r$ *and* $\ell(\lambda) \leq n\}$*;*
(iv) $\{p_\lambda : \lambda \vdash r$ *and* $\ell(\lambda) \leq n\}$*;*
(v) $\{p_\lambda : \lambda \vdash r$ *and* $\ell(\lambda') \leq n\}$*.*

Proof (i) If $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \vdash r$, then each monomial in e_λ is a product of the form

$$(x_{i_1} x_{i_2} \cdots x_{i_{\lambda_1}}) \cdot (x_{j_1} x_{j_2} \cdots x_{j_{\lambda_2}}) \cdots (x_{t_1} x_{t_2} \cdots x_{t_{\lambda_k}}) = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}, \qquad (4.11)$$

where $i_1 < i_2 < \cdots < i_{\lambda_1}$, $j_1 < j_2 < \cdots < j_{\lambda_2}$, $\ldots$, $t_1 < t_2 < \cdots < t_{\lambda_k}$. We can fill the boxes of the diagram of shape λ with the variables in the left-hand side of 4.11:

$$\begin{array}{l} x_{i_1}\ x_{i_2} \cdots\cdots\cdots x_{i_{\lambda_1}} \\ x_{j_1}\ x_{j_2} \cdots\cdots x_{j_{\lambda_2}} \\ \cdots\cdots\cdots \\ x_{t_1}\ x_{t_2} \cdots x_{t_{\lambda_k}} \end{array} \qquad (4.12)$$

Let $\lambda' = (\lambda'_1, \lambda'_2, \ldots, \lambda'_{\lambda_1})$ be the conjugate of λ. A look at (4.12) ensures that in (4.11) we have $a_1 \leq \lambda'_1$, $a_1 + a_2 \leq \lambda'_1 + \lambda'_2, \ldots$ In other words, if $\mu = (\mu_1, \mu_2, \ldots, \mu_t) \vdash r$, with $t \leq n$, and $x_1^{\mu_1} x_2^{\mu_2} \cdots x_t^{\mu_t}$ appears in e_λ, then necessarily $\mu \trianglelefteq \lambda'$; in particular, $x_1^{\lambda'_1} x_2^{\lambda'_2} \cdots x_{\lambda_1}^{\lambda'_{\lambda_1}}$ appears with coefficient equal to 1. Thus, we have

$$e_\lambda = m_{\lambda'} + \sum_{\substack{\mu \vdash r: \\ \mu \lhd \lambda' \\ \ell(\mu) \leq n}} a_{\lambda', \mu} m_\mu, \qquad (4.13)$$

where $a_{\lambda',\mu}$ are nonnegative coefficients. Then we can use the lexigographic order on the partitions and Proposition 3.6.5 to conclude that in (4.13) the matrix that expresses the e_λ's in terms of the m_λ's is triangular with 1's down the diagonal. By Corollary 4.1.3, this ends the proof.

(ii) We can write (i) in Lemma 4.1.9 in the following way:

$$e_k = \sum_{j=0}^{k-1} (-1)^{k-j-1} e_j h_{k-j}, \qquad k = 1, 2, \ldots, n. \qquad (4.14)$$

Using (4.14) and induction on k, one can easily prove that each e_k is a linear combination of polynomials h_μ, with $\mu \vdash k$ (and therefore $\ell(\mu') \leq k$). For instance, $e_1 = h_1$, $e_2 = -h_2 + h_1^2 = -h_2 + h_{1,1}$ and $e_3 = h_3 - 2h_2 h_1 + h_1^3 = h_3 - 2h_{2,1} + h_{1,1,1}$. Therefore, for each $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_t) \vdash r$, with $\ell(\lambda') \equiv \lambda_1 \leq n$, the elementary symmetric polynomial $e_\lambda = e_{\lambda_1} e_{\lambda_2} \cdots e_{\lambda_t}$ may be written as a linear combination of polynomials h_μ, with $\mu \vdash r$ and $\ell(\mu') \leq \ell(\lambda') \leq n$. Then (ii) follows from (i) and an obvious dimension argument.

(iii) This will be obtained as a consequence of Lemma 4.3.4 (Corollary 4.3.6) and, alternatively, of the Jacobi–Trudi identity (Corollary 4.3.19).

(iv) Suppose that $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ and $\mu = (\mu_1, \mu_2, \dots, \mu_t)$ are partitions of r, with $t, k \leq n$. If the monomial $x_1^{\mu_1} x_2^{\mu_2} \cdots x_t^{\mu_t}$ appears in

$$p_\lambda(x) = (x_1^{\lambda_1} + \cdots + x_n^{\lambda_1})(x_1^{\lambda_2} + \cdots + x_n^{\lambda_2}) \cdots (x_1^{\lambda_k} + \cdots + x_n^{\lambda_k})$$

then each μ_i is a sum of λ_j's, and therefore $\mu \trianglerighteq \lambda$. Then one can argue as in the proof of i.

(v) This may be proved arguing as in the proof of (ii), starting from (ii) in Lemma 4.1.9 and using (ii) in the present theorem (equivalently, starting from (iii) in Lemma 4.1.9 and using (i)). □

4.1.5 An involutive map

Consider the bases (i) and (ii) in Theorem 4.1.12. We can define a linear bijection $\omega \equiv \omega_n^r : \Lambda_n^r \to \Lambda_n^r$ by setting: $\omega(e_\lambda) = h_\lambda$, for each $\lambda \vdash r$ with $\ell(\lambda') \leq n$.

Proposition 4.1.13

(i) *If $f_1 \in \Lambda_n^r$ and $f_2 \in \Lambda_n^s$, then $\omega_n^{r+s}(f_1 f_2) = \omega_n^r(f_1)\omega_n^s(f_2)$.*

(ii) *The linear map ω is involutive: $\omega^2 = 1$.*

(iii) *For each $\lambda \vdash r$, we have: $\omega(p_\lambda) = (-1)^{r-\ell(\lambda)} p_\lambda$, that is, each p_λ is an eigenvector of ω and the corresponding eigenvalue is $(-1)^{r-\ell(\lambda)}$.*

Proof (i) It suffices to note that if $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ then

$$\omega(e_\lambda) = \omega(e_{\lambda_1})\omega(e_{\lambda_2}) \cdots \omega(e_{\lambda_k}).$$

(ii) Applying ω to Lemma 4.1.9.(i), we get $\sum_{j=0}^{k} (-1)^{k-j} h_j \omega(h_{k-j}) = 0$, for $k = 1, 2, \dots, n$, and therefore necessarily we have $\omega(h_{k-j}) = e_{k-j}$, because the conditions (i) in Lemma 4.1.9 determine the polynomials e_j (and certainly $\omega(h_0) = e_0$). We end by applying (i) in the present Lemma.

(iii) Applying ω to Lemma 4.1.9.(ii), and taking into account (ii) in the present lemma, we get: $ke_k = \sum_{j=1}^{k} \omega(p_j) e_{k-j}$, for $k = 1, 2, \dots, n$. Since the conditions (iii) in Lemma 4.1.9 determine the polynomials p_j, we can conclude that: $\omega(p_j) = (-1)^{j-1} p_j$. Again we can end by an application of (i) in the present lemma. Clearly, the roles of (ii) and (iii) in Lemma 4.1.9 may be interchanged. □

Remark 4.1.14 Set, as in [113], $\epsilon_\lambda = (-1)^{r-\ell(\lambda)}$. Then ϵ_λ coincides with the sign $\epsilon(\pi)$ of a permutation belonging to the conjugacy class $\mathcal{C}_\lambda$: indeed, $r - \ell(\lambda) = \lambda_1 - 1 + \lambda_2 - 1 + \cdots + \lambda_k - 1 \equiv \epsilon(\pi)$, if $k = \ell(\lambda)$ and $\pi \in \mathcal{C}_\lambda$. Applying the involution ω in Proposition 4.1.13 to the identity in Lemma

4.1.10, we deduce the following expression of the elementary function e_r in terms of the power sums symmetric polynomials:

$$e_r(x) = \sum_{\lambda \vdash r} \frac{\epsilon_\lambda}{z_\lambda} p_\lambda(x). \tag{4.15}$$

4.1.6 Antisymmetric polynomials

A polynomial $p(x_1, x_2 \dots, x_n)$ is *antisymmetric* if

$$p(x_{\pi(1)}, x_{\pi(2)}, \dots, x_{\pi(n)}) = \varepsilon(\pi) p(x_1, x_2, \dots, x_n),$$

for all $\pi \in \mathfrak{S}_n$. Suppose that $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ is a partition of r, with $k \leq n$. If the monomial $a x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_k^{\lambda_k}$, with $a \in \mathbb{C}$, appears in a homogeneous antisymmetric polynomial p of degree r, then also the following expression

$$a \sum_{\pi \in \mathfrak{S}_n} \varepsilon(\pi) x_{\pi(1)}^{\lambda_1} x_{\pi(2)}^{\lambda_2} \cdots x_{\pi(k)}^{\lambda_k} \tag{4.16}$$

appears in p. But if the parts of λ are not distinct (that is $\lambda_i = \lambda_{i+1}$ for some i), or if $k \leq n-2$, then the expression (4.16) is equal to zero. Therefore p must be the sum of polynomials of the form (4.16), with $k = n$ and $\lambda_1 > \lambda_2 > \cdots > \lambda_{n-1} > \lambda_n \geq 0$. We introduce the following notation: once n is fixed, δ denotes the following partition: $\delta = (n-1, n-2, \dots, 1, 0)$. Then the partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ satisfies the condition $\lambda_1 > \lambda_2 > \cdots > \lambda_{n-1} > \lambda_n \geq 0$ if and and only if it is of the form

$$\lambda = \mu + \delta := (\mu_1 + n - 1, \mu_2 + n - 2, \dots, \mu_{n-1} + 1, \mu_n),$$

where $\mu = (\mu_1, \mu_2, \dots, \mu_{n-1}, \mu_n)$ is an arbitrary n-parts partition, with possibly $\mu_i = \mu_{i+1} = \cdots = \mu_n = 0$ for some $1 \leq i \leq n$. The following proposition is an obvious consequence of the discussion above.

Proposition 4.1.15 *For each $\mu \vdash r$, with $\ell(\mu) \leq n$, set*

$$\begin{aligned} a_{\delta+\mu}(x) &= \sum_{\pi \in \mathfrak{S}_n} \varepsilon(\pi) x_{\pi(1)}^{\mu_1+n-1} x_{\pi(2)}^{\mu_2+n-2} \cdots x_{\pi(n)}^{\mu_n} \\ &\equiv \begin{vmatrix} x_1^{\mu_1+n-1} & x_2^{\mu_1+n-1} & \cdots & x_n^{\mu_1+n-1} \\ x_1^{\mu_2+n-2} & x_2^{\mu_2+n-2} & \cdots & x_n^{\mu_2+n-2} \\ \vdots & \vdots & & \vdots \\ x_1^{\mu_n} & x_2^{\mu_n} & \cdots & x_n^{\mu_n} \end{vmatrix} \end{aligned}$$

(where $\mu_j = 0$ *for* $j = \ell(\mu)+1, \ldots, n$*). Then* $\{a_{\delta+\mu} : \mu \vdash r, \ell(\mu) \le n\}$ *is a basis for the space of all homogeneous antisymmetric polynomials of degree* $\frac{n(n-1)}{2} + r$*.*

For $\mu = (0)$, a_δ is a well-known determinant, called the *Vandermonde determinant*, or the *discriminant*. To make induction on n, it is useful to introduce the following notation: $\overline{\delta} = (n-2, n-3, \ldots, 1, 0)$, that will be used also in the sequel of this section.

Lemma 4.1.16 (Vandermonde's determinant) *We have:*

$$a_\delta(x) \equiv \begin{vmatrix} x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1} \\ x_1^{n-2} & x_2^{n-2} & \cdots & x_n^{n-2} \\ \vdots & \vdots & & \vdots \\ x_1 & x_2 & \cdots & x_n \\ 1 & 1 & \cdots & 1 \end{vmatrix} = \prod_{1 \le i < j \le n} (x_i - x_j).$$

Proof We make induction on n. In the determinant a_δ, for $i = 1, 2, \ldots, n-1$, we can subtract from the ith row the $(i+1)$th row multiplied by x_n. In such a way, we get:

$$a_\delta(x_1, x_2, \ldots, x_n) = \begin{vmatrix} x_1^{n-2}(x_1 - x_n) & x_2^{n-2}(x_2 - x_n) & \cdots & x_{n-1}^{n-2}(x_{n-1} - x_n) & 0 \\ x_1^{n-3}(x_1 - x_n) & x_2^{n-3}(x_2 - x_n) & \cdots & x_{n-1}^{n-3}(x_{n-1} - x_n) & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ x_1 - x_n & x_2 - x_n & \cdots & x_{n-1} - x_n & 0 \\ 1 & 1 & \cdots & 1 & 1 \end{vmatrix}$$

$$= (x_1 - x_n)(x_2 - x_n) \cdots (x_{n-1} - x_n) a_{\overline{\delta}}(x_1, x_2, \ldots, x_{n-1})$$
$$= \prod_{1 \le i < j \le n} (x_i - x_j),$$

where the second equality is obtained by the Laplace expansion along the last column (and factoring out the polynomials $(x_i - x_n)$, $i = 1, 2, \ldots, n-1$), and the third equality by the inductive hypothesis. □

We end with a result that will be used in Section 4.2. The symbol $\mu \to \lambda$ has the meaning defined in Section 3.1.

Lemma 4.1.17 *For every* $\lambda \vdash n-1$, *we have:*

$$x_1x_2\cdots x_{n-1}(x_1+x_2+\cdots x_{n-1})a_{\overline{\delta}+\lambda}(x_1,x_2,\ldots,x_{n-1}) \\ = \sum_{\mu\vdash n:\mu\to\lambda} a_{\mu+\delta}(x_1,x_2,\ldots,x_n)|_{x_n=0}. \quad (4.17)$$

Proof Suppose that $\lambda=(\lambda_1,\lambda_2,\ldots,\lambda_k)$ and set $\lambda_{k+1}=\cdots=\lambda_n=0$. Then every $\mu\vdash n$ with $\mu\to\lambda$ is obtained by adding 1 to one λ_j with $\lambda_{j-1}>\lambda_j$. Therefore the right-hand side of (4.17) may be written in the form

$$\sum_{j:\lambda_{j-1}>\lambda_j}\sum_{\pi\in\mathfrak{S}_n}\varepsilon(\pi)x_{\pi(1)}^{\lambda_1+n-1}x_{\pi(2)}^{\lambda_2+n-2}\cdots x_{\pi(j)}^{\lambda_j+1+n-j}\cdots x_{\pi(n)}^{\lambda_n}|_{x_n=0};$$

since certainly $\lambda_n=0$, this expression is equal to

$$\sum_{j:\lambda_{j-1}>\lambda_j}\sum_{\pi\in\mathfrak{S}_{n-1}}\varepsilon(\pi)x_{\pi(1)}^{\lambda_1+n-1}x_{\pi(2)}^{\lambda_2+n-2}\cdots x_{\pi(j)}^{\lambda_j+1+n-j}\cdots x_{\pi(n-1)}^{\lambda_{n-1}+1}. \quad (4.18)$$

On the other hand, the left-hand side of (4.17) may be written in the form $\prod_{j=1}^{n-1}x_1x_2\cdots x_{n-1}\cdot x_j\cdot a_{\lambda+\overline{\delta}}(x_1,\ldots,x_{n-1})$, and this is equal to (4.18) too, because if $\lambda_{j-1}=\lambda_j$ then $x_1x_2\cdots x_{n-1}\cdot x_j\cdot a_{\lambda+\overline{\delta}}(x_1,\ldots,x_{n-1})=0$. □

4.1.7 The algebra of symmetric functions

In this section, we introduce the algebra of symmetric functions. Actually, these are *not* ordinary functions, but polynomials in an infinite number of variables $x_1,x_2,\ldots,x_n,\ldots$. A monomial in those variables is just a product of a *finite* number of variables: $x_{i_1}^{s_1}x_{i_2}^{s_2}\cdots x_{i_k}^{s_k}$; its degree is $s_1+s_2+\cdots+s_k$. A polynomial in the variables $x_1,x_2,\ldots,x_n,\ldots$ is a formal sum expression of the form

$$p(x)=\sum a_{i_1,s_1,i_2,s_2,\ldots,i_k,s_k}x_{i_1}^{s_1}x_{i_2}^{s_2}\cdots x_{i_k}^{s_k}$$

where the sum is over all monomials and the coefficients $a_{i_1,s_1,i_2,s_2,\ldots,i_k,s_k}$ are complex numbers. We say that p is homogeneous of degree r if only monomials of degree r appear in the sum. If $\lambda=(\lambda_1,\lambda_2,\ldots,\lambda_k)$ is a partition of r, the *monomial symmetric function* associated with λ is

$$m_\lambda=\sum x_{i_1}^{\lambda_1}x_{i_2}^{\lambda_2}\cdots x_{i_k}^{\lambda_k},$$

where the sum is over all k-sets $\{i_1,i_2,\ldots,i_k\}$ of natural numbers. The *algebra of symmetric functions* in the indeterminates $x_1,x_2,\ldots$, denoted by Λ, is the vector space whose basis is the set $\{m_\lambda:\lambda\vdash r, r=0,1,2,\ldots\}$ of all monomial symmetric functions. It is an algebra under the formal product. It is also a *graded algebra*:

$$\Lambda = \bigoplus_{r=0}^{\infty} \Lambda^r,$$

where Λ^r is the space of all symmetric functions of degree r (whose basis is $\{m_\lambda : \lambda \vdash r\}$). The term "graded algebra" simply means that if $f \in \Lambda^r$ and $g \in \Lambda^s$ then $f \cdot g \in \Lambda^{r+s}$. If $k \geq 0$, the *complete symmetric function* h_k is the sum of all monomials of degree k, the *elementary symmetric function* e_k is $e_k = \sum_{j_1 < j_2 < \cdots < j_k} x_{j_1} \cdots x_{j_k}$ and the *power sum symmetric function* p_k is $p_k = \sum_{j=1}^{\infty} x_j^k$. If $\lambda = (\lambda_1, \ldots, \lambda_k)$ is a partition, we set, as usual,

$$h_\lambda = h_{\lambda_1} h_{\lambda_2} \cdots h_{\lambda_k}, \quad e_\lambda = e_{\lambda_1} e_{\lambda_2} \cdots e_{\lambda_k}, \quad \text{and} \quad p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots p_{\lambda_k}.$$

Arguing as in the proofs of (i), (ii) and (iv) in Theorem 4.1.12, one can easily check that $\{h_\lambda : \lambda \vdash r\}$, $\{e_\lambda : \lambda \vdash r\}$ and $\{p_\lambda : \lambda \vdash r\}$ are all bases of Λ^r. Therefore $\{h_\lambda : \lambda \vdash r, r = 0, 1, 2, \ldots\}$, $\{e_\lambda : \lambda \vdash r, r = 0, 1, 2, \ldots\}$ and $\{p_\lambda : \lambda \vdash r, r = 0, 1, 2, \ldots\}$ are bases of Λ. We can also define a map $\omega : \Lambda \to \Lambda$ by requiring that $\omega(e_\lambda) = \omega(h_\lambda)$, for each partition λ. This is an *involutive automorphism* of Λ (see (i) in Proposition 4.1.13).

Exercise 4.1.18 Verify all the statements in this section.

For more details, we refer to the monographs by Macdonald [83], Sagan [108] and Stanley [113].

4.2 The Frobenius character formula

This section is devoted to the first main result of this chapter, the Frobenius character formula. It expresses the characters of the symmetric group as coefficients in the expansion of an antisymmetric polynomial. Our treatment is based on that of Simon [111], where the main ingredient is the branching rule for the characters of $\mathfrak{S}_n$. We also take into account the exposition in the monographs by Macdonald [83] and Fulton and Harris [43].

4.2.1 On the characters of the Young modules

Let λ be a partition of n. We denote by ψ^λ the character of the Young module M^λ. If μ is another partition of n, we denote by ψ^λ_μ the value of ψ^λ on the elements in the conjugacy class associated with μ.

Lemma 4.2.1 *Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ and $\mu = (1^{r_1}, 2^{r_2}, \dots, n^{r_n})$ be partitions of n. Then*

$$\psi_\mu^\lambda = \sum_{\{r_{i,j}\}} \prod_{i=1}^n \frac{r_i!}{r_{i,1}! r_{i,2}! \cdots r_{i,k}!}$$

where the sum runs over all collections $\{r_{i,j} : 1 \le j \le k,\ 1 \le i \le \lambda_j\}$ of nonnegative integers satisfying the conditions

$$\begin{cases} \sum_{i=1}^{\lambda_j} i r_{i,j} = \lambda_j & j = 1, 2, \dots, k \\ \sum_{j=1}^{k} r_{i,j} = r_i & i = 1, 2, \dots, n \end{cases} \tag{4.19}$$

(and we set $r_{i,j} = 0$ for $1 \le j \le k$ and $\lambda_j < i \le n$).

Proof Denote by $\mathcal{C}_\mu$ the conjugacy class associated with μ. Suppose that $\pi \in \mathcal{C}_\mu \cap \mathfrak{S}_\lambda$, where $\mathfrak{S}_\lambda$ is the Young subgroup associated with λ. Then $\pi = \pi_1 \pi_2 \cdots \pi_k$ with $\pi_j \in \mathfrak{S}_{\lambda_j}$ and if π_j has cycle type $(1^{r_{1,j}}, 2^{r_{2,j}}, \dots, \lambda_j^{r_{\lambda_j,j}})$, then the integers $r_{i,j}$ satisfy equations (4.19). Since $\mathcal{C}_\mu \cap \mathfrak{S}_\lambda$ is a union of conjugacy classes of $\mathfrak{S}_\lambda$, from Proposition 4.1.1 (applied to each such conjugacy class) it follows that

$$|\mathcal{C}_\mu \cap \mathfrak{S}_\lambda| = \sum_{\{r_{i,j}\}} \prod_{j=1}^k \frac{\lambda_j!}{\prod_{i=1}^{\lambda_j} r_{i,j}! i^{r_{i,j}}}.$$

Then the statement is an immediate consequence of the formula $\psi_\mu^\lambda = \frac{|\mathfrak{S}_n/\mathfrak{S}_\lambda|}{|\mathcal{C}_\mu|} |\mathcal{C}_\mu \cap \mathfrak{S}_\lambda|$ (see Corollary 1.3.11). □

Lemma 4.2.2 *For any $\mu \vdash n$ we have*

$$p_\mu = \sum_{\lambda \vdash n} \psi_\mu^\lambda m_\lambda.$$

Proof First note that the (multinomial) coefficient $\frac{r_i!}{r_{i,1}! r_{i,2}! \cdots r_{i,n}!}$, where $r_{i,1} + r_{i,2} + \cdots + r_{i,n} = r_i$, is equal to the coefficient of $y_1^{r_{i,1}} y_2^{r_{i,2}} \cdots y_n^{r_{i,n}}$ in $(y_1 + y_2 + \cdots + y_n)^{r_i}$. It follows that

$$\begin{aligned} p_\mu(x_1, x_2, \dots, x_n) &\equiv \prod_{i=1}^n (x_1^i + x_2^i + \cdots + x_n^i)^{r_i} \\ &= \sum_{\{r_{i,j}\}} \left[\prod_{j=1}^n x_j^{\sum_{i=1}^n i r_{i,j}} \right] \prod_{i=1}^n \frac{r_i!}{r_{i,1}! r_{i,2}! \cdots r_{i,n}!} \end{aligned}$$

where the sum runs over all collections $\{r_{i,j}\}$ of nonnegative integers satisfying $\sum_{j=1}^n r_{i,j} = r_i$ for all $i = 1, 2, \dots, n$.

Therefore, the coefficient of $x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_k^{\lambda_k}$ in p_μ is equal to the expression for ψ_μ^λ in Lemma 4.2.1. Since p_μ is a homogeneous polynomial of degree n, the statement follows. □

4.2.2 Cauchy's formula

Proposition 4.2.3 (Cauchy's formula) *Suppose that $y_1, y_2, \dots, y_n$ is another set of variables. Then the determinant of the matrix $\left(\frac{1}{1-x_i y_j}\right)_{i,j=1,2,\dots,n}$ is given by*

$$\det\left(\frac{1}{1-x_i y_j}\right) = \frac{a_\delta(x) a_\delta(y)}{\prod_{i,j=1}^n (1 - x_i y_j)},$$

where $a_\delta(x) = \prod_{1 \le i < j \le n}(x_i - x_j)$ is the Vandermonde determinant (see Lemma 4.1.16).

Proof The proof is by induction on n. For $n = 2$ it is an easy computation:

$$\begin{aligned}
\det\left(\frac{1}{1-x_i y_j}\right) &= \begin{vmatrix} \frac{1}{1-x_1 y_1} & \frac{1}{1-x_1 y_2} \\ \frac{1}{1-x_2 y_1} & \frac{1}{1-x_2 y_2} \end{vmatrix} \\
&= \frac{1}{1 - x_1 y_1} \frac{1}{1 - x_2 y_2} - \frac{1}{1 - x_1 y_2} \frac{1}{1 - x_2 y_1} \\
&= \frac{(1 - x_2 y_1)(1 - x_1 y_2) - (1 - x_1 y_1)(1 - x_2 y_2)}{(1 - x_1 y_1)(1 - x_1 y_2)(1 - x_2 y_1)(1 - x_2 y_2)} \\
&= \frac{1 - x_2 y_1 - x_1 y_2 + x_1 x_2 y_1 y_2 - (1 - x_1 y_1 - x_2 y_2 + x_1 x_2 y_1 y_2)}{(1 - x_1 y_1)(1 - x_1 y_2)(1 - x_2 y_1)(1 - x_2 y_2)} \\
&= \frac{-x_2 y_1 - x_1 y_2 + x_1 y_1 - x_2 y_2}{(1 - x_1 y_1)(1 - x_1 y_2)(1 - x_2 y_1)(1 - x_2 y_2)} \\
&= \frac{(x_1 - x_2)(y_1 - y_2)}{(1 - x_1 y_1)(1 - x_1 y_2)(1 - x_2 y_1)(1 - x_2 y_2)} \\
&= \frac{a_\delta(x) a_\delta(y)}{\prod_{i,j=1}^2 (1 - x_i y_j)}.
\end{aligned}$$

Note that

$$\frac{1}{1 - x_i y_j} - \frac{1}{1 - x_1 y_j} = \frac{x_i - x_1}{1 - x_1 y_j} \cdot \frac{y_j}{1 - x_i y_j} \tag{4.20}$$

and

$$\frac{y_j}{1 - x_i y_j} - \frac{y_1}{1 - x_i y_1} = \frac{y_j - y_1}{1 - x_i y_1} \cdot \frac{1}{1 - x_i y_j}. \tag{4.21}$$

Then we have:

$$\det\left(\frac{1}{1-x_iy_j}\right) =_{(*)} \prod_{i=2}^{n}(x_i-x_1)\prod_{j=1}^{n}\frac{1}{1-x_1y_j}\cdot \\ \cdot\begin{vmatrix} 1 & 1 & \cdots & 1 \\ \frac{y_1}{1-x_2y_1} & \frac{y_2}{1-x_2y_2} & \cdots & \frac{y_n}{1-x_2y_n} \\ \vdots & \vdots & & \vdots \\ \frac{y_1}{1-x_ny_1} & \frac{y_2}{1-x_ny_2} & \cdots & \frac{y_n}{1-x_ny_n} \end{vmatrix}$$

$$=_{(**)} \prod_{i=2}^{n}(x_i-x_1)\prod_{j=2}^{n}(y_j-y_1)\prod_{i=2}^{n}\frac{1}{1-x_iy_1}\cdot \\ \cdot\prod_{j=1}^{n}\frac{1}{1-x_1y_j}\cdot\begin{vmatrix} 1 & 0 & \cdots & 0 \\ y_1 & \frac{1}{1-x_2y_2} & \cdots & \frac{1}{1-x_2y_n} \\ \vdots & \vdots & & \vdots \\ y_1 & \frac{1}{1-x_ny_2} & \cdots & \frac{1}{1-x_ny_n} \end{vmatrix}$$

$$=_{(***)} \frac{a_\delta(x)a_\delta(y)}{\prod_{i,j=1}^{n}(1-x_iy_j)},$$

where: in $(*)$ we have subtracted the first row from each other row, we have used (4.20) and factored out the terms $\frac{1}{1-x_1y_j}$ (from each column) and $x_i - x_1$ (from rows $2, 3, \ldots, n$); in $(**)$ we have subtracted the first column from each other column, we have used (4.21) and factored out the terms $y_j - y_1$ and $\frac{1}{1-x_iy_1}$; finally, in $(***)$ using the Laplace expansion with respect to the first row, we have applied the inductive hypothesis and Lemma 4.1.16. □

4.2.3 Frobenius character formula

Let μ be a partition of n and consider the function $p_\mu(x)a_\delta(x)$, where p_μ is the power sum symmetric polynomial associated with μ and a_δ is the Vandermonde determinant. Since $p_\mu(x)a_\delta(x)$ is homogeneous antisymmetric of degree $n + \frac{n(n-1)}{2} = \frac{n(n+1)}{2}$, from Proposition 4.1.15 we deduce the existence of coefficients $\widetilde{\chi}^\lambda_\mu$ such that

$$p_\mu(x)a_\delta(x) = \sum_{\lambda\vdash n}\widetilde{\chi}^\lambda_\mu a_{\lambda+\delta}(x). \tag{4.22}$$

We want to show that the number $\widetilde{\chi}^\lambda_\mu$ is precisely the value of the character χ^λ at the conjugacy class of $\mathfrak{S}_n$ corresponding to the partition μ. For this, we need some preliminary results.

Proposition 4.2.4 *Let* $\lambda \vdash n$. *Let* ψ^ν *be the character of the Young module* M^λ. *Then there exist* integer *coefficients* $a_{\nu,\lambda}$, $\nu \vdash n$, *such that*

$$\widetilde{\chi}^\lambda_\mu = \sum_{\nu \vdash n} a_{\nu,\lambda} \psi^\nu_\mu$$

for all $\mu \vdash n$.

Proof Let m_ν be the monomial symmetric polynomial associated with ν. Since $m_\nu(x)a_\delta(x)$ is antisymmetric, there exist coefficients $a_{\nu,\lambda}$ such that

$$m_\nu(x)a_\delta(x) = \sum_{\mu \vdash n} a_{\nu,\lambda} a_{\lambda+\delta}(x) \tag{4.23}$$

(see Proposition 4.1.15). Moreover, $a_{\nu,\lambda}$ is equal to the coefficient of $x_1^{\lambda_1+n-1} \cdot x_2^{\lambda_2+n-2} \cdots x_n^{\lambda_n}$ in the polynomial $m_\nu(x)a_\delta(x)$, and therefore it is an integer.

By Lemma 4.2.2 and (4.23) we have

$$p_\mu(x)a_\delta(x) = \sum_{\nu \vdash n} \psi^\nu_\mu m_\nu(x)a_\delta(x) = \sum_{\lambda \vdash n} \left[\sum_{\nu \vdash n} \psi^\nu_\mu a_{\nu,\lambda} \right] a_{\lambda+\delta}(x).$$

Comparing the last equality with (4.22), the statement follows. □

Lemma 4.2.5 *The functions* $\widetilde{\chi}^\lambda_\mu$ *satisfy the orthogonal relations of the characters of irreducible representations of the symmetric groups (see* (4.3)*):*

$$\sum_{\mu \vdash n} \frac{1}{z_\mu} \widetilde{\chi}^\lambda_\mu \widetilde{\chi}^\nu_\mu = \delta_{\lambda,\nu}.$$

Proof From Cauchy's formula (Proposition 4.2.3) and Lemma 4.1.11 we get (recall that ε is the alternating representation of $\mathfrak{S}_n$):

$$\begin{aligned} \sum_{r=0}^{\infty} \sum_{\mu \vdash r} \frac{1}{z_\mu} p_\mu(x)p_\mu(y)a_\delta(x)a_\delta(y) &= \det\left(\frac{1}{1-x_i y_j}\right) \\ &= \sum_{\pi \in \mathfrak{S}_n} \varepsilon(\pi) \frac{1}{1-x_1 y_{\pi(1)}} \cdots \frac{1}{1-x_n y_{\pi(n)}} \\ &= \sum_{\pi \in \mathfrak{S}_n} \sum_{\alpha_1,\ldots,\alpha_n=0}^{\infty} \varepsilon(\pi) x_1^{\alpha_1} y_{\pi(1)}^{\alpha_1} \cdots x_n^{\alpha_n} y_{\pi(n)}^{\alpha_n}, \end{aligned} \tag{4.24}$$

where we have used the geometric series expansion. Note also that the exponents $\alpha_1, \ldots, \alpha_n$ can be taken to be all distinct, otherwise the corresponding monomial is in the kernel of $\sum_{\pi \in \mathfrak{S}_n} \varepsilon(\pi)\pi$.

We now consider the homogeneous component of degree $n(n+1)/2$ (both in the variables x and y) in the first and last term of (4.24). We get the equality

$$\sum_{\mu\vdash n}\frac{1}{z_\mu}p_\mu(x)a_\delta(x)p_\mu(y)a_\delta(y)$$
$$=\sum_{\substack{\alpha_1>\cdots>\alpha_n\\ \sum_{i=1}^n\alpha_i=n(n+1)/2}}\sum_{\pi,\sigma\in\mathfrak{S}_n}\varepsilon(\pi)x^{\alpha_1}_{\sigma(1)}y^{\alpha_1}_{\pi(\sigma(1))}\cdots x^{\alpha_n}_{\sigma(n)}y^{\alpha_n}_{\pi(\sigma(n))}.$$

Keeping into account (4.22) and the definition of $a_{\mu+\delta}$, this yields

$$\sum_{\lambda,\nu\vdash n}\left(\sum_{\mu\vdash n}\frac{1}{z_\mu}\widetilde{\chi}^\lambda_\mu\widetilde{\chi}^\nu_\mu\right)a_{\lambda+\delta}(x)a_{\nu+\delta}(y)=\sum_{\lambda\vdash n}a_{\lambda+\delta}(x)a_{\lambda+\delta}(y)$$

and the orthogonality relations follow. □

Lemma 4.2.6 *Let $\lambda,\mu\vdash n$. For $\lambda\neq(1^n)$ the coefficient $\widetilde{\chi}^\lambda_\mu$ satisfies the same branching rule of the character χ^λ of S^λ: if $\mu=(\mu_1,\mu_2,\dots,\mu_{t-1},1)$ and $\nu=(\mu_1,\mu_2,\dots,\mu_{t-1})$ then*

$$\widetilde{\chi}^\lambda_\mu=\sum_{\substack{\theta\vdash n-1:\\ \lambda\to\theta}}\widetilde{\chi}^\theta_\nu.$$

Proof First of all, note that if $\lambda\neq(1^n)$ then $\lambda_n=0$ and therefore (setting $\overline{\delta}=(n-2,n-3,\dots,1,0)$)

$$\begin{aligned}a_{\lambda+\delta}(x)|_{x_n=0}&=\sum_{\pi\in\mathfrak{S}_n}\varepsilon(\pi)x^{\lambda_1+n-1}_{\pi(1)}x^{\lambda_2+n-2}_{\pi(2)}\cdots x^{\lambda_{n-1}+1}_{\pi(n-1)}|_{x_n=0}\\&=\sum_{\pi\in\mathfrak{S}_{n-1}}\varepsilon(\pi)x^{\lambda_1+n-2}_{\pi(1)}x^{\lambda_2+n-3}_{\pi(2)}\cdots x^{\lambda_{n-1}}_{\pi(n-1)}x_1x_2\cdots x_{n-1}\\&=x_1x_2\cdots x_{n-1}a_{\lambda+\overline{\delta}}(x_1,x_2,\dots,x_{n-1}).\end{aligned}$$

This implies that the polynomials $\{a_{\lambda+\delta}(x)|_{x_n=0}:\lambda\vdash n,\lambda\neq(1^n)\}$ are linearly independent. Note also that for $\lambda=(1^n)$ one clearly has $a_{\lambda+\delta}(x)|_{x_n=0}=0$.

Let μ and ν as in the statement. On the one hand, we have, by (4.22),

$$p_\mu(x)a_\delta(x)|_{x_n=0}=\sum_{\lambda\vdash n}\widetilde{\chi}^\lambda_\mu a_{\lambda+\delta}(x)|_{x_n=0}. \tag{4.25}$$

On the other hand, computing $p_\mu(x)|_{x_n=0}$ and $a_\delta(x)|_{x_n=0}$, and then applying (4.22) again, we get

$$\begin{aligned} p_\mu(x)a_\delta(x)|_{x_n=0} &= x_1x_2\cdots x_{n-1}a_{\overline{\delta}}(x_1,x_2,\ldots,x_{n-1})\cdot \\ &\quad\cdot(x_1+x_2+\cdots+x_{n-1})p_\nu(x_1,x_2,\ldots,x_{n-1}) \\ &= \sum_{\theta\vdash n-1} x_1x_2\cdots x_{n-1}(x_1+x_2+\cdots+x_{n-1})\cdot \\ &\quad\cdot\widetilde{\chi}^\theta_\nu a_{\theta+\overline{\delta}}(x_1,x_2,\ldots,x_{n-1}) \\ &=_{(*)} \sum_{\theta\vdash n-1}\widetilde{\chi}^\theta_\nu \sum_{\substack{\lambda\vdash n:\\ \lambda\to\theta}} a_{\lambda+\delta}(x)|_{x_n=0} \\ &= \sum_{\substack{\lambda\vdash n:\\ \lambda\neq(1^n)}}\left(\sum_{\substack{\theta\vdash n-1:\\ \lambda\to\theta}}\widetilde{\chi}^\theta_\nu\right)a_{\lambda+\delta}(x)|_{x_n=0}. \end{aligned} \tag{4.26}$$

where $(*)$ follows from Lemma 4.1.17. Equating (4.25) and (4.26), and keeping in mind the fact that $\{a_{\lambda+\delta}(x)|_{x_n=0} : \lambda\vdash n, \lambda\neq(1^n)\}$ is a linearly independent set, the branching rule follows. □

Lemma 4.2.7 *For each $\lambda\vdash n$, we have $\widetilde{\chi}^\lambda_{(1^n)}>0$.*

Proof We first show that $\widetilde{\chi}^{(1^n)}_{(1^n)}=1$. Indeed, we have

$$(x_1+x_2+\cdots+x_n)a_\delta(x)=\sum_{\pi\in\mathfrak{S}_n}\varepsilon(\pi)x^n_{\pi(1)}x^{n-2}_{\pi(2)}x^{n-3}_{\pi(3)}\cdots x_{\pi(n-1)}$$

because no monomials with two x_i's with the same power can (effectively) appear in an antisymmetric polynomial. Similarly,

$$(x_1+x_2+\cdots+x_n)^2a_\delta(x)=\sum_{\pi\in\mathfrak{S}_n}\varepsilon(\pi)x^n_{\pi(1)}x^{n-1}_{\pi(2)}x^{n-3}_{\pi(3)}\cdots x_{\pi(n-1)}+P(x)$$

where $P(x)$ is the sum of all momomials containing some x_i with degree larger than n.

Continuing this way, we get the expression

$$(x_1+x_2+\cdots+x_n)^na_\delta(x)=a_{\delta+(1^n)}(x)+Q(x) \tag{4.27}$$

where, again, $Q(x)$ is the sum of all terms containing some x_i with degree larger than n.

Now, $p_{(1^n)}=(x_1+x_2+\cdots+x_n)^n$ and (4.22) for $\mu=(1^n)$ becomes

$$(x_1+x_2+\cdots+x_n)^na_\delta(x)=\widetilde{\chi}^{(1^n)}_{(1^n)}a_{\delta+(1^n)}(x)+\sum_{\substack{\lambda\vdash n:\\ \lambda\neq(1^n)}}\widetilde{\chi}^\lambda_{(1^n)}a_{\lambda+\delta}(x). \tag{4.28}$$

Recalling that $\{a_{\lambda+\delta}(x) : \lambda \vdash n\}$ is a basis (cf. Proposition 4.1.15), and equating (4.28) with (4.27), it follows that $\widetilde{\chi}^{(1^n)}_{(1^n)} = 1 > 0$.

To show that $\widetilde{\chi}^{\lambda}_{(1^n)} > 0$ for all $\lambda \vdash n$ we argue by induction on n. Suppose that $n = 1$. Then (4.22) reduces to

$$p_1(x_1) \equiv x_1 = \widetilde{\chi}^{(1)}_{(1)} x_1$$

and therefore $\widetilde{\chi}^{(1)}_{(1)} = 1$ (which is indeed $\chi^{(1^n)}_{(1^n)} = 1$ for $n = 1$). Using this as the base of the induction, and keeping into account the particular case $\lambda = (1^n)$ one proves the statement using the branching rule in Lemma 4.2.6 for the inductive step. □

Theorem 4.2.8 (Frobenius character formula) *For all partitions $\lambda, \mu \vdash n$ we have $\widetilde{\chi}^{\lambda}_{\mu} = \chi^{\lambda}_{\mu}$. In other words,*

$$p_\mu(x) a_\delta(x) = \sum_{\lambda \vdash n} \chi^{\lambda}_{\mu} a_{\lambda+\delta}(x)$$

for all $\mu \vdash n$.

Proof Consider the map $\widetilde{\chi}^{\lambda} : \mu \mapsto \widetilde{\chi}^{\lambda}_{\mu}$ as a central function defined on $\mathfrak{S}_n$ (that is, $\widetilde{\chi}^{\lambda}(\pi) = \widetilde{\chi}^{\lambda}_{\mu}$ for all $\pi \in \mathcal{C}_\mu$). From Proposition 4.2.4, we deduce that $\widetilde{\chi}^{\lambda}$ is a linear combination with *integer* coefficients of irreducible characters:

$$\widetilde{\chi}^{\lambda} = \sum_{\rho \vdash n} b_{\lambda,\rho} \chi^{\rho}$$

with $b_{\lambda,\rho} \in \mathbb{Z}$. From the orthogonality relations (in Lemma 4.2.5 for $\widetilde{\chi}^{\lambda}$, and in (4.3) for χ^{ρ}) we get

$$1 = \sum_{\mu \vdash n} \frac{1}{z_\mu} (\widetilde{\chi}^{\lambda}_{\mu})^2 = \sum_{\rho,\theta \vdash n} b_{\lambda,\rho} b_{\lambda,\theta} \sum_{\mu \vdash n} \frac{1}{z_\mu} \chi^{\rho}_{\mu} \chi^{\theta}_{\mu} = \sum_{\rho \vdash n} (b_{\lambda,\rho})^2.$$

As the $b_{\lambda,\rho}$'s are integers, this forces $\widetilde{\chi}^{\lambda} = \pm \chi^{\rho}$ for some ρ. Observing that $\mathcal{C}_{(1^n)} = \{1_{\mathfrak{S}_n}\}$, so that $\chi^{\rho}_{(1^n)} = \dim \rho > 0$, the condition $\widetilde{\chi}^{\lambda}_{(1^n)} > 0$ (see Lemma 4.2.7) gives $\widetilde{\chi}^{\lambda} = \chi^{\rho}$. This shows that every $\widetilde{\chi}^{\lambda}$ is the character of an irreducible representation of $\mathfrak{S}_n$.

We now use induction to show that $\widetilde{\chi}^{\lambda} = \chi^{\lambda}$. First of all, let $n = 2$. Then, formula (4.22) gives, for $\mu = (2)$,

$$(x_1^2 + x_2^2)(x_1 - x_2) = \widetilde{\chi}^{(2)}_{(2)} (x_1^3 - x_2^3) + \widetilde{\chi}^{1,1}_{(2)} (x_1^2 x_2 - x_1 x_2^2),$$

so that $\widetilde{\chi}^{(2)}_{(2)} = 1$ and $\widetilde{\chi}^{1,1}_{(2)} = -1$, and, for $\mu = (1, 1)$,

$$(x_1 + x_2)^2 (x_1 - x_2) = \widetilde{\chi}^{(2)}_{1,1} (x_1^3 - x_2^3) + \widetilde{\chi}^{1,1}_{1,1} (x_1^2 x_2 - x_1 x_2^2)$$

so that $\widetilde{\chi}^{(2)}_{1,1} = \widetilde{\chi}^{1,1}_{1,1} = 1$.

Therefore $\widetilde{\chi}^{(2)} = \chi^{(2)}$ and $\widetilde{\chi}^{1,1} = \chi^{1,1}$ are precisely the characters of $\mathfrak{S}_2$.

Using the branching rule in Lemma 4.2.6 and the fact that, for $n \geq 2$, each partition $\lambda \vdash n$ is uniquely determined by the set $\{\theta \vdash n-1 : \lambda \to \theta\}$, we can apply induction. Finally, the case $\widetilde{\chi}^{(1^n)} = \chi^{(1^n)}$ follows, by exclusion, once we have proved (by induction) that $\widetilde{\chi}^{\lambda} = \chi^{\lambda}$ for all $\lambda \vdash n$, $\lambda \neq (1^n)$. □

We now introduce the following notation.

Given a polynomial $p(x_1, x_2, \ldots, x_n)$ and a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ with $\ell(\lambda) \leq n$, we denote by $[p(x_1, x_2, \ldots, x_n)]_\lambda$ the coefficient of the monomial $x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_k^{\lambda_k}$ in $p(x_1, x_2, \ldots, x_n)$. Clearly, the operator $p \mapsto [p]_\lambda$ is linear: if $\alpha, \beta \in \mathbb{C}$ and p, q are polynomials then $[\alpha p + \beta q]_\lambda = \alpha [p]_\lambda + \beta [q]_\lambda$.

The Frobenius character formula can be given the following alternative formulation.

Proposition 4.2.9 *Let $\lambda, \mu \vdash n$ and set $k = \ell(\lambda)$. Then*

$$\chi_\mu^\lambda = [p_\mu(x_1, x_2, \ldots, x_k) a_{\widetilde{\delta}}]_{\lambda + \widetilde{\delta}}, \tag{4.29}$$

where $\widetilde{\delta} = (k-1, k-2, \ldots, 1, 0)$.

Proof From Theorem 4.2.8 we know that $\chi_\mu^\lambda = [p_\mu(x_1, x_2, \ldots, x_n) a_\delta]_{\lambda+\delta}$, where $\delta = (n-1, n-2, \ldots, 1, 0)$. Setting $x_{k+1} = x_{k+2} = \cdots = x_n = 1$ we get

$$\begin{aligned} a_\delta(x_1, x_2, \ldots, x_k, 1, \ldots, 1) \\ = a_{\widetilde{\delta}}(x_1, x_2, \ldots, x_k)(x_1 x_2 \cdots x_k)^{n-k} + q_1(x_1, x_2, \ldots, x_k) \end{aligned}$$

and

$$p_\mu(x_1, x_2, \ldots, x_k, 1, \ldots, 1) = p_\mu(x_1, x_2, \ldots, x_k) + q_2(x_1, x_2, \ldots, x_k)$$

where q_1 and q_2 are polynomials of lower degree. Since

$$x_1^{\lambda_1+n-1} x_2^{\lambda_2+n-2} \cdots x_k^{\lambda_k+n-k} = x_1^{\lambda_1+k-1} x_2^{\lambda_2+k-2} \cdots x_k^{\lambda_k} (x_1 x_2 \cdots x_k)^{n-k},$$

(4.29) follows immediately. □

4.2.4 Applications of Frobenius character formula

Recall that d_λ denotes the dimension of the irreducible representation S^λ.

Proposition 4.2.10 *Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \vdash n$ and set $\mu = \lambda + \delta$, where $\delta = (k-1, k-2, \ldots, 1, 0)$. Then*

$$d_\lambda = \frac{n!}{\mu_1! \mu_2! \cdots \mu_k!} a_\delta(\mu_1, \mu_2, \ldots, \mu_k). \tag{4.30}$$

Proof First of all, note that

$$\begin{aligned} p_n(x_1, x_2, \ldots, x_k) a_\delta(x_1, x_2, \ldots, x_k) &= (x_1 + x_2 + \cdots + x_k)^n \cdot \\ &\cdot \sum_{\pi \in \mathfrak{S}_k} \varepsilon(\pi) x_k^{\pi(1)-1} x_{k-1}^{\pi(2)-1} \cdots x_1^{\pi(k)-1} \\ &= \sum_{\pi \in \mathfrak{S}_k} \sum_{\substack{a_1, a_2, \ldots, a_k: \\ a_1 + a_2 + \cdots + a_k = n}} \frac{n!}{a_1! a_2! \cdots a_k!} \varepsilon(\pi) \cdot \\ &\cdot x_k^{\pi(1)-1+a_k} x_{k-1}^{\pi(2)-1+a_{k-1}} \cdots x_1^{\pi(k)-1+a_1}. \end{aligned} \tag{4.31}$$

From Proposition 4.2.9, it follows that d_λ is precisely the coefficient of $x_1^{\mu_1} x_2^{\mu_2} \cdots x_k^{\mu_k}$ in $p_n(x_1, x_2, \ldots, x_k) a_\delta(x_1, x_2, \ldots, x_k)$. Therefore, if we consider the terms in (4.31) with $a_j = \mu_j - \pi(k - j + 1) + 1$, for $j = 1, 2, \ldots, k$, we find

$$d_\lambda = \sum \varepsilon(\pi) \frac{n!}{(\mu_1 - \pi(k) + 1)!(\mu_2 - \pi(k-1) + 1)! \cdots (\mu_k - \pi(1) + 1)!}, \tag{4.32}$$

where the sum runs over all $\pi \in \mathfrak{S}_k$ such that $\mu_j - \pi(k - j + 1) + 1 \geq 0$, for $j = 1, 2, \ldots, k$. Since

$$\frac{1}{(\mu_j - \pi(k - j + 1) + 1)!} = \frac{\mu_j(\mu_j - 1) \cdots (\mu_j - \pi(k - j + 1) + 2)}{\mu_j!}$$

(with the right-hand side equal to 0 if $\mu_j - \pi(k - j + 1) + 1 < 0$) we can write (4.32) in the following way:

$$\begin{aligned} d_\lambda &= \frac{n!}{\mu_1! \mu_2! \cdots \mu_k!} \sum \varepsilon(\pi) \prod_{j=1}^{k} \mu_j(\mu_j - 1) \cdots (\mu_j - \pi(k - j + 1) + 2) \\ &= \frac{n!}{\mu_1! \mu_2! \cdots \mu_k!} \begin{vmatrix} 1 & \mu_k & \mu_k(\mu_k - 1) & \cdots \\ 1 & \mu_{k-1} & \mu_{k-1}(\mu_{k-1} - 1) & \cdots \\ \vdots & \vdots & \cdots & \\ 1 & \mu_1 & \mu_1(\mu_1 - 1) & \cdots \end{vmatrix} \\ &= \frac{n!}{\mu_1! \mu_2! \cdots \mu_k!} a_\delta(\mu_1, \mu_2, \ldots, \mu_k) \end{aligned}$$

where the last equality may be obtained by adding the second column to the third one, then -2 times the second column and 3 times the third column to the fourth one and so on, until the determinant reduces to the usual Vandermonde. □

Now we compute the value of χ^λ, $\lambda \vdash n$, on a cycle of length m, where $1 \leq m \leq n$.

Proposition 4.2.11 *Let* $\lambda \vdash n$, *set* $\mu = \lambda + \delta$ *(cf. Proposition 4.2.10) and define* $\phi(x) = \prod_{i=1}^k (x - \mu_i)$. *For* $1 \leq m \leq n$ *and* $\rho = (m, 1^{n-m})$, *we have*

$$\chi^\lambda_\rho = -\frac{d_\lambda}{m^2|\mathcal{C}_\rho|} \sum_{j=1}^k \frac{\mu_j(\mu_j - 1)\cdots(\mu_j - m + 1)}{\phi'(\mu_j)} \phi(\mu_j - m).$$

Proof Clearly, χ^λ_ρ is equal to the coefficient of $x_1^{\mu_1} x_2^{\mu_2} \cdots x_k^{\mu_k}$ in $(x_1 + x_2 + \cdots + x_k)^{n-m}(x_1^m + x_2^m + \cdots + x_k^m) a_\delta(x_1, x_2, \ldots, x_k)$. Arguing as in the proof of Proposition 4.2.10, we can deduce that χ^λ_ρ is equal to

$$\sum_{\substack{j=1 \\ \mu_j - m > 0}}^k \frac{(n-m)! a_\delta(\mu_1, \mu_2, \ldots, \mu_j - m, \ldots, \mu_{k-1}, \mu_k)}{\mu_1! \mu_2! \cdots (\mu_j - m)! \cdots \mu_{k-1}! \mu_k!}.$$

From Proposition 4.2.10, it follows that

$$\begin{aligned}
\frac{\chi^\lambda_\rho}{d_\lambda} &= \frac{(n-m)!}{n!} \sum_{\substack{j=1 \\ \mu_j - m > 0}}^k \frac{\mu_j!}{(\mu_j - m)!} \prod_{i \neq j} \frac{\mu_j - \mu_i - m}{\mu_j - \mu_i} \\
&= -\frac{1}{m^2|\mathcal{C}_\rho|} \sum_{j=1}^k \frac{\mu_j(\mu_j - 1)\cdots(\mu_j - m + 1)}{\phi'(\mu_j)} \phi(\mu_j - m),
\end{aligned}$$

because $|\mathcal{C}_\rho| = n!/(n-m)!m$ (cf. Proposition 4.1.1). □

Exercise 4.2.12 Set $A_j = \mu_j(\mu_j - 1)\cdots(\mu_j - m + 1)\frac{\phi(\mu_j - m)}{\phi'(\mu_j)}$.

(1) Prove that, for $h \geq 1$, $\sum_{j=1}^k A_j(\mu_j)^{h-1}$ is the coefficient of $1/x^h$ in the Taylor series at infinity (i.e. the expansion in descending powers of x) of the rational function $x(x-1)\cdots(x-m+1)\phi(x-m)/\phi(x)$.

(2) Deduce that $|\mathcal{C}_\rho|\frac{\chi^\lambda_\rho}{d_\lambda}$ is equal to the coefficient of $1/x$ in the expansion of $x(x-1)\cdots(x-m+1)\phi(x-m)/-m^2\phi(x)$.

Hint: We have $A_j = \lim_{x\to\mu_j}(x-\mu_j)\,[x(x-1)\cdots(x-m+1)\phi(x-m)/\phi(x)]$ and therefore $x(x-1)\cdots(x-m+1)\phi(x-m)/\phi(x) = \sum_{j=1}^{k}\frac{A_j}{x-\mu_j} + q(x)$, where $q(x)$ is a polynomial in x. Then one can use $\frac{A_j}{x-\mu_j} = \sum_{h=1}^{\infty} A_j(\mu_j)^{h-1}/x^h$.

The formulae in Proposition 4.2.11 and Exercise 4.2.12 are due to Frobenius. A classical exposition is in Murnaghan's monograph [95]; several particular cases were worked out in [62].

In Section 5.2 we will treat the expansion in Exercise 4.2.12 in great detail.

Exercise 4.2.13 Prove the formula in Theorem 3.4.10 by using the Frobenius character formula.

Consider the diagram of a partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$, and let $\lambda' = (\lambda'_1, \lambda'_2, \dots, \lambda'_{\lambda_1})$ be its conjugate. The *hook length* of a box of coordinates (i, j) is defined as $h_{i,j} := (\lambda_i - j) + (\lambda'_j - i) + 1$. Clearly, $\lambda_i - j$ equals the number of boxes on the same row of and on the right of (i, j), while $\lambda'_j - i$ equals the number of boxes on the same column of and below (i, j). For instance, in the diagram of $\lambda = (7, 4, 3, 2)$ the hook length of $(2, 2)$ is $h_{2,2} = 2 + 2 + 1 = 5$ (Figure 4.1).

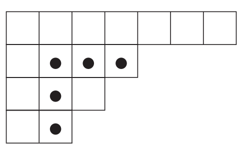


Figure 4.1 The hook length of (2, 2) equals the number of •'s.

Theorem 4.2.14 (Hook length formula) *For each $\lambda \vdash n$ we have*

$$d_\lambda = \frac{n!}{\prod_{(i,j)} h_{i,j}},$$

where the product runs over all boxes (i, j) of the diagram of λ.

Proof First note that, with the notation in Proposition 4.2.10, $\mu_1, \mu_2, \dots, \mu_k$ are precisely the hook lengths of the boxes in the first column (from top to bottom), and that, setting $t = \lambda'_2$, then $\widetilde{\mu}_j = \mu_j - k + t - 1$, for $j = 1, 2, \dots, t$, are the hook lengths of boxes in the second column. Note also that, for $j = t+1, t+2, \dots, k$, we have $\mu_j = k - j + 1$ and therefore $\prod_{i=1}^{j-1}(\mu_i - \mu_j) = [\prod_{i=1}^{t}(\mu_i - k + j - 1)](j - t - 1)!$; moreover, $(\mu_{t+1} - 1)! \cdots$

$(\mu_k - 1)! = (k-t-1)!(k-t-2)!\cdots 1!0!$. It follows that

$$\prod_{j=t+1}^{k}\prod_{i=1}^{j-1}(\mu_i - \mu_j)$$
$$= (\mu_{t+1}-1)!\cdots(\mu_k-1)!\prod_{i=1}^{t}(\mu_i-1)(\mu_i-2)\cdots[\mu_i-(k-t)],$$

and therefore, the formula in Proposition 4.2.10 yields

$$\begin{aligned}\frac{d_\lambda}{n!} &= \frac{a_\delta(\mu_1,\mu_2,\ldots,\mu_k)}{\mu_1!\mu_2!\cdots\mu_k!}\\ &= \frac{1}{\mu_1\mu_2\cdots\mu_k}\cdot\frac{\prod_{1\le i<j\le t}(\widetilde{\mu}_i-\widetilde{\mu}_j)}{\widetilde{\mu}_1!\widetilde{\mu}_2!\cdots\widetilde{\mu}_t!}.\end{aligned} \tag{4.33}$$

Then, we can apply (4.33) and use induction on the number of columns in the diagram of λ. □

The hook length formula was proved by Frame, Robinson, and Thrall [41]. The story of its discovery is narrated in Section 3.1 of Sagan's book [108], where one may also find the probabilistic proof given by Greene, Nijenhuis and Wilf [52].

We end this section by deriving the Murnaghan–Nakayama formula in the original form obtained by Murnaghan (see [94]). We need an extension of the definition of the alternant $a_{\lambda+\delta}$. For $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k) \in \mathbb{Z}^k$ and $\delta = (k-1, k-2, \ldots, 1, 0)$, we set

$$\begin{aligned}a_{\alpha+\delta}(x_1,x_2,\ldots,x_k) &= \sum_{\pi\in\mathfrak{S}_k}\varepsilon(\pi)x_{\pi(1)}^{\alpha_1+k-1}x_{\pi(2)}^{\alpha_2+k-2}\cdots x_{\pi(k)}^{\alpha_k}\\ &= \begin{vmatrix} x_1^{\alpha_1+k-1} & x_2^{\alpha_1+k-1} & \cdots & x_k^{\alpha_1+k-1}\\ x_1^{\alpha_2+k-2} & x_2^{\alpha_2+k-2} & \cdots & x_k^{\alpha_2+k-2}\\ \vdots & \vdots & & \vdots\\ x_1^{\alpha_k} & x_2^{\alpha_k} & \cdots & x_k^{\alpha_k}\end{vmatrix}\end{aligned}$$

see Proposition 4.1.15. If an integer in $\alpha+\delta$ is negative, then we set $a_{\alpha+\delta} = 0$. Obviously, if $\alpha' = (\alpha_1, \alpha_2, \ldots, \alpha_{j-1}, \alpha_{j+1}-1, \alpha_j+1, \alpha_{j+2}, \ldots, \alpha_k)$ then

$$a_{\alpha+\delta} = -a_{\alpha'+\delta}. \tag{4.34}$$

Indeed, $a_{\alpha'+\delta}$ is obtained from $a_{\alpha+\delta}$ by switching the jth and the $j+1$st rows:

$$a_{\alpha+\delta} = \begin{vmatrix} \cdots & \cdots & \cdots & \cdots \\ x_1^{\alpha_j+k-j} & x_2^{\alpha_j+k-j} & \cdots & x_k^{\alpha_j+k-j} \\ x_1^{\alpha_{j+1}+k-j-1} & x_2^{\alpha_{j+1}+k-j-1} & \cdots & x_k^{\alpha_{j+1}+k-j-1} \\ \cdots & \cdots & \cdots & \cdots \end{vmatrix}$$

and

$$a_{\alpha'+\delta} = \begin{vmatrix} \cdots & \cdots & \cdots & \cdots \\ x_1^{\alpha_{j+1}+k-j-1} & x_2^{\alpha_{j+1}+k-j-1} & \cdots & x_k^{\alpha_{j+1}+k-j-1} \\ x_1^{\alpha_j+k-j} & x_2^{\alpha_j+k-j} & \cdots & x_k^{\alpha_j+k-j} \\ \cdots & \cdots & \cdots & \cdots \end{vmatrix}.$$

Using the rule (4.34), it is easy to see that $a_{\alpha+\delta} = 0$ or $a_{\alpha+\delta} = \pm a_{\lambda+\delta}$ for a partition λ of $\alpha_1 + \alpha_2 + \cdots + \alpha_k$. Indeed, if j is the first index such that $\alpha_{j+1} > \alpha_j$ then

- if $\alpha_{j+1} = \alpha_j + 1$ then clearly $a_{\alpha+\delta} = 0$ because the jth row coincides with the $(j+1)$st row;
- if $\alpha_j > \alpha_{j+1} + 1$ then we can use $a_{\alpha+\delta} = a_{\alpha'+\delta}$

Iterating this procedure, $a_{\alpha+\delta} = 0$ or we get a partition λ of $\alpha_1 + \alpha_2 + \cdots + \alpha_k$ such that $a_{\alpha+\delta} = \pm a_{\lambda+\delta}$. We can also use Proposition 4.2.9 to define, for $\alpha \in \mathbb{Z}^k$ with $\sum_{j=1}^k \alpha_j = n$, and $\mu \vdash n$

$$\chi_\mu^\alpha = [p_\mu(x_1, x_2, \ldots, x_k)a_\delta]_{\alpha+\delta}, \tag{4.35}$$

where $[Q]_{\alpha+\delta}$ now stands for the coefficient of the antisymmetric polynomials $a_{\alpha+\delta}$ in Q (and $[Q]_{\alpha+\delta} = 0$ if $a_{\alpha+\delta} = 0$). Clearly (4.34) gives

$$\chi^\alpha = -\chi^{\alpha'} \tag{4.36}$$

so that $\chi^\alpha = 0$ or $\chi^\alpha = \pm\chi^\lambda$. The function $\mu \mapsto \chi_\mu^\alpha$ is a *virtual character* of $\mathfrak{S}_n$. Alternatively, we may use the determinantal formula (Theorem 4.3.25) to define χ^α.

We need the following notation: if $\mu = (\mu_1, \mu_2, \ldots, \mu_\ell) \vdash n$ and p is a part of μ, that is $\mu_i = p$ for some i, we set $\mu \setminus p = (\mu_1, \mu_2, \ldots, \mu_{i-1}, \mu_{i+1}, \ldots, \mu_\ell)$. If $r \in \mathbb{Z}$ and $\alpha \in \mathbb{Z}^k$ then $\alpha \pm r\epsilon_i = (\alpha_1, \alpha_2, \ldots, \alpha_i \pm r, \ldots, \alpha_k)$ (that is, $\epsilon_i = (0, 0, \ldots, 1, \ldots, 0)$ with 1 in position i). Now we can state and prove Murnaghan's rule:

Theorem 4.2.15 (Murnaghan rule) *Let λ, μ be two partitions of n. Suppose that r is a part of μ and denote by k the length of λ. Then*

$$\chi_\mu^\lambda = \sum_{j=1}^k \chi_{\mu\setminus r}^{\lambda - r\epsilon_j}.$$

Proof Since $p_\mu = p_{\mu\setminus r} p_r$, from Proposition 4.1.15 and Proposition 4.2.9 we deduce

$$\begin{aligned}
\chi_\mu^\lambda &= [p_\mu a_\delta]_{\lambda+\delta} \\
&= [p_{\mu\setminus r} p_r a_\delta]_{\lambda+\delta} \\
&= \sum_{j=1}^k \sum_{\pi\in\mathfrak{S}_k} [\varepsilon(\pi) p_{\mu\setminus r} x_k^{\pi(1)-1} \cdots x_j^{\pi(k-j+1)-1+r} \cdots x_1^{\pi(k)-1}]_{\lambda+\delta} \\
&= \sum_{j=1}^k \sum_{\pi\in\mathfrak{S}_k} [\varepsilon(\pi) p_{\mu\setminus r} x_k^{\pi(1)-1} \cdots x_j^{\pi(k-j+1)-1} \cdots x_1^{\pi(k)-1}]_{\lambda - r\epsilon_j+\delta} \\
&= \sum_{j=1}^k [p_{\mu\setminus r} a_\delta]_{\lambda - r\epsilon_j+\delta} \\
&= \sum_{j=1}^k \chi_{\mu\setminus r}^{\lambda - r\epsilon_j}.
\end{aligned}$$

□

4.3 Schur polynomials

4.3.1 Definition of Schur polynomials

Theorem 4.3.1 *For $\lambda \vdash r$, $\ell(\lambda) \le n$, set*

$$s_\lambda(x_1, x_2, \dots, x_n) = \frac{a_{\lambda+\delta}(x_1, x_2, \dots, x_n)}{a_\delta(x_1, x_2, \dots, x_n)}.$$

Then, each s_λ is a symmetric polynomial, and the set $\{s_\lambda : \lambda \vdash r, \ell(\lambda) \le n\}$ is another basis of Λ_n^r.

Proof First of all, note that if the polynomial $p(x_1, x_2, \dots, x_n)$ is antisymmetric, then $p(x_1, x_2, \dots x_i \dots, x_i, \dots, x_n) = 0$. It follows that $p(x)$ is divisible by $(x_i - x_j)$ and therefore by $a_\delta(x)$. Indeed, if $ax_1^{\mu_1} x_2^{\mu_2} \cdots x_i^{\mu_i} \cdots x_j^{\mu_j} \cdots x_n^{\mu_n}$ is a monomial in $p(x)$, so that $\mu_1, \mu_2, \dots, \mu_n$ are all distinct and, say $\mu_j < \mu_i$, then $p(x)$ also contains the monomial $-ax_1^{\mu_1} x_2^{\mu_2} \cdots x_j^{\mu_j} \cdots x_i^{\mu_i} \cdots x_n^{\mu_n}$, and the sum of all these monomials is divisible by $(x_i - x_j)$.

Denote by $\mathcal{A}_n^r$ the space of all antisymmetric polynomials of degree $r + n(n-1)/2$ in the variables $x_1, x_2, \dots, x_n$. From the above considerations, we have that the map

$$\begin{array}{rcl} \mathcal{A}_n^r & \to & \Lambda_n^r \\ p(x) & \mapsto & \frac{p(x)}{a_\delta(x)} \end{array}$$

is a linear isomorphism. Then, s_λ is precisely the image of $a_{\lambda+\delta}$ and therefore the statement follows from Proposition 4.1.15. □

The polynomials s_λ are called *Schur polynomials*.

We now give another expansion for the product $\frac{1}{\prod_{i,j}(1-x_i y_j)}$ (see Lemma 4.1.11 and Proposition 4.2.3) in terms of Schur polynomials.

Lemma 4.3.2 *We have*

$$\frac{1}{\prod_{i,j}(1 - x_i y_j)} = \sum_{r=0}^{\infty} \sum_{\substack{\lambda \vdash r: \\ \ell(\lambda) \leq n}} s_\lambda(x) s_\lambda(y).$$

Proof First of all, note that (using the geometric series expansion)

$$\begin{aligned} \det\left(\frac{1}{1 - x_i y_j}\right) &= \sum_{\pi \in \mathfrak{S}_n} \varepsilon(\pi) \prod_{i=1}^n \frac{1}{1 - x_i y_{\pi(i)}} \\ &= \sum_{u_{i,j}=0}^{\infty} \sum_{\pi \in \mathfrak{S}_n} \varepsilon(\pi) \prod_{i=1}^n x_i^{u_{i,\pi(i)}} y_{\pi(i)}^{u_{i,\pi(i)}} \end{aligned}$$

and therefore if $\lambda \vdash r$, $\ell(\lambda) \leq n$, the coefficient of $x_1^{\lambda_1+n-1} x_2^{\lambda_2+n-2} \cdots x_n^{\lambda_n}$ in $\det\left(\frac{1}{1-x_i y_j}\right)$ is $a_{\lambda+\delta}(y_1, y_2, \dots, y_n)$. Since $\det\left(\frac{1}{1-x_i y_j}\right)$ is antisymmetric, both in the x's and the y's, it follows that it is the sum of all polynomials of the form $a_{\lambda+\delta}(x) a_{\lambda+\delta}(y)$:

$$\frac{1}{\prod_{i,j}(1 - x_i y_j)} = \sum_{r=0}^{\infty} \sum_{\substack{\lambda \vdash r: \\ \ell(\lambda) \leq n}} a_{\lambda+\delta}(x) a_{\lambda+\delta}(y).$$

Then, the statement follows from Cauchy's identity (Proposition 4.2.3). □

We end this subsection by evaluating a Schur polynomial for $x_1 = x_2 = \cdots x_n = 1$. This result will be used only in Section 4.4.4.

Theorem 4.3.3 *Suppose again that λ is a partition of r of length k and that $n \geq k$. Then setting $x_1 = x_2 = \cdots x_n = 1$ in the Schur polynomial s_λ we have:*

$$s_\lambda(1, 1, \dots, 1) = \frac{d_\lambda}{r!} \prod_{i=1}^{k} \prod_{j=1}^{\lambda_i} (j - i + n). \tag{4.37}$$

Proof First of all, note that if we set $x_j = x^{j-1}$, $j = 0, 1, \dots, n-1$ in $a_{\lambda+\delta}$ (where x^{j-1} are the powers of a single indeterminate x), then

$$a_{\lambda+\delta}(1, x, \dots, x^{n-1}) = \det[(x^{\lambda_i+n-i})^{j-1}]$$

is a Vandermonde determinant whose ith row is made up of the powers $(x^{\lambda_i+n-i})^{j-1}$, $j = 1, 2, \dots, n$, of x^{λ_i+n-1}. It follows that

$$a_{\lambda+\delta}(1, x, \dots, x^{n-1}) = \prod_{1 \leq i < j \leq n} (x^{\lambda_i+n-i} - x^{\lambda_j+n-j}), \tag{4.38}$$

and therefore

$$\begin{aligned} s_\lambda(1, x, \dots, x^{n-1}) &= \frac{\prod_{1 \leq i < j \leq n} (x^{\lambda_i+n-i} - x^{\lambda_j+n-j})}{\prod_{1 \leq i < j \leq n} (x^{n-i} - x^{n-j})} \\ &= \prod_{j=2}^{n} x^{\lambda_j(j-1)} \cdot \frac{\prod_{1 \leq i < j \leq n} (x^{\lambda_i-\lambda_j+j-i} - 1)}{\prod_{1 \leq i < j \leq n} (x^{j-i} - 1)}. \end{aligned}$$

Now we can take the limit for $x \to 1$ and apply (4.30), getting (recall that $\lambda_{k+1} = \cdots \lambda_n = 0$):

$$\begin{aligned} s_\lambda(1, 1, \dots, 1) = \lim_{x \to 1} s_\lambda(1, x, \dots, x^{n-1}) &= \frac{\prod_{1 \leq i < j \leq n} (\lambda_i - \lambda_j + j - i)}{\prod_{1 \leq i < j \leq n} (j - i)} \\ &= \frac{\prod_{1 \leq i \leq k < j \leq n} (\lambda_i + j - i)}{\prod_{1 \leq i \leq k < j \leq n} (j - i)} \cdot \frac{\prod_{1 \leq i < j \leq k} (\lambda_i - \lambda_j + j - i)}{\prod_{1 \leq i < j \leq k} (j - i)} \\ \text{(by (4.30))} \quad &= \frac{\prod_{1 \leq i \leq k < j \leq n} (\lambda_i + j - i)}{\prod_{i=1}^{k} \prod_{j=i+1}^{n} (j - i)} \cdot \frac{d_\lambda}{r!} (\lambda_1 + k - 1)! (\lambda_2 + k - 2)! \cdots \lambda_k! \\ &= \frac{d_\lambda}{r!} \prod_{i=1}^{k} \frac{(\lambda_i + n - i)!}{(n - i)!} \\ &= \frac{d_\lambda}{r!} \prod_{i=1}^{k} \prod_{j=1}^{\lambda_i} (j - i + n). \end{aligned}$$

□

The polynomial $c(t) = \prod_{i=1}^{k} \prod_{j=1}^{\lambda_i} (j - i + t)$ is often called *the content polynomial* associated with the partition λ; see [83] for more details.

The following expression, obtained in the above proof, is also noteworthy:

$$s_\lambda(1, 1, \ldots, 1) = \prod_{1 \le i < j \le n} \frac{\lambda_i - \lambda_j + j - i}{j - i}. \tag{4.39}$$

4.3.2 A scalar product

We introduce a scalar product in Λ_n^r just by requiring that $\{s_\lambda : \lambda \vdash r, \ell(\lambda) \le n\}$ is an orthonormal basis.

Lemma 4.3.4

(i) *The basis* $\{m_\lambda : \lambda \vdash r, \ell(\lambda) \le n\}$ *and the set* $\{h_\lambda : \lambda \vdash r, \ell(\lambda) \le n\}$ *are dual to each other with respect to this scalar product, that is,*

$$\langle h_\lambda, m_\mu \rangle = \delta_{\lambda,\mu}$$

for all $\lambda, \mu \vdash r, \ell(\lambda), \ell(\mu) \le n$.

(ii) *If* $n \ge r$, *the basis* $\{p_\lambda : \lambda \vdash r\}$ *is orthogonal with respect to the scalar product:*

$$\langle p_\lambda, p_\mu \rangle = z_\lambda \delta_{\lambda,\mu}$$

for all $\lambda, \mu \vdash r$, *where* z_λ *is as in Proposition 4.1.1.*

Proof (i) Define the coefficients $a_{\lambda,\mu}$ and $b_{\lambda,\rho}$ by requiring that

$$h_\lambda(x) = \sum_\mu a_{\lambda,\mu} s_\mu(x) \quad \text{and} \quad m_\lambda(x) = \sum_\rho b_{\lambda,\rho} s_\rho(x).$$

From (4.10) and Lemma 4.3.2, we get the equality

$$\sum_\lambda h_\lambda(x) m_\lambda(y) = \sum_\lambda s_\lambda(x) s_\lambda(y).$$

Since, on the other hand,

$$\sum_\lambda h_\lambda(x) m_\lambda(y) = \sum_{\mu,\rho} \left(\sum_\lambda a_{\lambda,\mu} b_{\lambda,\rho} \right) s_\mu(x) s_\rho(y)$$

we get $\sum_\lambda a_{\lambda,\mu} b_{\lambda,\rho} = \delta_{\mu,\rho}$. Then we also have $\sum_\mu a_{\lambda,\mu} b_{\theta,\mu} = \delta_{\lambda,\theta}$ and therefore

$$\begin{aligned}\langle h_\lambda, m_\theta \rangle &= \sum_{\mu,\rho} a_{\lambda,\mu} b_{\theta,\rho} \langle s_\mu, s_\rho \rangle \\ &= \sum_{\mu} a_{\lambda,\mu} b_{\theta,\mu} \\ &= \delta_{\lambda,\theta}.\end{aligned}$$

The proof of (ii) is analogous: one uses (4.9) in place of (4.10). □

Corollary 4.3.5 *If $n \geq r$, then the involution $\omega : \Lambda_n^r \to \Lambda_n^r$ defined by $\omega(e_\lambda) = h_\lambda$ for all $\lambda \vdash r$ (see Section 4.1.5), is an isometry.*

Proof From (iii) in Proposition 4.1.13 we have that the polynomials p_λ, with $\lambda \vdash r$ are eigenvectors of ω with corresponding eigenvalues of modulus one. On the other hand, by (ii) in the previous proposition, these vectors constitute an orthogonal basis for Λ_n^r, and this ends the proof. □

Corollary 4.3.6 *The set $\{h_\lambda : \lambda \vdash r \text{ and } \ell(\lambda) \leq n\}$ is a basis for Λ_n^r (see Theorem 4.1.12(iii)).*

Exercise 4.3.7 Show that in Λ_2^3 we have: $p_{(3)} = m_{(3)} = 2h_{(3)} - h_{2,1}$, $p_{2,1} = m_{(3)} + m_{2,1} = h_{(3)}$ and $p_{1,1,1} = m_{(3)} + 3m_{2,1} = 2h_{2,1} - h_{(3)}$. From these identities it follows that $\langle p_{2,1}, p_{(3)} \rangle = 1 \neq 0$, $\langle p_{2,1}, p_{1,1,1} \rangle = 1 \neq 0$ (and $\langle p_{(3)}, p_{1,1,1} \rangle = -1 \neq 0$) and therefore if $n < r$, then the power sums do not yield an orthogonal basis.

4.3.3 The characteristic map

We introduce some notation: if $\pi \in \mathfrak{S}_n$, we denote by $c(\pi)$ the cycle type of π (which is a partition of n). Moreover, we denote by $p_c(x)$ the function $\mathfrak{S}_n \ni \pi \mapsto p_{c(\pi)}(x)$, where $p_{c(\pi)}$ is the power sum symmetric polynomial associated with $c(\pi)$. We denote by $\mathcal{C}(\mathfrak{S}_n)$ the algebra of central functions on $\mathfrak{S}_n$. Also, for $\phi \in \mathcal{C}(\mathfrak{S}_n)$ and $\lambda \vdash n$, we denote by ϕ_λ the value of ϕ on $\pi \in \mathcal{C}_\lambda$. Finally, for $0 < r \leq n$, we define the *characteristic map* $\mathrm{ch} : \mathcal{C}(\mathfrak{S}_r) \to \Lambda_n^r$ by setting

$$\mathrm{ch}(\phi) = \frac{1}{r!} \sum_{\pi \in \mathfrak{S}_r} \phi(\pi) p_{c(\pi)} = \sum_{\lambda \vdash r} \frac{1}{z_\lambda} \phi_\lambda p_\lambda \equiv \frac{1}{r!} \langle \phi, p_c \rangle_{L(\mathfrak{S}_r)}$$

for every $\phi \in \mathcal{C}(\mathfrak{S}_r)$.

Proposition 4.3.8 *The map* $\sqrt{r!}\,\mathrm{ch}$ *is a linear isometry.*

Proof Let $\phi, \psi \in \mathcal{C}(\mathfrak{S}_r)$. Then

$$\begin{aligned}
\langle \sqrt{r}!\mathrm{ch}(\phi), \sqrt{r}!\mathrm{ch}(\psi)\rangle_{\Lambda_n^r} &= \langle \sum_{\lambda \vdash r} \frac{\sqrt{r!}}{z_\lambda} \phi_\lambda p_\lambda(x), \sum_{\mu \vdash r} \frac{\sqrt{r!}}{z_\mu} \psi_\mu p_\mu(x)\rangle_{\Lambda_n^r} \\
\text{(by (ii) in Lemma 4.3.4)} &= \sum_{\lambda \vdash r} \frac{r!}{z_\lambda} \phi_\lambda \overline{\psi_\lambda} \\
&= \sum_{\pi \in \mathfrak{S}_r} \phi(\pi)\overline{\psi(\pi)} \\
&= \langle \phi, \psi \rangle_{L(\mathfrak{S}_r)}.
\end{aligned}$$

□

Theorem 4.3.9 *Let* $n \geq r$. *Then*

$$s_\lambda = \mathrm{ch}(\chi^\lambda)$$

for all $\lambda \vdash r$.

Proof Recalling that for $\lambda, \mu \vdash r$ one has, by definition, $(\chi^\lambda)_\mu = \chi_\mu^\lambda$, we have

$$\mathrm{ch}(\chi^\lambda) = \sum_{\mu \vdash r} \frac{1}{z_\mu} \chi_\mu^\lambda p_\mu(x).$$

On the other hand, the Frobenius character formula (cf. Theorem 4.2.8) may be written in the form

$$p_\mu(x) = \sum_{\lambda \vdash r} \chi_\mu^\lambda s_\lambda(x).$$

Therefore

$$\langle s_\lambda, \frac{1}{\sqrt{z_\mu}} p_\mu \rangle_{\Lambda_n^r} = \chi_\mu^\lambda \frac{1}{\sqrt{z_\mu}} = \langle \mathrm{ch}(\chi^\lambda), \frac{1}{\sqrt{z_\mu}} p_\mu \rangle_{\Lambda_n^r}.$$

Since $\{\frac{1}{\sqrt{z_\mu}} p_\mu : \mu \vdash r\}$ is an orthonormal basis for Λ_n^r, this ends the proof. □

Remark 4.3.10 We will also use the following form of the Frobenius formula:

$$p_\mu(x) = \sum_{\substack{\lambda \vdash r: \\ \ell(\lambda) \leq n}} \chi_\mu^\lambda s_\lambda(x). \tag{4.40}$$

The case $n \geq r$ was obtained in the above proof. For $n < r$, we can start from the case $r = n$ and use the following identities:

$$p_\mu(x_1, x_2, \ldots, x_r)|_{[x_{n+1} = \cdots = x_r = 0]} \equiv p_\mu(x_1, x_2, \ldots, x_n),$$

while

$$s_\lambda(x_1, x_2, \ldots, x_r)|_{[x_{n+1}=\cdots=x_r=0]} = \begin{cases} 0 & \text{if } \ell(\lambda) > n \\ s_\lambda(x_1, x_2, \ldots, x_n) & \text{if } \ell(\lambda) \le n. \end{cases}$$

To get the last identity, we may apply repeatedly the following fact (see the expression in Proposition 4.1.15): if $\lambda_n = 0$ then $a_{\delta+\lambda}(x_1, x_2, \ldots, x_n)|_{x_n=0} = x_1 x_2 \cdots x_{n-1} a_{\overline{\delta}+\lambda}(x_1, x_2 \ldots, x_{n-1})$, where $\overline{\delta} = (n-2, n-1, \ldots, 1, 0)$.

We now introduce a useful notation. For $\lambda \vdash r$ and $\mu \vdash m$ we denote by $\chi^\lambda \circ \chi^\mu$ the character of the representation $\mathrm{Ind}_{\mathfrak{S}_r \times \mathfrak{S}_m}^{\mathfrak{S}_{r+m}}(S^\lambda \boxtimes S^\mu)$. We then extend the operation $\circ$ to the whole $\mathcal{C}(\mathfrak{S}_r) \times \mathcal{C}(\mathfrak{S}_m)$ by setting, for all $\phi = \sum_{\lambda \vdash r} a_\lambda \chi^\lambda$ and $\psi = \sum_{\mu \vdash m} b_\mu \chi^\mu$, where $a_\lambda, b_\mu \in \mathbb{C}$,

$$\phi \circ \psi = \sum_{\lambda \vdash r} \sum_{\mu \vdash m} a_\lambda b_\mu \chi^\lambda \circ \chi^\mu \in \mathcal{C}(\mathfrak{S}_{r+m}).$$

It is clear that if ϕ is the character of a representation ρ of $\mathfrak{S}_r$ and ψ is the character of a representation σ of $\mathfrak{S}_m$ (ρ and σ not necessarily irreducible), then $\phi \circ \psi$ is the character of $\mathrm{Ind}_{\mathfrak{S}_r \times \mathfrak{S}_m}^{\mathfrak{S}_{r+m}}(\rho \boxtimes \sigma)$.

Proposition 4.3.11 *Suppose that $n \ge m + r$ and let $\phi \in \mathcal{C}(\mathfrak{S}_r)$ and $\psi \in \mathcal{C}(\mathfrak{S}_m)$. Then*

$$\mathrm{ch}(\phi \circ \psi) = \mathrm{ch}(\phi)\mathrm{ch}(\psi). \tag{4.41}$$

Proof By linearity, it suffices to check (4.41) when $\phi = \chi^\lambda$ and $\psi = \chi^\mu$. Let $\pi = \pi_1 \pi_2 \in \mathfrak{S}_r \times \mathfrak{S}_m$, with $\pi_1 \in \mathfrak{S}_r$ and $\pi_2 \in \mathfrak{S}_m$. Then $p_{c(\pi)}(x) = p_{c(\pi_1)}(x) p_{c(\pi_2)}(x)$. We express this fact as follows

$$p_c(x)|_{\mathfrak{S}_r \times \mathfrak{S}_m} = p_{c_1}(x) p_{c_2}(x).$$

Similarly, $(\chi^\lambda \chi^\mu)(\pi) = \chi^\lambda(\pi_1)\chi^\mu(\pi_2)$ is the character of $S^\lambda \boxtimes S^\mu$ (cf. (v) in Proposition 1.3.4). Then we have

$$\begin{aligned} \mathrm{ch}(\chi^\lambda \circ \chi^\mu) &= \frac{1}{(r+m)!} \langle \chi^\lambda \circ \chi^\mu, p_c(x) \rangle_{L(\mathfrak{S}_{r+m})} \\ \text{(by Frobenius reciprocity)} \quad &= \frac{1}{r!m!} \langle \chi^\lambda \chi^\mu, p_{c_1}(x) p_{c_2}(x) \rangle_{L(\mathfrak{S}_r \times \mathfrak{S}_m)} \\ &= \frac{1}{r!} \langle \chi^\lambda, p_{c_1}(x) \rangle_{L(\mathfrak{S}_r)} \frac{1}{m!} \langle \chi^\mu, p_{c_2}(x) \rangle_{L(\mathfrak{S}_m)} \\ &= \mathrm{ch}(\chi^\lambda)\mathrm{ch}(\chi^\mu). \end{aligned}$$

□

Corollary 4.3.12 *Let $\lambda \vdash r$ and denote by ψ^λ the character of M^λ as in Lemma 4.2.1. Then we have*

$$\mathrm{ch}(\psi^\lambda) = h_\lambda.$$

Proof Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$. Then $\psi^\lambda = \psi^{\lambda_1} \circ \psi^{\lambda_2} \circ \cdots \circ \psi^{\lambda_k}$ and therefore

$$\mathrm{ch}(\psi^\lambda) = \mathrm{ch}(\psi^{\lambda_1})\mathrm{ch}(\psi^{\lambda_2}) \cdots \mathrm{ch}(\psi^{\lambda_k}) = h_{\lambda_1} h_{\lambda_2} \cdots h_{\lambda_k} = h_\lambda,$$

where the second equality follows from

$$\mathrm{ch}(\psi^{\lambda_i}) = \sum_{\mu \vdash \lambda_i} \frac{1}{z_\mu} \psi^{\lambda_i}_\mu p_\mu(x) = \sum_{\mu \vdash \lambda_i} \frac{1}{z_\mu} p_\mu(x) = h_{\lambda_i}$$

(in the last equality we have used Lemma 4.1.10). □

We can now use the characteristic map to translate, in the language of Schur's polynomials, the Young rule (Corollary 3.7.11) and the Pieri rule (Corollary 3.5.14).

Corollary 4.3.13 (Young's rule) *Let $\lambda \vdash r$ and suppose $n \geq r$. Then*

$$h_\lambda = \sum_{\mu \vdash r} K(\mu, \lambda) s_\mu$$

where $K(\mu, \lambda) \equiv |\mathrm{STab}(\mu, \lambda)|$ is the number of semistandard tableaux of shape μ and content λ.

Corollary 4.3.14 (Pieri's rule) *Let $\lambda \vdash r$ and suppose $n \geq r + m$. Then*

$$s_\lambda s_{(m)} = \sum_\mu s_\mu$$

where the sum runs over all $\mu \vdash r + m$ such that $\lambda \preceq \mu$ and μ/λ is totally disconnected.

Exercise 4.3.15 ([43]) (1) Prove the following identity:

$$a_{\lambda+\delta}(x) \prod_{j=1}^{n} \frac{1}{1 - x_j} = \sum_\mu a_{\mu+\delta}(x)$$

where the sum is over all partitions μ such that $\lambda \preceq \mu$ and μ/λ is totally disconnected.

(2) From (1) deduce the Pieri rule in Corollary 4.3.14.

(3) From the Pieri rule in Corollary 4.3.14 deduce the Young rule in Corollary 4.3.13.

Pieri's rule for Schur's polynomial was proved in a geometrical setting by Mario Pieri in [102].

Exercise 4.3.16 Consider the algebra Λ of all symmetric functions in infinitely many variables (see Section 4.1.7).

(1) Show that, for each partition $\lambda \vdash r$, there exists a unique $s_\lambda \in \Lambda$ such that $s_\lambda(x_1, x_2, \ldots, x_n, 0, 0, \ldots)$ is the Schur polynomial in $x_1, x_2, \ldots, x_n$, for all $n \geq \ell(\lambda)$.
Hint. Prove it for $n+1$ variables: $s_\lambda(x_1, x_2, \ldots, x_n, 0) = s_\lambda(x_1, x_2, \ldots, x_n)$.

(2) Prove that the characteristic map may be extended to a map

$$\mathrm{ch} : \bigoplus_{n=0}^{\infty} \mathcal{C}(\mathfrak{S}_n) \to \Lambda$$

that is also an algebra isomorphism ($\bigoplus_{n=0}^{\infty} \mathcal{C}(\mathfrak{S}_n)$ is an algebra with respect to $\circ$). See the books of Macdonald [83] and Sagan [108] for more details.

Following the notation introduced in order to state and prove Theorem 4.2.15, given $\alpha \in \mathbb{Z}^k$, with $\alpha_1 + \alpha_2 + \cdots + \alpha_k = n$, we can define a corresponding Schur's function by setting $s_\alpha = a_{\alpha+\delta}/a_\delta$. We have $s_\alpha = \pm s_{\alpha'}$ as in (4.34) and

$$s_\alpha = \sum_{\mu \vdash r} \frac{1}{z_\mu} \chi^\alpha_\mu p_\mu \equiv \mathrm{ch}(\chi^\alpha).$$

4.3.4 Determinantal identities

In this section we give two determinantal expressions for the Schur polynomials. The first one, known as the *Jacobi–Trudi identity*, was given without proof by Karl Gustav Jacobi in [63] and subsequently proved by Nicola Trudi, a student of Jacobi, in [119].

The second one is called the *dual Jacobi–Trudi identity* . Both identities were rediscovered, in a geometrical setting, by Giovanni Zeno Giambelli in [47].

In what follows, we extend the parameterization of the elementary and complete polynomials e_k and h_k to nonpositive values of k, by setting $e_0 = h_0 = 1$ and $e_k = h_k = 0$ for $k < 0$. We also set $e_k = 0$ for $k > n$ (the number of variables). We shall deduce the Jacobi–Trudi identities as a consequence of the following expansion for $\prod_{i,j=1}^n \frac{1}{1-x_i y_j}$.

Lemma 4.3.17 *We have*

$$\prod_{i,j=1}^{n} \frac{1}{1 - x_i y_j} = \sum_{r=0}^{\infty} \sum_{\substack{\lambda \vdash r: \\ \ell(\lambda) \leq n}} \det(h_{\lambda_i - i + j}(x)) s_\lambda(y)$$

where $(h_{\lambda_i-i+j}(x))$ *is the* $n\times n$ *matrix whose* (i,j)*-entry is* $h_{\lambda_i-i+j}(x)$ *(and we set* $\lambda_{\ell(\lambda)+1}=\lambda_{\ell(\lambda)+2}=\cdots=\lambda_n=0$*).*

Proof We follow the proof in [84]. Let $\alpha=(\alpha_1,\alpha_2,\dots,\alpha_n)$ be a sequence of nonnegative integers. We set $h_\alpha=h_{\alpha_1}h_{\alpha_2}\cdots h_{\alpha_n}$ and $a_{\alpha+\delta}$ to be the determinant of the $n\times n$ matrix whose (i,j)-entry is $(x_j^{\alpha_i+n-i})$. Clearly, $a_{\alpha+\delta}=0$ if the integers $\alpha_1+n-1,\alpha_2+n-2,\dots,\alpha_n$ are not distinct, while, if they are all distinct, there exists a unique $\pi\in\mathfrak{S}_n$ such that $\alpha+\delta=\pi(\lambda+\delta)$, with λ a partition and $\pi(\lambda+\delta)=(\lambda_{\pi(1)}+n-\pi(1),\lambda_{\pi(2)}+n-\pi(2),\dots,\lambda_{\pi(n)}+n-\pi(n))$. Clearly, $a_{\alpha+\delta}(x)=\varepsilon(\pi)a_{\lambda+\delta}(x)$.

Let $H(t)$ be the generating function of the complete symmetric polynomials (see Lemma 4.1.7). We have

$$
\begin{aligned}
\left(\prod_{i,j=1}^{n}\frac{1}{1-x_iy_j}\right)a_\delta(y) &= \left(\prod_{j=1}^{n}H(y_j)\right)a_\delta(y)\\
&= \det\left(y_j^{n-i}H(y_j)\right)\\
&= \det\left(\sum_{\alpha_i=0}^{\infty}h_{\alpha_i}(x)y_j^{\alpha_i+n-i}\right)\\
&= \sum_{\pi\in\mathfrak{S}_n}\varepsilon(\pi)\prod_{j=1}^{n}\sum_{\alpha_{\pi(j)}=0}^{\infty}h_{\alpha_{\pi(j)}}(x)y_j^{\alpha_{\pi(j)}+n-\pi(j)}\\
&= \sum_{\alpha_1,\alpha_2,\dots,\alpha_n=0}^{\infty}h_\alpha(x)\sum_{\pi\in\mathfrak{S}_n}\varepsilon(\pi)\prod_{j=1}^{n}y_j^{\alpha_{\pi(j)}+n-\pi(j)}\\
&= \sum_{\alpha_1,\alpha_2,\dots,\alpha_n=0}^{\infty}h_\alpha(x)a_{\alpha+\delta}(y)\\
(\alpha+\delta=\pi(\lambda+\delta))\quad &= \sum_{r=0}^{\infty}\sum_{\substack{\lambda\vdash r:\\ \ell(\lambda)\le n}}\sum_{\pi\in\mathfrak{S}_n}h_{\pi(\lambda+\delta)-\delta}(x)a_{\pi(\lambda+\delta)}(y)\\
&= \sum_{r=0}^{\infty}\sum_{\substack{\lambda\vdash r:\\ \ell(\lambda)\le n}}a_{\lambda+\delta}(y)\sum_{\pi\in\mathfrak{S}_n}\varepsilon(\pi)h_{\pi(\lambda+\delta)-\delta}(x)\\
&= \sum_{r=0}^{\infty}\sum_{\substack{\lambda\vdash r:\\ \ell(\lambda)\le n}}\det\left(h_{\lambda_i-i+j}(x)\right)a_{\lambda+\delta}(y).
\end{aligned}
$$

Dividing by $a_\delta(y)$, one gets the desired identity. □

Corollary 4.3.18 (Jacobi–Trudi identity) *Let $\lambda \vdash n$ with $\ell(\lambda) \le n$ and set $\lambda_{\ell(\lambda)+1} = \lambda_{\ell(\lambda)+2} = \cdots = \lambda_n = 0$. Then*

$$s_\lambda(x) = \begin{vmatrix} h_{\lambda_1}(x) & h_{\lambda_1+1}(x) & \cdots & h_{\lambda_1+n-1}(x) \\ h_{\lambda_2-1}(x) & h_{\lambda_2}(x) & \cdots & h_{\lambda_2+n-2}(x) \\ \vdots & \vdots & & \vdots \\ h_{\lambda_n-n+1}(x) & h_{\lambda_n-n+2}(x) & \cdots & h_{\lambda_n}(x) \end{vmatrix}. \tag{4.42}$$

Proof The statement follows from the expansions in Lemma 4.3.2 and Lemma 4.3.17. □

Note that, if $\ell(\lambda) < n$, then the determinant in (4.42) is equal to the determinant of the $\ell(\lambda) \times \ell(\lambda)$ submatrix at the upper left corner.

Corollary 4.3.19 *The set $\{h_\lambda : \lambda \vdash r$ and $\ell(\lambda) \le n\}$ is a basis for Λ_n^r (see Theorem 4.1.12(iii)).*

Lemma 4.3.20 *For $k \ge 1$, let $\mathcal{H}_k$ and $\mathcal{E}_k$ denote the $k \times k$ matrices whose (i, j) entries are h_{i-j} and $(-1)^{i-j} e_{i-j}$, respectively. Then $\mathcal{H}_k$ and $\mathcal{E}_k$ are both lower triangular with 1's along the diagonal, and $(\mathcal{H}_k)^{-1} = \mathcal{E}_k$.*

Proof This statement is an immediate consequence of (i) in Lemma 4.1.9. □

Exercise 4.3.21 We sketch an alternative proof of the Jacobi–Trudi identity taken from the book of Fulton and Harris [43].

(1) Prove that

$$\sum_{k=0}^{n} e_k(x) x_j^{s-k} (-1)^{s-k} = 0$$

for $j = 1, 2, \ldots, n$ and $s \ge n$.

Hint. $(\sum_{k=0}^{\infty} (-1)^k x_j^k t^k)(\sum_{s=0}^{n} e_s(x) t^s) = \prod_{\substack{i=1 \\ i \ne j}}^{n} (1 + x_i t)$.

(2) Show that there exist polynomials $A(s, k)$ in $e_1, e_2 \ldots, e_n$ such that

$$x_j^s = A(s, 1) x_j^{n-1} + A(s, 2) x_j^{n-2} + \cdots + A(s, n) \tag{4.43}$$

for $j = 1, 2, \ldots, n$ and $s > n$, and

$$h_{s-m} = A(s, 1) h_{n-m-1}(x) + A(s, 2) h_{n-m-2}(x) + \cdots + A(s, n) h_{-m}(x) \tag{4.44}$$

for $0 \le m < n$ and $s \ge n$.

Hint. To show (4.44), use Lemma 4.1.9.

(3) For $\lambda \vdash r$ and $\ell(\lambda) \le n$ (and $\lambda_{\ell(\lambda)+1} = \lambda_{\ell(\lambda)+2} = \cdots = \lambda_n = 0$), let $\mathcal{X}_n$, $\mathcal{X}_\lambda$, $\mathcal{A}_\lambda$ and $\mathcal{H}_\lambda$ be the $n \times n$ matrices whose (i, j) entries are x_j^{n-i}, $x_j^{\lambda_i+n-i}$,

$A(\lambda_i + n - i, j)$ and $h_{\lambda_i+j-i}(x)$, respectively. Prove that

$$\mathcal{X}_\lambda = \mathcal{H}_\lambda \mathcal{E}_n \mathcal{X}_n \tag{4.45}$$

where $\mathcal{E}_n$ is as in Lemma 4.3.20.

Hint. Write (4.43) and (4.44) in the form $\mathcal{X}_\lambda = \mathcal{A}_\lambda \mathcal{X}_n$ and $\mathcal{H}_\lambda = \mathcal{A}_\lambda \mathcal{H}_n$, respectively, and use Lemma 4.3.20.

Taking the determinants in (4.45), one immediately gets the Jacobi–Trudi identity.

In order to prove the dual Jacobi–Trudi identity, we need the following identity for minors and cofactors of matrices. We follow the exposition in the book by Fulton and Harris [43].

Lemma 4.3.22 *Let $A = (a_{i,j})$ be an invertible $m \times m$ matrix and denote by $B = (b_{i,j})$ its inverse A^{-1}. Take σ and π in $\mathfrak{S}_m$, and, for $k \in \{1, 2, \dots, m\}$ set $\widetilde{A} = (a_{\pi(i),\sigma(j)})_{i,j=1}^k$ and $\widetilde{B} = (b_{\sigma(i),\pi(j)})_{i,j=k+1}^m$. Then*

$$\det \widetilde{A} = \varepsilon(\pi)\varepsilon(\sigma) \det A \det \widetilde{B}. \tag{4.46}$$

Proof There exist $m \times m$ permutation matrices P and Q such that

$$PAQ = \begin{pmatrix} \widetilde{A} & A_1 \\ A_2 & A_3 \end{pmatrix} \quad \text{and} \quad Q^{-1}BP^{-1} = \begin{pmatrix} B_1 & B_2 \\ B_3 & \widetilde{B} \end{pmatrix},$$

for suitable matrices A_1, A_2, A_3, B_1, B_2 and B_3 (note that $P^{-1} = P^t$ and $Q^{-1} = Q^t$). We then have

$$\begin{pmatrix} \widetilde{A} & A_1 \\ A_2 & A_3 \end{pmatrix} \begin{pmatrix} I_k & B_2 \\ 0 & \widetilde{B} \end{pmatrix} = \begin{pmatrix} \widetilde{A} & 0 \\ A_2 & I_{m-k} \end{pmatrix}.$$

This implies that $\det(PAQ) \det \widetilde{B} = \det \widetilde{A}$. Since $\det(P) = \varepsilon(\pi)$ and $\det(Q) = \varepsilon(\sigma)$, the identity 4.46 follows. □

We also need another lemma on partitions.

Lemma 4.3.23 *Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ be a partition and denote by $\lambda' = (\lambda'_1, \lambda'_2, \dots, \lambda'_t)$ its conjugate. Then $\{1, 2, \dots, k+t\} = \{\lambda_i + k + 1 - i : 1 \le i \le k\} \coprod \{k + j - \lambda'_j : 1 \le j \le t\}$.*

Proof In the diagram of λ, consider the segments at the end of the rows and at the end of the columns. Label consecutively all these segments with the numbers $1, 2, \dots, k+t$, starting from the bottom. For example, in Figure 4.2, if $\lambda = (5, 3, 2, 2, 1)$, so that $\lambda' = (5, 4, 2, 1, 1)$, we have

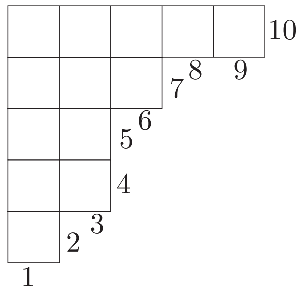


Figure 4.2

Then, the labels of the segments at the end of the rows are precisely $\{\lambda_i + k - i + 1 : 1 \le i \le k\}$, and the labels at the end of the columns are $\{k + j - \lambda'_j : 1 \le j \le t\}$. □

Theorem 4.3.24 (Dual Jacobi–Trudi identity) *Let $\lambda \vdash n$ with $\ell(\lambda) \le n$, set $\lambda_{\ell(\lambda)+1} = \lambda_{\ell(\lambda)+2} = \cdots = \lambda_n = 0$ and denote by $\lambda' = (\lambda'_1, \lambda'_2, \ldots, \lambda'_n)$ the conjugate partition. Then*

$$\begin{vmatrix} h_{\lambda_1}(x) & h_{\lambda_1+1}(x) & \cdots & h_{\lambda_1+n-1}(x) \\ h_{\lambda_2-1}(x) & h_{\lambda_2}(x) & \cdots & h_{\lambda_2+n-2}(x) \\ \vdots & \vdots & & \vdots \\ h_{\lambda_n-n+1}(x) & h_{\lambda_n-n+2}(x) & \cdots & h_{\lambda_n}(x) \end{vmatrix} = \begin{vmatrix} e_{\lambda'_1}(x) & e_{\lambda'_1+1}(x) & \cdots & e_{\lambda'_1+n-1}(x) \\ e_{\lambda'_2-1}(x) & e_{\lambda'_2}(x) & \cdots & e_{\lambda'_2+n-2}(x) \\ \vdots & \vdots & & \vdots \\ e_{\lambda'_n-n+1}(x) & e_{\lambda'_n-n+2}(x) & \cdots & e_{\lambda'_n}(x) \end{vmatrix} \tag{4.47}$$

and therefore the Schur polynomial s_λ is also equal to the second determinant.

Proof Set $k = \ell(\lambda)$, $t = \ell(\lambda')$ and consider the matrices $\mathcal{H}_{k+t}$ and $\mathcal{E}_{k+t}$ as in Lemma 4.3.20. Let $\sigma, \pi \in \mathfrak{S}_{k+t}$ be defined by:

$$\begin{cases} \pi(i) = \lambda_i + k - i + 1, & i = 1, 2, \ldots, k \\ \pi(j) = j - \lambda'_j, & j = k+1, k+2, \ldots, k+t \end{cases}$$

(cf. Lemma 4.3.23) and

$$\begin{cases} \sigma(i) = k - i + 1, & i = 1, 2, \ldots, k \\ \sigma(j) = j, & j = k+1, k+2, \ldots, k+t. \end{cases}$$

Applying Lemma 4.3.22 in this setting, we get

$$\det(\widetilde{\mathcal{H}}_{k+t}) = \varepsilon(\pi)\varepsilon(\sigma)\det(\mathcal{H}_{k+t})\det(\widetilde{\mathcal{E}}_{k+t}). \tag{4.48}$$

Notice that $\widetilde{\mathcal{H}}_{k+t}$ is the $k \times k$ matrix whose (i, j)-entry is h_{λ_i+j-i} (indeed, $\pi(i) - \sigma(j) = \lambda_i + k - i + 1 - (k - j + 1) = \lambda_i + j - i$), and $\widetilde{\mathcal{E}}_{k+t}$ is the $t \times t$ matrix whose (i, j)-entry is $e_{\lambda'_j+i-j}(-1)^{\lambda'_j+i-j}$, so that

$$\begin{aligned}\det\left(\widetilde{\mathcal{E}}_{k+t}\right) &= \det\left(e_{\lambda'_j+i-j}\right) \cdot (-1)^{\sum(\lambda'_j-j)}(-1)^{\sum i}\\ &= \det\left(e_{\lambda'_j+i-j}\right)(-1)^{|\lambda'|}.\end{aligned}$$

Moreover

$$\varepsilon(\pi)\varepsilon(\sigma) = (-1)^{\sum(\lambda_i+k-i)}(-1)^{\sum(k-i)} = (-1)^{|\lambda|} \equiv (-1)^{|\lambda'|}$$

(to compute $\varepsilon(\pi)$, note that $(\pi(1), \pi(2), \ldots, \pi(k+t)) = (\lambda_1 + k, \lambda_2 + k - 1, \ldots, \lambda_k + 1, k - \lambda'_1 + 1, k - \lambda'_2 + 2, \ldots, k - \lambda'_t + t))$. Finally, if $\lambda_{k+1} = \lambda_{k+2} = \cdots = \lambda_n = 0$ then the determinant on the left-hand side of (4.47) is equal to the determinant of its upper left $k \times k$ submatrix; similarly for the right-hand side of (4.47).

Taking into account (4.48) and the above remarks we get immediately the identity (4.47). □

Extend the definition of ψ^α (see Lemma 4.2.1) to all sequences $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ of integers as follows. We set $\psi^\alpha = 0$ if any of the α_j's are negative, and then we define ψ^α to be the character of M^α, if α is a composition. Note that, in the notation of Proposition (4.3.11), we have

$$\psi^\alpha = \psi^{\alpha^1} \circ \psi^{\alpha^2} \circ \cdots \circ \psi^{\alpha^k}, \tag{4.49}$$

where ψ^{α^j} is the character of the trivial representation of $\mathfrak{S}_{\alpha_j}$ if $\alpha_j \geq 1$, $\psi^0 = 1$ and $\psi^{\alpha_j} = 0$ if $\alpha_j < 0$. Then we have:

Theorem 4.3.25 (Determinantal formula for χ^λ) *For any partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ of n, the irreducible character χ^λ of $\mathfrak{S}_n$ is given by*

$$\chi^\lambda = \sum_{\pi \in \mathfrak{S}_k} \varepsilon(\pi)\psi^{(\lambda_1+\pi(1)-1, \lambda_2+\pi(2)-2, \cdots, \lambda_k+\pi(k)-k)}. \tag{4.50}$$

Proof It is an immediate consequence of the Jacobi–Trudi identity (Corollary 4.3.18) and of Theorem 4.3.9 and Corollary 4.3.12. □

The formula (4.50) may be also written as a formal determinant:

$$\chi^\lambda = \begin{vmatrix} \psi^{\lambda_1} & \psi^{\lambda_1+1} & \dots & \psi^{\lambda_1+k-1} \\ \psi^{\lambda_2-1} & \psi^{\lambda_2} & \dots & \psi^{\lambda_2+k-2} \\ \vdots & \vdots & & \vdots \\ \psi^{\lambda_k-k+1} & \psi^{\lambda_k-k+2} & \dots & \psi^{\lambda_k} \end{vmatrix}$$

keeping into account of Proposition 4.3.11 and (4.49). As a Corollary, we get a stronger version of Proposition 4.2.4.

Corollary 4.3.26 *Every irreducible character of $\mathfrak{S}_n$ may be written as a linear combination, with coefficients in $\{-1, 1\}$, of the permutation characters ψ^λ.*

Corollary 4.3.27 *The set $\{\psi^\lambda : \lambda \vdash n\}$ is a vector space basis for the center of $\mathfrak{S}_n$.*

4.4 The Theorem of Jucys and Murphy

The YJM elements were introduced independently by Jucys [70] and Murphy [97] in order to prove the following remarkable result: every element in the center of the group algebra $L(\mathfrak{S}_n)$ may be written as a symmetric polynomial in the YJM elements $X_2, X_3, \dots, X_n$. This was also rediscovered by G. Moran [93].

In Section 4.4.2 we reproduce Murphy's original proof which is elementary and similar to the proof of Olshanskii's theorem (Theorem 3.2.6). In Section 4.4.4 we give an alternative proof due to Adriano Garsia, that is more constructive and gives an explicit expression of the characters of $\mathfrak{S}_n$ as symmetric polynomials in the YJM elements.

4.4.1 Minimal decompositions of permutations as products of transpositions

In this section, we present a technical but interesting result on the minimal decomposition of a permutation as a product of transpositions.

Lemma 4.4.1 *Let $\theta, \pi \in \mathfrak{S}_n$ and suppose that there exist distinct $i, j, c \in \{1, 2, \dots, n\}$ such that (i) $\theta = (j \to c \to j)\pi(i \to c \to i)$ and (ii) $\theta(c) = c$. Then,*

(1) *if $\pi(c) = c$, then $\theta = \pi$;*
(2) *if $\pi(c) = k \neq c$, then $k \neq j$ and $\theta = (j \to k \to j)(k \to c \to k)\pi$.*

Proof First of all, note that, in any case, (i) and (ii) imply that (iii) $\pi(i) = j$.

Suppose that $\pi(c) = c$. Then, we also have (iv) $\theta(i) = j$. Moreover, if $h \neq c, i$, then (by (ii) and (iv)) $\theta(h) \neq c, j$, so that $\pi(h) = [(j \to c \to j)\theta(i \to c \to i)](h) = \theta(h)$. It then follows that $\pi \equiv \theta$ and (1) is proved.

Suppose now that $\pi(c) = k \neq c$. Set $\sigma = (j \to k \to j)(k \to c \to k)\pi$. Observe that (iii) implies that $k \neq j$, so that $\theta(i) = k$. We then deduce that

$$\sigma(i) = [(j \to k \to j)(k \to c \to k)\pi](i) = [(j \to k \to j)(k \to c \to k)](j) = k \\ = \theta(i)$$

and

$$\sigma(c) = [(j \to k \to j)(k \to c \to k)\pi](c) = [(j \to k \to j)(k \to c \to k)](k) = c \\ = \theta(c).$$

Moreover, if $h \neq i, c$, then $r := \pi(h) \neq j, k$ and therefore

$$\begin{aligned} \sigma(h) &= [(j \to k \to j)(k \to c \to k)\pi](h) \\ &= [(j \to k \to j)(k \to c \to k)](r) \\ &= \begin{cases} r & \text{if } r \neq c \\ j & \text{if } r = c \end{cases} \end{aligned}$$

and, similarly

$$\begin{aligned} \theta(h) &= [(j \to c \to j)\pi(i \to c \to i)](h) \\ &= [(j \to c \to j)\pi](h) \\ &= (j \to c \to j)(r) \\ &= \begin{cases} r & \text{if } r \neq c \\ j & \text{if } r = c. \end{cases} \end{aligned}$$

In conclusion, $\sigma \equiv \theta$ and also (2) is proved. □

Example 4.4.2

(i) For $i = 1$, $j = 2$ and $c = 3$, with $\pi = (1 \to 2 \to 4 \to 1)$ and $\theta = (2 \to 3 \to 2)(1 \to 2 \to 4 \to 1)(1 \to 3 \to 1)$, we have

$$(2 \to 3 \to 2)(1 \to 2 \to 4 \to 1)(1 \to 3 \to 1) = (1 \to 2 \to 4 \to 1)$$

(note that $\theta(c) = \theta(3) = 3 = c$. Moreover, $\pi(c) = \pi(3) = 3 = c$: we are in case (1), $\theta = \pi$).

(ii) If i, j, k and c are all distinct and $\pi = (k \to c \to k)(i \to j \to i)$, $\theta = (i \to k \to j \to i)$, then we have $\theta = (j \to c \to j)\pi(i \to c \to i) = (j \to k \to j)(k \to c \to k)\pi$: we are in case (2).

Proposition 4.4.3 *Let* $t_1, t_2, \dots, t_m \in \mathfrak{S}_n$ *be distinct transpositions and set* $\sigma = t_1 t_2 \cdots t_m$. *Suppose that there exist* $c \in \{1, 2, \dots, n\}$ *and* $\ell = \ell(\sigma) \in \{1, 2, \dots, m\}$ *such that* $\sigma(c) = c$ *and* $t_\ell = (i \to c \to i)$. *Then, there exist an integer* $h \geq 1$ *and distinct transpositions* $t'_1, t'_2, \dots, t'_{m-2h} \in \mathfrak{S}_n$ *such that*

$$\sigma = t'_1 t'_2 \cdots t'_{m-2h}. \tag{4.51}$$

In other words, if a permutation σ *is a product of* m *distinct transpositions and a fixed point of* σ *appears in one of these transpositions (and therefore, necessarily, also in a second one), then* σ *can be expressed as a product of* $m - 2h$ *(distinct) transpositions, with* $h \geq 1$.

Proof First of all, observe that given an expression $\sigma = t'_1 t'_2 \cdots t'_m$, with $t'_1, t'_2, \dots, t'_m$ transpositions and $t'_a = t'_b$ for some $1 \leq a < b \leq m$, then

$$\sigma = t'_1 t'_2 \cdots t'_{a-1} t''_{a+1} t''_{a+2} \cdots t''_{b-1} t'_{b+1} \cdots t'_m \tag{4.52}$$

where $t''_{a+1}, t''_{a+2}, \dots, t''_{b-1}$ are the conjugates of $t'_{a+1}, t'_{a+2}, \dots, t'_{b-1}$ by t'_a. Moreover, (4.52) may be used repeatedly until we reach an expression with all distinct transpositions.

From our hypotheses, it follows that necessarily there exist $q = q(\sigma) \in \{1, 2, \dots, m\}$ such that $t_q = (j \to c \to j)$ with i, j, c distinct (recall that, by assumption, the t_s's are all distinct). To fix the ideas, suppose that $q < \ell$ and that q and ℓ are minimal and maximal, respectively, that is, $t_s \neq (f \to c \to f)$ for all $1 \leq s < q$ and $\ell < k < t$ and $f \in \{1, 2, \dots, n\}$. Set $\theta = t_q t_{q+1} \cdots t_\ell$ and observe that $\theta(c) = c$. Given a product of transpositions $\rho = s_1 s_2 \cdots s_N$ we denote by $n_c(\rho)$ the number of transpositions s_i's containing c. Also set $\pi = t_{q+1} t_{q+2} \cdots t_{\ell-1}$. Note that $n_c(\pi) = n_c(\theta) - 2 = n_c(\sigma) - 2$. We distinguish two cases.

First case: $\pi(c) = c$. This holds, for instance, if $n_c(t_{q+1} t_{q+2} \cdots t_{\ell-1}) = 0$. In this case, Lemma 4.4.1.(1) guarantees that $\pi = \theta$ and therefore

$$\begin{aligned} \sigma &= t_1 t_2 \cdots t_{q-1} \theta t_{\ell+1} \cdots t_m \\ &= t_1 t_2 \cdots t_{q-1} \pi t_{\ell+1} \cdots t_m \\ &= t_1 t_2 \cdots t_{q-1} (t_{q+1} t_{q+2} \cdots t_{\ell-1}) t_{\ell+1} \cdots t_m. \end{aligned}$$

Thus, in this case, we are done.

Second case: $\pi(c) \neq c$. Now, by applying Lemma 4.4.1.(2), we have $\theta = (j \to k \to j)(k \to c \to k)\pi$ for a suitable $k \neq j, c$, and therefore

$$\begin{aligned} \sigma &= t_1 t_2 \cdots t_{q-1} \theta t_{\ell+1} \cdots t_m \\ &= t_1 t_2 \cdots t_{q-1} [(j \to k \to j)(k \to c \to k)\pi] t_{\ell+1} \cdots t_m \\ &= t_1 t_2 \cdots t_{q-1} [(j \to k \to j)(k \to c \to k) t_{q+1} t_{q+2} \cdots t_{\ell-1}] t_{\ell+1} \cdots t_m. \end{aligned} \tag{4.53}$$

Denote by

$$\begin{aligned} \sigma' &= t_1 t_2 \cdots t_{q-1} [(j \to k \to j)(k \to c \to k) t_{q+1} t_{q+2} \cdots t_{\ell-1}] t_{\ell+1} \cdots t_m \\ &=: t_1' t_2' \cdots t_m' \end{aligned}$$

the new expression of σ given by (4.53). If $t_1', t_2', \dots, t_m'$ are not distinct we may use (4.52) repeatedly, until we reach (4.51).

Otherwise, denoting by $q' = q(\sigma')$ and $\ell' = \ell(\sigma')$ the corresponding indices in $\{1, 2, \dots, m\}$, we have $q' = q + 1$ (in fact $t_{q'} = (k \to c \to k)$) and $\ell' \leq \ell$. Note that $n_c(\sigma') = n_c(\sigma) - 1$. Therefore, if $\theta' = t_{q'} t_{q'+1} \cdots t_{\ell'}$ and $\pi' = t_{q'+1} t_{q'+2} \cdots t_{\ell'-1}$, we have $n_c(\pi') = n_c(\theta') - 2 = n_c(\sigma') - 2 = (n_c(\sigma) - 1) - 2 = n_c(\pi) - 1$. We can therefore apply Lemma 4.4.1.(2) to π' and θ'. Eventually, after b steps in which we follow the procedure, we arrive at a new expression for σ given by $\sigma^{(b)} = t_1^{(b)} t_2^{(b)} \cdots t_m^{(b)}$, with a corresponding $\pi^{(b)}$ satisfying $\pi^{(b)}(c) = c$ (for instance, if $n_c(\pi^{(b)}) = 0$), so that we are in the first case and the proof is complete. □

Theorem 4.4.4 *Let $\sigma \in \mathfrak{S}_n$. Suppose that σ has cycle structure $\mu = (\mu_1, \mu_2, \dots, \mu_r) \vdash n$, in other words, $\sigma = \omega_1 \omega_2 \cdots \omega_r$, with each ω_j a cycle of length μ_j, and $\omega_1, \omega_2, \dots, \omega_r$ all disjoint. Let σ also be expressed as $\sigma = t_1 t_2 \cdots t_m$ with t_i's transpositions. Then, $m \geq n - \ell(\mu)$. Moreover, if $m = n - \ell(\mu)$, then the transpositions $t_1, t_2, \dots, t_m$ may be rearranged in such a way that*

$$\begin{aligned} \omega_1 &= t_1 t_2 \cdots t_{\mu_1 - 1}, \\ \omega_2 &= t_{\mu_1} t_{\mu_1 + 1} \cdots t_{\mu_1 + \mu_2 - 2}, \\ \omega_3 &= t_{\mu_1 + \mu_2 - 1} t_{\mu_1 + \mu_2} \cdots t_{\mu_1 + \mu_2 + \mu_3 - 3}, \\ &\cdots\cdots \\ \omega_r &= t_{m - \mu_r + 2} t_{m - \mu_r + 3} \cdots t_m \end{aligned}$$

(note that $m - \mu_r + 2 = \mu_1 + \mu_2 + \cdots + \mu_{r-1} - r + 2$ and $\mu_1 + \mu_2 + \cdots + \mu_r - r = n - r = m$).

Proof We proceed by induction on $n - \ell(\mu)$. For $n - \ell(\mu) = 1$ and 2, the statement is trivial. It is also trivial when $\mu_1 = 2$, that is, the cycle decomposition of σ consists of disjoint transpositions. Indeed, the product of m transpositions moves at most $2m$ elements and the maximum is achieved when such transpositions are all disjoint.

For the general case, we use the following decomposition. Let $c \in \{1, 2, \ldots, n\}$ be moved by σ, say $\sigma(c) = k$, with $k \neq c$, and let $i = \sigma^{-1}(c)$. Observe that c is contained in a cycle of length $\mu_j \geq 2$ of the form $(\cdots \to i \to c \to k \to \cdots)$. Setting $\omega = \sigma(i \to c \to i)$, we have that $\omega(c) = c$ and ω has cycle type $\overline{\mu} = (\mu_1, \mu_2, \ldots, \mu_{j-1}, \mu_j - 1, \mu_{j+1}, \ldots, \mu_r, 1)$ (indeed, the jth cycle becomes $(\cdots \to i \to k \to \cdots)$ of length $\mu_j - 1$ and we have an extra cycle (c) of length one. We may also have $i = k$, so that $\mu_j = 2$ and $\mu_j - 1 = 1$ is another trivial cycle). Therefore, $n - \ell(\overline{\mu}) = n - (\ell(\mu) + 1) < n - \ell(\mu)$.

We first prove that $m \geq n - \ell(\mu)$. Suppose, by contradiction, that $\sigma = t_1 t_2 \cdots t_m$ with $t_1, t_2, \ldots, t_m$ distinct transpositions, and $m < n - \ell(\mu)$ (so that, necessarily, $m \leq n - \ell(\mu) - 2$: to respect the parity of σ). Then,

$$\omega = t_1 t_2 \cdots t_m (i \to c \to i)$$

with c fixed by ω. By applying Proposition 4.4.3, we obtain an expression of ω of the form $\omega = t'_1 t'_2 \cdots t'_{m-2} t'_{m-2h+1}$, contradicting the minimality of $n - \ell(\overline{\mu})$ for ω.

Suppose now that $m = n - \ell(\mu)$. In this case, we may suppose that c appears only in one transposition, say t_h. Indeed, $m = n - \ell(\mu) = \mu_1 + \mu_2 + \cdots + \mu_s - s$ (if $\mu_s \geq 2$ and $\mu_{s+1} = \mu_{s+2} = \cdots = \mu_r = 1$) and σ moves exactly $\mu_1 + \mu_2 + \cdots + \mu_s$ numbers in $\{1, 2, \ldots, n\}$, so that such a c (moved by σ and appearing only in one transposition) exists.

Let $t_h = (j \to c \to j)$. Set again $\omega = t_1 t_2 \cdots t_{h-1} (j \to c \to j) t_{h+1} \cdots t_m \cdot \cdot (i \to c \to i)$, where $i = \sigma^{-1}(c)$; now $t_1, t_2, \ldots, t_{h-1}, t_{h+1}, \ldots, t_m$ do not move c. Again, ω is of cycle type $\overline{\mu}$ as above. We distinguish two cases.

First case: $i = j$. In this case, if one of the $t_{h+1}, t_{h+2}, \ldots, t_m$ moves i, setting $t'_\ell = (i \to c \to i) t_\ell (i \to c \to i)$ for $\ell = h+1, h+2, \ldots, m$, we get

$$\omega = t_1 t_2 \cdots t_{h-1} t'_{h+1} t'_{h+2} \cdots t'_m \tag{4.54}$$

By Proposition 4.4.3, (4.54) contradicts the inductive hypothesis (at least one of the $t'_{h+1}, t'_{h+2}, \ldots, t'_m$ moves c). Therefore, none of the $t_{h+1}, t_{h+2}, \ldots, t_m$ can move i and we must have

$$\omega = t_1 t_2 \cdots t_{h-1} t_{h+1} t_{h+2} \cdots t_m. \tag{4.55}$$

By the inductive hypothesis, $t_1, t_2, \ldots t_{h-1}, t_{h+1}, t_{h+2} \ldots t_m$ may be grouped respecting the cycle structure of ω, and therefore this also holds for all the $t_1, t_2 \ldots, t_m$ and σ.

Second case: $i \neq j$. Now, we may apply Lemma 4.4.1.(1), obtaining again (4.55). We then conclude as before. □

4.4.2 The Theorem of Jucys and Murphy

The following is the theorem of Jucys–Murphy. The proof that we present is based on Murphy's paper [97], but with more details with respect to the original source.

Theorem 4.4.5 (Theorem of Jucys and Murphy) *The center $Z(n)$ of the group algebra of the symmetric group $\mathfrak{S}_n$ coincides with the algebra of all symmetric polynomials in the YJM elements $X_2, X_3, \dots, X_n$.*

Proof First of all, we show that if p is a symmetric polynomial, then $p(X_2, X_3, \dots, X_n) \in Z(n)$. By the fundamental theorem on symmetric polynomials 4.1.12, it suffices to show that

$$s_i e_k(X_2, X_3, \dots, X_n) s_i = e_k(X_2, X_3, \dots, X_n) \tag{4.56}$$

for each elementary symmetric polynomial e_k and each adjacent transposition s_i. From the commutation relations (3.30) and (3.31) we get

$$s_i(X_i + X_{i+1})s_i = X_{i+1} - s_i + X_i + s_i = X_i + X_{i+1}$$

and

$$\begin{aligned} s_i X_i X_{i+1} s_i &= s_i X_i s_i \cdot s_i X_{i+1} s_i \\ &= (X_{i+1} - s_i)(X_i + s_i) \\ &= X_i X_{i+1} - s_i X_i + X_{i+1} s_i - 1 \\ &= X_i X_{i+1}. \end{aligned}$$

Moreover, from (3.29) we get that s_i commutes with every X_j with $j \neq i, i+1$.

Now, we can write each elementary symmetric function $e_k(x_2, x_3, \dots, x_n)$ in the form

$$e_k(x_2, x_3, \dots, x_n) = f_1 + (x_i + x_j) f_2 + x_i x_j f_3,$$

where f_1, f_2, f_3 do not contain the variables x_i and x_{i+1} (for instance, we have $e_4(x_2, x_3, x_4, x_5, x_6) = (x_2 + x_3)x_4x_5x_6 + x_2x_3(x_4x_5 + x_4x_6 + x_5x_6)$).

From this, and from the commutation relations proved above, we easily get (4.56) and this implies that every symmetric polynomial in $X_2, X_3, \dots, X_n$ belongs to $Z(n)$.

We now prove that the elements $p(X_2, X_3, \dots, X_n)$, with p symmetric polynomial, *span* the whole $Z(n)$. Recall that $\dim Z(n)$ equals the number of conjugacy classes in $\mathfrak{S}_n$ and therefore, by Proposition 3.1.3, it is equal to the number of partitions of n. Therefore, it suffices to construct a set of linearly independent symmetric polynomials in $X_2, X_3, \dots X_n$ indexed by the partitions of n. For each $\mu = (\mu_1, \mu_2, \dots, \mu_t) \vdash n$ (with $\mu_t > 0$), set

$$X_\mu = \sum X_{i_1}^{\mu_1 - 1} X_{i_2}^{\mu_2 - 1} \cdots X_{i_t}^{\mu_t - 1}$$

where the sum runs over all *distinct* monomials $X_{i_1}^{\mu_1 - 1} X_{i_2}^{\mu_2 - 1} \cdots X_{i_t}^{\mu_t - 1}$, with $i_1, i_2, \dots, i_t$ distinct indices in $\{2, 3, \dots, n\}$. This means that if a monomial of this form may be obtained in two different ways corresponding to different choices of the indices $i_1, i_2, \dots, i_t$, then it appears in X_μ with coefficients equals to 1. For instance, if $\mu = (a, a) \vdash n = 2a$ we have

$$X_{a,a} = \sum_{2 \le i < j \le n} X_i^{a-1} X_j^{a-1}.$$

We also set $X_{(1^n)} = 1$.

Let $\mu = (\mu_1, \mu_2, \dots, \mu_t) \vdash n$. Suppose that $\mu_s > 1$ and that $\mu_{s+1} = \mu_{s+2} = \cdots = \mu_t = 1$.

Claim 1 *Among all the permutations $\sigma \in \mathfrak{S}_n$ in the support of the group algebra element $X_\mu \in L(\mathfrak{S}_n)$, those with the smallest number of fixed points have the form*

$$\begin{aligned} &(i_1 \to a_1 \to a_2 \to \cdots \to a_{\mu_1 - 1} \to i_1) \\ &\quad \cdot (i_2 \to b_1 \to b_2 \to \cdots \to b_{\mu_2 - 1} \to i_2) \cdots \\ &\quad \cdots (i_s \to c_1 \to c_2 \to \cdots \to c_{\mu_s - 1} \to i_s) \end{aligned} \tag{4.57}$$

where the numbers

$$i_1, i_2, \dots, i_s, a_1, a_2, \dots, a_{\mu_1 - 1}, b_1, b_2, \dots, b_{\mu_2 - 1}, \dots, c_1, c_2, \dots, c_{\mu_s - 1} \tag{4.58}$$

are all distinct. Moreover, each permutation minimizing the number of fixed points appears in X_μ with coefficient equal to 1.

Proof of Claim 1 Let $\sigma \in \mathfrak{S}_n$ belong to the support of X_μ. Then σ belongs to the support of a monomial, say $X_{i_1}^{\mu_1 - 1} X_{i_2}^{\mu_2 - 1} \cdots X_{i_s}^{\mu_s - 1} = \prod_{j=1}^s X_{i_j}^{\mu_j - 1}$. Now,

$$X_{i_j}^{\mu_j - 1} = \big((i_j \to 1 \to i_j) + (i_j \to 2 \to i_j) + \cdots + (i_j \to i_j - 1 \to i_j)\big)^{\mu_j - 1},$$

and any $\pi_j \in \mathfrak{S}_n$ in the support of $X_{i_j}^{\mu_j-1}$ is therefore of the form

$$(i_j \to d_{\mu_j-1} \to i_j) \cdot (i_j \to d_{\mu_j-2} \to i_j) \cdots (i_j \to d_1 \to i_j), \tag{4.59}$$

where $d_1, d_2, \ldots, d_{\mu_j-1} \in \{1, 2, \ldots, i_j - 1\}$. Multiplying these π_j's, we then deduce that σ may be written in the form

$$\begin{aligned}(i_1 \to a_{\mu_1-1} \to i_1) \cdot (i_1 \to a_{\mu_1-2} \to i_1) \cdots (i_1 \to a_1 \to i_1) \cdot \\ \cdot (i_2 \to b_{\mu_2-1} \to i_2) \cdot (i_2 \to b_{\mu_2-2} \to i_2) \cdots (i_2 \to b_1 \to i_2) \cdots \\ \cdots (i_s \to c_{\mu_s-1} \to i_s) \cdot (i_s \to c_{\mu_s-2} \to i_s) \cdots (i_s \to c_1 \to i_s).\end{aligned} \tag{4.60}$$

We then deduce that σ can move at most $\mu_1 + \mu_2 + \cdots + \mu_s$ numbers, and this maximum is attained when the numbers in (4.58) are all distinct. In this case, the product of transpositions (4.59) equals the cycle

$$(i_j \to d_1 \to d_2 \to \cdots \to d_{\mu_j-1} \to i_j)$$

and (4.60) yields exactly (4.57); moreover, (4.60) is the unique way to write (4.57) as a product of $\mu_1 - 1$ transpositions coming from X_{i_1}, $\mu_2 - 1$ transpositions from $X_{i_2}, \cdots$, and $\mu_s - 1$ transpositions from X_{i_s}. In this case, the coefficient of σ in X_μ is equal to one. □

We introduce some notation from [97]. Let $\mu = (\mu_1, \mu_2, \ldots, \mu_t) \vdash n$, with $\mu_s > 1$ and $\mu_{s+1} = \mu_{s+2} = \cdots = \mu_t = 1$. We set $\hat{\mu}_0 = 0$ and, for $i = 1, 2, \ldots, t$,

$$\hat{\mu}_i = \mu_1 + \mu_2 + \cdots + \mu_i.$$

Claim 2 *Consider the following permutation of type* (4.57)*:*

$$\sigma_\mu = \prod_{i=1}^{s} (n - \hat{\mu}_{i-1} \to n - \hat{\mu}_{i-1} - 1 \to \cdots \to n - \hat{\mu}_i + 1 \to n - \hat{\mu}_{i-1}). \tag{4.61}$$

Then σ_μ belong to the support of X_μ, with coefficient 1.

Proof of Claim 2 Let us show that the permutation σ_μ belongs to the support of X_μ. It suffices to show that it belongs to the support of the monomial

$$(X_n)^{\mu_1-1}(X_{n-\hat{\mu}_1})^{\mu_2-1}(X_{n-\hat{\mu}_2})^{\mu_3-1} \cdots (X_{\mu_t})^{\mu_t-1}.$$

Now we have, for $i = 1, 2, \ldots, s$,

$$
\begin{aligned}
(X_{n-\hat{\mu}_{i-1}})^{\mu_i-1} &= \{(n-\hat{\mu}_{i-1} \to 1 \to n-\hat{\mu}_{i-1}) + (n-\hat{\mu}_{i-1} \to 2 \to n-\hat{\mu}_{i-1}) + \cdots \\
&\cdots + (n-\hat{\mu}_{i-1} \to n-\hat{\mu}_i \to n-\hat{\mu}_{i-1}) + \\
&+ [(n-\hat{\mu}_{i-1} \to n-\hat{\mu}_i + 1 \to n-\hat{\mu}_{i-1}) + (n-\hat{\mu}_{i-1} \to n-\hat{\mu}_i + 2 \to n-\hat{\mu}_{i-1}) + \cdots \\
&\cdots + (n-\hat{\mu}_{i-1} \to n-\hat{\mu}_{i-1} - 1 \to n-\hat{\mu}_{i-1})]\}^{\mu_i-1} \\
&= (n-\hat{\mu}_{i-1} \to n-\hat{\mu}_i + 1 \to n-\hat{\mu}_{i-1}) \cdot (n-\hat{\mu}_{i-1} \to n-\hat{\mu}_i + 2 \to n-\hat{\mu}_{i-1}) \cdots \\
&\cdots (n-\hat{\mu}_{i-1} \to n-\hat{\mu}_{i-1} - 1 \to n-\hat{\mu}_{i-1}) + \text{other terms} \\
&= (n-\hat{\mu}_{i-1} \to n-\hat{\mu}_{i-1} - 1 \to n-\hat{\mu}_{i-1} - 2 \to \cdots \to n-\hat{\mu}_i + 1 \to n-\hat{\mu}_{i-1}) + \\
&+ \text{other terms}
\end{aligned}
$$

Note that all the numbers in $\{1, 2, \ldots, n\}$ appearing in the cycles factoring the permutation σ_μ are all distinct (thus (4.61) is the cycle decomposition of σ_μ), so that, an application of the second part of Claim 1 ends the proof of Claim 2. □

We now introduce an order relation between the partitions of n which is a slight modification of the lexicographic order (cf. end of Section 3.6.1). Let $\lambda, \mu \vdash n$ and denote by $\ell(\lambda)$ and $\ell(\mu)$ their lengths (cf. Section 4.1.1). We write $\lambda \gtrsim \mu$ if $\ell(\lambda) > \ell(\mu)$ or $\ell(\lambda) = \ell(\mu)$ and $\lambda > \mu$ in the lexicographic order. Clearly, $\gtrsim$ is a total order on the set of all partitions of n.

Claim 3 *Let $\lambda, \mu \vdash n$ and suppose that σ_λ appears in X_μ. Then $\lambda \gtrsim \mu$.*

Proof of Claim 3 Each permutation which appears in X_μ is the product of at most $n - \ell(\mu)$ transpositions. Therefore, from Theorem 4.4.4, if σ_λ appears in X_μ, then $n - \ell(\lambda) \le n - \ell(\mu)$, that is $\ell(\mu) \le \ell(\lambda)$. If $\ell(\lambda) > \ell(\mu)$, we are done.

Suppose that $\ell(\lambda) = \ell(\mu)$, say $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_t)$ and $\mu = (\mu_1, \mu_2, \ldots, \mu_t)$. If σ_λ appears in the monomial $X_{i_1}^{\mu_1-1} X_{i_2}^{\mu_2-1} \cdots X_{i_t}^{\mu_t-1}$ of X_μ, then Theorem 4.4.4 ensures that all the μ_j transpositions taken from X_{i_j} contribute to a single cycle of σ_λ (the cycle of σ_λ containing i_j). This means that each part of λ is the sum of one or more parts of μ. Therefore, $\lambda \trianglerighteq \mu$ and, by Proposition 3.6.5, $\lambda \ge \mu$. This ends the proof of the claim. □

Note that if $\lambda \ne \mu, \ell(\lambda) = \ell(\mu)$ and σ_λ appears in X_μ, then also the following condition must be satisfied: the number of fixed points in σ_λ must be strictly greater than the number of fixed points in σ_μ ($\equiv$ the minimum number of fixed points for a permutation in X_μ). This property follows from Claim 1, but it will not used any more in the sequel. From Claim 2 and Claim 3, it follows that the X_μ's are linearly independent and this ends the proof of the theorem. □

4.4.3 Bernoulli and Stirling numbers

In this section, we give some basic notions on Bernoulli numbers, falling factorials and Stirling numbers.

Let $a_0, a_1, \dots, a_n$ be a sequence of numbers (or polynomials). The associated *exponential generating function* is given by

$$\sum_{k=0}^{\infty} a_k \frac{z^k}{k!}$$

for $z \in \mathbb{C}$.

The *Bernoulli numbers* are the numbers recursively defined by setting $B_0 = 1$ and, for $m \geq 1$,

$$\sum_{k=0}^{m} \binom{m+1}{k} B_k = 0. \tag{4.62}$$

Exercise 4.4.6 Show that $B_1 = -1/2$, $B_2 = 1/6$, $B_3 = 0$, $B_4 = -1/30$, $B_5 = 0$, $B_6 = 1/42$, $B_7 = 0$, $B_8 = -1/30$, $B_9 = 0$ and $B_{10} = 5/66$.

All the B_{2m+1}'s are equal to zero, for $m \geq 1$: this will be proved in Exercise 4.4.9.

We also define the functions $R_n(x)$, $n = 0, 1, 2, \dots$ of a real variable x, by means of the following exponential generating function:

$$\sum_{n=0}^{\infty} \frac{z^n}{n!} R_n(x) = e^z \frac{e^{zx} - 1}{e^z - 1}. \tag{4.63}$$

Note that by power series expansion, the $R_n(x)$'s are polynomials.

In the following lemma we collect the basic properties of the Bernoulli numbers, of the polynomials $R_n(x)$ and their relations with the sums of powers of consecutive integers. For more on Bernoulli numbers and their applications, see the monographs by Graham, Knuth and Patashnik [51] and Jordan [68].

Lemma 4.4.7

(i) *The Bernoulli numbers have the following exponential generating function:*

$$\sum_{k=0}^{\infty} B_k \frac{z^k}{k!} = \frac{z}{e^z - 1}.$$

(ii) *For $n \geq 1$, the polynomials $R_n(x)$ may be expressed in terms of the Bernoulli numbers by the following explicit formula:*

$$R_n(x) = \frac{1}{n+1} \sum_{k=0}^{n} \binom{n+1}{k} (x+1)^{n-k+1} B_k.$$

(iii) *For $m \geq 1$ one has $R_0(m) = m$ and, for $n \geq 1$,*

$$R_n(m) = \sum_{t=1}^{m} t^n.$$

Proof (i) Setting $n = m + 1$ and adding B_n to both sides, (4.62) may be written in the form

$$\sum_{k=0}^{n} \binom{n}{k} B_k = \begin{cases} B_n & n = 2, 3, \dots \\ 1 + B_1 & n = 1 \\ B_0 & n = 0. \end{cases} \tag{4.64}$$

Set $b(z) = \sum_{k=0}^{\infty} \frac{B_k}{k!} z^k$. Computing the exponential generating function of the left-hand side of (4.64), we get

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{z^n}{n!} \sum_{k=0}^{n} \binom{n}{k} B_k &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{z^{n-k}}{(n-k)!} \frac{B_k}{k!} z^k \\ \text{(Cauchy product)} \quad &= \left(\sum_{m=0}^{\infty} \frac{z^m}{m!} \right) \cdot \left(\sum_{k=0}^{\infty} \frac{B_k}{k!} z^k \right) \\ &= e^z b(z). \end{aligned}$$

On the other hand, the exponential generating function of the right-hand side of (4.64) is simply $\sum_{n=0}^{\infty} B_n \frac{z^n}{n!} + z = z + b(z)$. Therefore, $z + b(z) = e^z b(z)$, that is,

$$b(z) = \frac{z}{e^z - 1}.$$

(ii) From (i) and the power series expansion of e^z we immediatley get

$$\begin{aligned} e^z \frac{e^{zx} - 1}{e^z - 1} &= b(z) \frac{e^{z(x+1)} - e^z}{z} \\ &= \left(\sum_{k=0}^{\infty} \frac{B_k}{k!} z^k \right) \cdot \left(\sum_{h=0}^{\infty} \frac{(x+1)^{h+1} - 1}{(h+1)!} z^h \right) \\ \text{(Cauchy product)} \quad &= \sum_{n=0}^{\infty} \frac{z^n}{n!} \left[\sum_{k=0}^{n} \frac{n!}{k!(n-k+1)!} ((x+1)^{n-k+1} - 1) B_k \right] \end{aligned}$$

and therefore, from the definition of $R_n(x)$, we have

$$\begin{aligned} R_n(x) &= \frac{1}{n+1}\sum_{k=0}^{n}\binom{n+1}{k}((x+1)^{n-k+1}-1)B_k \\ \text{(by (4.62))} \quad &= \frac{1}{n+1}\sum_{k=0}^{n}\binom{n+1}{k}(x+1)^{n-k+1}B_k. \end{aligned}$$

(iii) Again, taking the exponential generating function of the sequence $\{\sum_{t=1}^{m} t^n\}_{n=0,1,2,\ldots}$ we get

$$\begin{aligned} \sum_{n=0}^{\infty}\frac{z^n}{n!}\sum_{t=1}^{m} t^n &= \sum_{t=1}^{m}\sum_{n=0}^{\infty}\frac{(zt)^n}{n!} \\ &= \sum_{t=1}^{m} e^{zt} \\ &= e^z\frac{e^{zm}-1}{e^z-1}, \end{aligned}$$

and therefore $R_n(m) = \sum_{t=1}^{m} t^n$. □

Exercise 4.4.8 Show that

$$\begin{aligned} R_0(x) &= x \\ R_1(x) &= \frac{x(x+1)}{2} \\ R_2(x) &= \frac{x(x+1)(2x+1)}{6} \\ R_3(x) &= \frac{x^2(x+1)^2}{4} \\ R_4(x) &= \frac{x(x+1)(2x+1)(3x^2+3x-1)}{30}. \end{aligned}$$

Exercise 4.4.9 Show that $B_{2m+1} = 0$ for all $m \geq 1$.

Hint: $\frac{z}{e^z-1} = -\frac{z}{2} + \frac{z}{2}\frac{e^z+1}{e^z-1}$.

Exercise 4.4.10 Set $\sigma_n(m) = \sum_{t=1}^{m} t^n$. Show that

$$\sum_{k=1}^{n}\binom{n}{k}\sigma_{n-k}(m) = (m+1)^n - 1.$$

Use this formula to compute the formulae in Exercise 4.4.8.
Hint: Compute $(m+1)^n - 1 = \sum_{t=1}^{m}[(t+1)^n - t^n]$ by means of the binomial formula $(t+1)^n - t^n = \sum_{k=1}^{n}\binom{n}{k}t^{n-k}$.

For $z \in \mathbb{C}$ and n a nonnegative integer, the *falling factorial* is defined by $[z]_n = z(z-1)\cdots(z-n+1)$, and $[z]_0 = 1$. The *raising factorial* (or

Pochhammer's symbol) is defined by $(z)_n = z(z+1)\cdots(z+n-1)$ and $(z)_0 = 1$. It is obvious that $(z)_n = [z+n-1]_n$. Clearly, the set $\{[z]_n : n \geq 0\}$ constitutes a basis for the vector space of all complex polynomials in the indeterminate z. Therefore, there exists connection coefficients $\{S(n,k) : 0 \leq k \leq n\}$ and $\{s(n,k) : 0 \leq k \leq n\}$ such that

$$[z]_n = \sum_{k=0}^{n} s(n,k) z^k \qquad n = 0, 1, 2, \ldots$$

and

$$z^n = \sum_{k=0}^{n} S(n,k)[z]_k \qquad n = 0, 1, 2, \ldots \tag{4.65}$$

The numbers $s(n,k)$ (resp. $S(n,k)$) are called the *Stirling numbers of the first* (resp. *second*) *kind*. We also set $S(n,k) = s(n,k) = 0$ for $k \leq 0$ or $k \geq n+1$. It is also clear that

$$S(n,n) = s(n,n) = 1 \quad \text{for all } n \geq 0,$$
$$S(n,0) = s(n,0) = 0 \quad \text{for all } n \geq 1.$$

In the following exercise, we give some basic properties of the Stirling numbers. Our main goal is to give another expression for the Bernoulli numbers.

Exercise 4.4.11

(1) Show that $s(n+1,k) = s(n,k-1) - ns(n,k)$ and $S(n+1,k) = S(n,k-1) + kS(n,k)$.

(2) Show that

$$\sum_{k=0}^{n} S(n,k)s(k,m) = \delta_{n,m}.$$

(3) Set $(Ef)(x) = f(x+1)$ and $(If)(x) = f(x)$, where f is any function of one real variable. Show that

$$\left[(E-I)^k x^n\right|_{x=0} = \sum_{j=0}^{n} \binom{k}{j} (-1)^{k-j} j^n$$

and

$$(E-I)^k [x]_h |_{x=0} = h! \delta_{h,k}.$$

Deduce the following formula:

$$S(n,k) = \frac{1}{k!} \sum_{j=0}^{n} \binom{k}{j} (-1)^{k-j} j^n.$$

(4) Using (3), prove that the exponential generating function for the sequence $\{S(n,k)\}_{n\in\mathbb{N}}$ is given by

$$\sum_{n=0}^{\infty} S(n,k)\frac{z^n}{n!} = \frac{(e^z-1)^k}{k!}.$$

(5) Use (2) and (4), to show that that the exponential generating function for the sequence $\{s(n,k)\}_{n\in\mathbb{N}}$ is given by

$$\sum_{n=0}^{\infty} s(n,k)\frac{z^n}{n!} = \frac{[\log(1+z)]^k}{k!}.$$

(6) Use (4) to deduce the following expression for the Bernoulli numbers:

$$B_n = \sum_{k=0}^{n}(-1)^k k!\frac{S(n,k)}{k+1} \tag{4.66}$$

for all $n \geq 0$.

Hint: (1) Use $[z]_{n+1} = (z-n)[z]_n$ and $z^{n+1} = zz^n$.

(3) For $(E-I)^k x^n$, apply the binomial expansion to $(E-I)^k$.

(6) Show that the exponential generating function of the right-hand side of (4.66) is just $\frac{z}{e^z-1}$.

In the following tables we give the Stirling numbers of the first and second kind, up to $n = 10$. These numbers will be used in the subsequent section to write explicit formulae for the characters of the symmetric group.

Table of Stirling numbers of the first kind $s(n,k)$.

$n\backslash k$	1	2	3	4	5	6	7	8	9	10
1	1									
2	−1	1								
3	2	−3	1							
4	−6	11	−6	1						
5	24	−50	35	−10	1					
6	−120	274	−225	85	−15	1				
7	720	−1764	1624	−735	175	−21	1			
8	−5040	13068	−13132	6769	−1960	322	−28	1		
9	40320	−109584	118124	−67284	22449	−4536	546	−36	1	
10	−362880	1026576	−1172700	723680	−269325	63273	−9450	870	−45	1

Table of Stirling numbers of the second kind $S(n, k)$.

$n\backslash k$	1	2	3	4	5	6	7	8	9	10
1	1									
2	1	1								
3	1	3	1							
4	1	7	6	1						
5	1	15	25	10	1					
6	1	31	90	65	15	1				
7	1	63	301	350	140	21	1			
8	1	127	966	1701	1050	266	28	1		
9	1	255	3025	7770	6951	2646	462	36	1	
10	1	511	9330	34105	42525	22827	5880	750	45	1

Both tables may be obtained using the recurrence relation in (1), Exercise 4.4.11.

We refer to the books by Cameron [17], Stanley [112], Graham, Knuth and Patashnik [51], Jordan [68] and Riordan [107] for more on Stirling numbers, their combinatorial meaning and their applications.

4.4.4 Garsia's expression for χ_λ

In this section, we present a formula due to Garsia which gives the characters of $\mathfrak{S}_n$ as symmetric polynomials in the YJM elements. As a byproduct, we obtain an alternative and more constructive proof of the theorem of Jucys and Murphy (cf. Theorem 4.4.5).

In the sequel, for $T \in \text{Tab}(n)$, we use the notation $a_T(j)$ to denote the jth component in $C(T)$, so that $C(T) = (a_T(1), a_T(2), \ldots, a_T(n))$ (see Section 3.1.5). Then the spectral analysis of the YJM elements may be summarized by the identity: $X_k w_T = a_T(k) w_T$ for all $T \in \text{Tab}(n)$, $k = 1, 2, \ldots, n$ (see 3.26 and Theorem 3.3.7). This fact has the following obvious generalization (recall that $X_1 \equiv 0$).

Proposition 4.4.12 *Let $P(x_2, x_3, \ldots, x_n)$ be a polynomial in the variables $x_2, x_3, \ldots, x_n$. Then*

$$P(X_2, X_3, \ldots, X_n) w_T = P(a_T(2), a_T(3), \ldots, a_T(n)) w_T$$

for all $T \in \text{Tab}(n)$.

As an immediate consequence of this we have the following corollary which was already proved in the first part of the theorem of Jucys and Murphy (Theorem 4.4.5).

Corollary 4.4.13 *Let $P(x_2, x_3, \dots, x_n)$ be a symmetric polynomial in the variables $x_2, x_3, \dots, x_n$. Then $P(X_2, X_3, \dots, X_n)$ belongs to the center of the group algebra of $\mathfrak{S}_n$.*

Proof If $P(x_2, x_3, \dots, x_n)$ then the eigenvalue $P(a_T(2), a_T(3), \dots, a_T(n))$ of $P(X_2, X_3, \dots, X_n)$ depends only on the shape of T, and therefore it is the same for all $T \in \mathrm{Tab}(\lambda)$. □

Note that, in the notation introduced in Chapter 2, $P(X_2, X_3, \dots, X_n)w_T$ is the Fourier transform of $P(X_2, X_3, \dots, X_n)$ applied to w_T, so that in the proof above we implicitly use Corollary 1.5.16.

The theorem of Jucys and Murphy suggests the existence of a family $\{Q_\lambda : \lambda \vdash n\}$ of symmetric polynomials such that $\chi^\lambda = Q_\lambda(X_2, X_3, \dots, X_n)$ for each partition $\lambda \vdash n$.

Proposition 4.4.14 *Let $\{Q_\lambda : \lambda \vdash n\}$ be a family of symmetric polynomials in $n - 1$ variables such that*

$$Q_\lambda(a_T(2), a_T(3), \dots, a_T(n)) = \begin{cases} \frac{n!}{d_\lambda} & \text{if } T \in \mathrm{Tab}(\lambda) \\ 0 & \text{if } T \in \mathrm{Tab}(\mu),\ \mu \neq \lambda. \end{cases}$$

Then,

$$\chi^\lambda = Q_\lambda(X_2, X_3, \dots, X_n).$$

Proof This follows immediately from the fact that χ^λ and $Q_\lambda(X_2, X_3, \dots, X_n)$ have the same Fourier transform (see the expression of the Fourier transform of a character (Corollary 1.3.14) and Proposition 4.4.12, respectively). □

Following [45], we shall give an explicit expression for a family of symmetric polynomials Q_λ satisfying the condition of Proposition 4.4.14. As a byproduct we obtain an alternative, more constructive proof of the theorem of Jucys and Murphy.

Consider the polynomials $\pi_{n,k}(x)$ (here $x = (x_1, x_2, \dots, x_n)$) defined by setting

$$\pi_{n,k}(x) = R_k(n-1) + \sum_{t=1}^{k} \binom{k}{t} [n^t - (n-1)^t] p_{k-t}(x)$$

where $p_k(x) = x_1^k + x_2^k + \cdots + x_n^k$ are the usual power sum symmetric polynomials and $R_k(n-1)$ is as in Section 4.4.3.

Note that $\pi_{n,k}$ is a symmetric polynomial of degree k in the variables $x_1, x_2, \ldots, x_n$, but it is not homogeneous.

We introduce the operator π_n on the vector space of all symmetric polynomials in n variables by setting $(\pi_n p_k)(x) = \pi_{n,k}(x)$ for every power sum symmetric polynomial, and then, if $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_h)$

$$(\pi_n p_\lambda)(x) = \pi_{n,\lambda_1}(x)\pi_{n,\lambda_2}(x) \cdots \pi_{n,\lambda_h}(x)$$

(we recall that $p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots p_{\lambda_h}$). Finally, given a symmetric polynomial P we have, for suitable $a_\lambda \in \mathbb{C}$, $P = \sum_{\lambda \vdash n} a_\lambda p_\lambda$ (cf. Theorem 4.1.12) and we set

$$(\pi_n P)(x) = \sum_{\lambda \vdash n} a_\lambda (\pi_n p_\lambda)(x).$$

Clearly, π_n is linear and multiplicative: $\pi_n(PQ) = \pi_n(P)\pi_n(Q)$ if P and Q are symmetric polynomials in n variables.

We need a little more notation. If P is a polynomial in n variables and $a = (a_1, a_2, \ldots, a_n)$ is an n-parts composition, then we simply write $P(a)$ for $P(a_1, a_2, \ldots, a_n)$. If, in addition, P is symmetric and $T \in \mathrm{Tab}(\lambda)$, $\lambda \vdash n$, we set

$$P[C(\lambda)] = P(a_T(1), a_T(2), \ldots, a_T(n)). \tag{4.67}$$

In other words, the right-hand side of (4.67) does not depend on the particular tableau $T \in \mathrm{Tab}(\lambda)$ (because P is symmetric) and $C(\lambda)$ denotes the content of the shape of λ. We also recall that if $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_h) \vdash n$ then $\lambda + \delta = (\lambda_1 + n - 1, \lambda_2 + n - 2, \ldots, \lambda_h + h - 1, h - 2, \ldots, 1)$ (see Section 4.1.6). We also set $p_0 \equiv 1$ and $p_0[C(\lambda)] = n$.

Theorem 4.4.15 (Garsia's replacement theorem) *Let P be a symmetric polynomial in n variables. Then for all $\lambda \vdash n$ we have*

$$(\pi_n P)[C(\lambda)] = P(\lambda + \delta).$$

Proof Clearly, it suffices to prove that

$$p_k(\lambda + \delta) = \pi_{n,k}[C(\lambda)] \tag{4.68}$$

for all $\lambda \vdash n$ and $k = 0, 1, 2, \ldots$

Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n) \vdash n$ with, possibly, $\lambda_{h+1} = \lambda_{h+2} = \cdots \lambda_n = 0$ for some $1 \le h \le n$. First of all, note that for $k \ge 1$ we have

$$\begin{aligned} p_k[C(\lambda)] &= \sum_{i=1}^{n}\sum_{j=1}^{\lambda_i}(j-i)^k \\ &= \sum_{i=1}^{n}\sum_{j=1}^{\lambda_i}\sum_{m=0}^{k}\binom{k}{m}(-i)^{k-m}j^m \\ &= \sum_{i=1}^{n}\sum_{m=0}^{k}\binom{k}{m}(-i)^{k-m}R_m(\lambda_i). \end{aligned}$$

Observe that this formula is also valid for $k = 0$ because

$$p_0[C(\lambda)] = n = \lambda_1 + \lambda_2 + \cdots + \lambda_n = \sum_{i=1}^{n} R_0(\lambda_i).$$

We now apply the above formula to the exponential generating function associated with the sequence $\{p_k[C(\lambda)]\}_{k=0,1,2,\dots}$.

$$\begin{aligned} \sum_{k=0}^{\infty}\frac{p_k[C(\lambda)]}{k!}z^k &= \sum_{i=1}^{n}\sum_{m=0}^{\infty}\frac{R_m(\lambda_i)}{m!}z^m\sum_{k=m}^{\infty}\frac{(-i)^{k-m}}{(k-m)!}z^{k-m} \\ &= \sum_{i=1}^{n}\sum_{m=0}^{\infty}\frac{R_m(\lambda_i)}{m!}z^m e^{-iz} \\ \text{(by (4.63))}\quad &= \sum_{i=1}^{n} e^z\frac{e^{\lambda_i z}-1}{e^z-1}e^{-iz} \\ &= \frac{e^{-nz}}{1-e^{-z}}\sum_{i=1}^{n}\left(e^{(\lambda_i+n-i)z} - e^{(n-i)z}\right) \\ &= \frac{e^{-nz}}{1-e^{-z}}\sum_{i=1}^{n}\sum_{k=1}^{\infty}[(\lambda_i+n-i)^k - (n-i)^k]\frac{z^k}{k!} \\ &= \frac{e^{-nz}}{1-e^{-z}}\sum_{k=1}^{\infty}[p_k(\lambda+\delta) - R_k(n-1)]\frac{z^k}{k!} \end{aligned}$$

and therefore

$$\sum_{k=1}^{\infty} p_k(\lambda+\delta)\frac{z^k}{k!} = \sum_{k=1}^{\infty} R_k(n-1)\frac{z^k}{k!} + [e^{nz} - e^{(n-1)z}]\sum_{k=0}^{\infty} p_k[C(\lambda)]\frac{z^k}{k!}. \tag{4.69}$$

Since $e^{nz} - e^{(n-1)z} = \sum_{t=1}^{\infty}[n^t - (n-1)^t]\frac{z^t}{t!}$, we have

$$[e^{nz} - e^{(n-1)z}]\sum_{h=0}^{\infty} p_h[C(\lambda)]\frac{z^h}{h!} = \sum_{k=1}^{\infty}\sum_{t=1}^{k}\binom{k}{t}[n^t - (n-1)^t]p_{k-t}[C(\lambda)]\frac{z^k}{k!},$$

and therefore (4.69) yields

$$p_k(\lambda+\delta) = R_k(n-1) + \sum_{t=1}^{k}\binom{k}{t}[n^t - (n-1)^t]p_{k-t}[C(\lambda)]$$

which is precisely (4.68). □

Let $\lambda \vdash n$ be a partition. Define the symmetric function $\Xi_{\lambda+\delta}$ by setting

$$\Xi_{\lambda+\delta}(x_1, x_2, \ldots, x_n) = \frac{\det\left([x_i]_{\lambda_j+n-j}\right)_{i,j}}{a_\delta(x)}.$$

Lemma 4.4.16 *For $\lambda, \mu \vdash n$ we have*

$$\Xi_{\lambda+\delta}(\mu+\delta) = \begin{cases} \frac{n!}{d_\lambda} & \text{if } \mu = \lambda \\ 0 & \text{otherwise.} \end{cases}$$

Proof Setting $x_i = \mu_i + n - i$, the determinant in the numerator of $\Xi_{\lambda+\delta}$ becomes

$$\sum_{\sigma\in\mathfrak{S}_n}\varepsilon(\sigma)\prod_{i=1}^{n}(\mu_i + n - i)_{\lambda_{\sigma(i)}+n-\sigma(i)}.$$

Since $[n]_k = 0$ whenever $k > n$, in the above sum, the only nonzero terms are those for which

$$\mu_i + n - i \geq \lambda_{\sigma(i)} + n - \sigma(i) \tag{4.70}$$

for all $i = 1, 2, \ldots, n$. But

$$\sum_{i=1}^{n}(\mu_i + n - i) = n + \frac{n(n-1)}{2} = \sum_{j=1}^{n}(\lambda_j + n - j)$$

and $\mu_1 + n - 1 > \mu_2 + n - 2 > \cdots \mu_n$ and $\lambda_1 + n - 1 > \lambda_2 + n - 2 > \cdots > \lambda_n$. Therefore, (4.70) is satisfied if and only if $\lambda = \mu$ and σ is the identity. This means that $\Xi_{\lambda+\delta}(\mu+\delta) = 0$ if $\mu \neq \lambda$, while

$$\Xi_{\lambda+\delta}(\lambda+\delta) = \frac{(\lambda_1+n-1)!(\lambda_2+n-2)!\cdots\lambda_n!}{a_\delta(\lambda_1+n-1, \lambda_2+n-2, \ldots, \lambda_n)} = \frac{n!}{d_\lambda},$$

where the last equality follows from Proposition (4.2.10). □

For $\lambda \vdash n$ define the polynomial Q_λ by setting

$$Q_\lambda(x_1, x_2, \ldots, x_n) = (\pi_n\Xi_{\lambda+\delta})(x_1, x_2, \ldots, x_n).$$

Theorem 4.4.17 (Garsia's formula for χ^λ) *For $\lambda, \mu \vdash n$ we have*

$$Q_\lambda[C(\mu)] = \begin{cases} \frac{n!}{d_\lambda} & \text{if } \mu = \lambda \\ 0 & \text{otherwise.} \end{cases} \tag{4.71}$$

and

$$\chi^\lambda = Q_\lambda(0, X_2, X_3, \dots, X_n). \tag{4.72}$$

Proof (4.71) is a consequence of Garsia's replacement theorem (Theorem 4.4.15) and Lemma 4.4.16. Then (4.72) follows from (4.71) and Proposition 4.4.14. □

Theorem 4.4.17, combined with Corollary 4.4.13 gives, as we mentioned before, an alternative, more constructive proof of the theorem of Jucys and Murphy (Theorem 4.4.5).

We end this section by giving another fundamental result of Jucys on the evaluations of the elementary symmetric polynomials at the YJM elements. This was a key ingredient in Jucys' proof in [70] of the theorem of Jucys and Murphy. Again, we follow Garsia's notes [45] for its proof. We recall that $\mathcal{C}_\mu$ is the $\mathfrak{S}_n$-conjugacy class associated with $\mu \vdash n$.

Theorem 4.4.18 (Jucys, [70]) *For $s = 0, 1, \dots, n$, we have*

$$e_s(X_1, X_2 \dots, X_n) = \sum_{\substack{\mu \vdash n: \\ \ell(\mu) = n-s}} \mathcal{C}_\mu.$$

Proof We use the results in Section 4.3 with n replaced by N and r replaced by n. From the characteristic map (Theorem 4.3.9) we get

$$s_\lambda(1, 1, \dots, 1) = \sum_{\mu \vdash n} \frac{1}{z_\mu} \chi^\lambda_\mu p_\mu(1, 1, \dots, 1) = \sum_{\mu \vdash n} \frac{1}{z_\mu} \chi^\lambda_\mu N^{\ell(\mu)}.$$

Combining this result with Theorem 4.3.3 and (4.2) we get the identity

$$\prod_{i=1}^{\ell(\lambda)} \prod_{j=1}^{\lambda_i} (j - i + N) = \sum_{\mu \vdash n} \frac{|\mathcal{C}_\mu|}{d_\lambda} \chi^\lambda_\mu N^{\ell(\mu)},$$

which is valid for all $N \geq n$, and therefore it may be transformed into a polynomial identity (t is a real or complex indeterminate):

$$\prod_{i=1}^{\ell(\lambda)} \prod_{j=1}^{\lambda_i} (j - i + t) = \sum_{\mu \vdash n} \frac{|\mathcal{C}_\mu|}{d_\lambda} \chi^\lambda_\mu t^{\ell(\mu)}. \tag{4.73}$$

On the other hand, from the spectral analysis of the YJM elements in Section 2.3, if $T \in \mathrm{Tab}(\lambda)$ and w_T is the associated Gelfand–Tsetlin vector, we have $(t + X_k)w_T = (t + j - i)w_T$, where $j - i$ is the content of the box of T containing k. Combining this fact with (4.73), we get:

$$\sum_{k=1}^{n}(t + X_k)w_T = \left[\prod_{i=1}^{\ell(\lambda)}\prod_{j=1}^{\lambda_i}(t + j - i)\right] w_T$$
$$= \left\{\sum_{\mu\vdash n}\left[\frac{|\mathcal{C}_\mu|}{d_\lambda}\chi^\lambda_\mu\right] t^{\ell(\mu)}\right\} w_T$$
$$= \sum_{s=0}^{n}\left\{\left[\sum_{\substack{\mu\vdash n:\\ \ell(\mu)=n-s}}\frac{|\mathcal{C}_\mu|}{d_\lambda}\chi^\lambda_\mu\right] w_T\right\} t^{n-s}.$$

Since

$$\sum_{k=1}^{n}(t + X_k)w_T = \sum_{s=0}^{n}[e_s(X_1, X_2, \ldots, X_n)w_T]t^{n-s},$$

from the arbitrarity of t we deduce that

$$e_s(X_1, X_2, \ldots, X_n)w_T = \sum_{\substack{\mu\vdash n:\\ \ell(\mu)=n-s}}\frac{|\mathcal{C}_\mu|}{d_\lambda}\chi^\lambda_\mu w_T.$$

But $e_s(X_1, X_2, \ldots, X_n)$ belongs to the center of $L(S_n)$ (easy part of Theorem 4.4.5), and therefore

$$e_s(X_1, X_2, \ldots, X_n)\chi^\lambda = \left(\sum_{\substack{\mu\vdash n:\\ \ell(\mu)=n-s}}\frac{|\mathcal{C}_\mu|}{d_\lambda}\chi^\lambda_\mu\right)\chi^\lambda. \tag{4.74}$$

From Corollary 1.5.12 we deduce that

$$\left(\sum_{\substack{\mu\vdash n:\\ \ell(\mu)=n-s}}\mathcal{C}_\mu\right)\chi^\lambda = \left(\sum_{\substack{\mu\vdash n:\\ \ell(\mu)=n-s}}\frac{|\mathcal{C}_\mu|}{d_\lambda}\chi^\lambda_\mu\right)\chi^\lambda \tag{4.75}$$

(following the notation introduced in Section 3.2.1 in the left-hand sides of the last two formulae we have a convolution in the group algebra of the symmetric group). Since λ is arbitrary, from (4.74) and (4.75) we get the desired identity. □

From Theorem 4.4.18 and the fundamental theorem on symmetric functions (see Theorem 4.1.12.(i)) we can deduce immediately a famous result of Farahat–Higman [35].

Theorem 4.4.19 *For* $t = 1, 2, \ldots, n$ *set*

$$Z_t = \sum_{\substack{\mu \vdash n: \\ \ell(\mu)=t}} \mathcal{C}_t.$$

Then the center of the group algebra $L(S_n)$ *is generated by the elements* $Z_1, Z_2, \ldots, Z_n$.

Actually, Jucys derived Theorem 4.4.5 from Theorem 4.4.18 and the Farahat–Higman result.

5

Content evaluation and character theory of the symmetric group

This chapter is devoted to an exposition of the results contained in the papers [81] and [24], together with some auxiliary results from [79, 80]. The papers [81] and [24] give expressions for the characters of the symmetric group in terms of the contents of the corresponding partitions. The results in [81] are more explicit and we first derive them. Subsequently, we deduce those in [24]. We have arranged all the material in order to give a self-contained and elementary exposition which only requires the results already developed in the previous chapters of the present book.

5.1 Binomial coefficients

First of all, we give some basic properties of binomial coefficients and we introduce a generalization due to Lassalle [79]. We also present a new family of symmetric functions from [80].

5.1.1 Ordinary binomial coefficients: basic identities

In this section, we recall some basic facts on the ordinary binomial coefficients and present some more specific technical identities.

For $z \in \mathbb{C}$ and $k \in \mathbb{N}$, the ordinary binomial coefficient $\binom{z}{k}$ is given by the formula:

$$\binom{z}{k} = \frac{z(z-1)\cdots(z-k+1)}{k!}.$$

It is useful to also set $\binom{z}{0} = 1$ and $\binom{z}{k} = 0$ if k is a negative integer.

Note that, if $n \in \mathbb{N}$ and $0 \leq k \leq n$, then $\binom{n}{k}$ is the number of k-subsets of an n-set, and $\binom{n}{k} = \frac{n!}{(n-k)!k!}$.

Let n, m and k integers, with $n \geq 0$ and $z, w \in \mathbb{C}$. The following identities are elementary but quite useful:

$$\binom{n}{k} = \binom{n}{n-k} \quad \text{(symmetry property)} \tag{5.1}$$

$$\binom{z}{k} = (-1)^k \binom{k-z-1}{k} \quad \text{(superior negation)} \tag{5.2}$$

$$\binom{z}{k} = \frac{z}{k}\binom{z-1}{k-1} \quad \text{if } k \neq 0 \ \text{(absorption)} \tag{5.3}$$

$$\binom{z}{m}\binom{m}{k} = \binom{z}{k}\binom{z-k}{m-k} \quad \text{(trinomial revision)} \tag{5.4}$$

$$\sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i} = (x+y)^n \quad \text{(binomial theorem)} \tag{5.5}$$

$$\sum_{i=0}^{n} \binom{z}{i}\binom{w}{n-i} = \binom{z+w}{n} \quad \text{(Chu–Vandermonde convolution).} \tag{5.6}$$

For $z = a$, $w = b$ with a, b positive integers such that $0 \leq n \leq a + b$, the Chu–Vandermonde identity has the following nice combinatorial interpretation. Let A and B be two disjoint finite sets with $|A| = a$ and $|B| = b$. Then $\binom{a}{i}\binom{b}{n-i}$ equals the number of n-subsets $C \subseteq A \coprod B$ such that $|A \cap C| = i$ and $|B \cap c| = n - i$. It follows that both sides in

$$\sum_{i=0}^{n} \binom{a}{i}\binom{b}{n-i} = \binom{a+b}{n}$$

equal the number of all n-subsets of $A \coprod B$. Then the general case $z, w \in \mathbb{C}$ follows because both sides of (5.6) are polynomials of degree n. For $z = 1$ (5.6) reduces to the well-known

$$\binom{w}{n} + \binom{w}{n-1} = \binom{w+1}{n} \quad \text{(addition formula).} \tag{5.7}$$

Zhu Shijie (Wade–Giles: Chu Shih-Chieh) was a Chinese mathematician that discovered (5.6) in 1303. It was rediscovered by Alexandre Vandermonde in 1700.

Exercise 5.1.1 Prove (5.6) when $z = a \in \mathbb{N}$, by induction on a. Extend (5.6) for n negative integer.

In the following proposition, we give two elementary variations of the Chu–Vandermonde formula; for further variations see the book by Graham, Knuth and Patashnik [51].

Proposition 5.1.2
(i) *For n, k integers, with $k \le n$, and $r \in \mathbb{C}$, we have*

$$\sum_{i=0}^{n-k}(-1)^i\binom{r}{i}\binom{n-i-1}{n-k-i} = \binom{n-r-1}{n-k}. \tag{5.8}$$

(ii) *For q, ℓ, m, n integers, with $\ell, m \ge 0$ and $n \ge q \ge 0$ we have*

$$\sum_{k=0}^{\ell}\binom{\ell-k}{m}\binom{q+k}{n} = \binom{\ell+q+1}{m+n+1}. \tag{5.9}$$

Proof (i) We have:

$$\begin{aligned}
\sum_{i=0}^{n-k}(-1)^i\binom{r}{i}\binom{n-i-1}{n-k-i} &= \sum_{i=0}^{n-k}(-1)^{n-k}\binom{r}{i}\binom{-k}{n-k-i} \\
\text{(by (5.6))} \quad &= (-1)^{n-k}\binom{r-k}{n-k} \\
&= \binom{n-r-1}{n-k}
\end{aligned}$$

where the first equality and the last equality follow from (5.2).
(ii) We have

$$\begin{aligned}
\sum_{k=0}^{\ell}\binom{\ell-k}{m}\binom{q+k}{n} &= \sum_{k=0}^{\ell}(-1)^{\ell-m+q-n}\binom{-m-1}{\ell-k-m}\binom{-n-1}{q+k-n} \\
&= (-1)^{\ell-m+q-n}\sum_{k=n-q}^{\ell-m}\binom{-m-1}{\ell-k-m}\binom{-n-1}{q+k-n} \\
\text{(set } k'=q+k-n) \quad &= (-1)^{\ell-m+q-n}\sum_{k'=0}^{q+\ell-m-n}\binom{-m-1}{(\ell-m+q-n)-k'}\binom{-n-1}{k'} \\
\text{(by (5.6))} \quad &= (-1)^{\ell-m+q-n}\binom{-m-n-2}{\ell-m+q-n} \\
&= \binom{\ell+q+1}{m+n+1}
\end{aligned}$$

where the first inequality and the last inequality follow from (5.2) and (5.1). □

Exercise 5.1.3 Prove that $\sum_{k=0}^{n}\binom{m+k}{k}=\binom{n+m+1}{n}$ and $\sum_{k=0}^{n}\binom{k}{m}=\binom{n+1}{m+1}$ where n, m are non negative integers.

Hint: Use (5.7) and induction.

The generating function for the binomial coefficients is given by the binomial power series of elementary calculus (or elementary complex analysis):

$$(1+z)^{\alpha}=\sum_{k=0}^{\infty}\binom{\alpha}{k}z^{k}$$

for $z, \alpha \in \mathbb{C}$ and $|z|<1$.

In particular, for $\alpha = n \in \mathbb{N}$ the series is a finite sum (the binomial theorem); for $\alpha = -m-1$ (negative integer) the binomial power series, (5.2) and (5.1) give

$$\begin{aligned}\frac{1}{(1-z)^{m+1}}&=\sum_{k=0}^{\infty}\binom{-m-1}{k}(-1)^{k}z^{k}=\sum_{k=0}^{\infty}\binom{k+m}{k}z^{k}\\&=\sum_{k=0}^{\infty}\binom{k+m}{m}z^{k}.\end{aligned}\tag{5.10}$$

From this we deduce, for $q \in \mathbb{N}$, $0 \le q \le m$,

$$\frac{z^{m-q}}{(1-z)^{m+1}}=\sum_{k=0}^{\infty}\binom{k+m}{m}z^{k+m-q}=\sum_{k'=m-q}^{\infty}\binom{k'+q}{m}z^{k'},\tag{5.11}$$

where in the last equality we have used the substitution $k' = k+m-q$.

Exercise 5.1.4 Use the last formulae to give an alternative proof of (5.6), (5.8) and (5.9).

5.1.2 Binomial coefficients: some technical results

In this section, we collect three particular results involving binomial coefficients that will play a fundamental role in the sequel. We begin with a formula that involves the Stirling numbers of the first kind (see Section 4.4.3). From now on, we shall often use the following convention: in a sum we omit the lower and upper limits of summation and assume that index runs over all possible value for which the coefficients are defined and nonzero. For instance, we can write Chu–Vandermonde in the form $\sum_k \binom{z}{m+k}\binom{w}{n-k}=\binom{z+w}{m+n}$.

Proposition 5.1.5 *Given integers i, r, p, with $p \geq 0$ and $0 \leq i \leq r + p + 1$ we have:*

$$\sum_{j=\max\{0,r+1-i\}}^{p+r+1-i} (-1)^j \binom{i+j-1}{i-1} s(p, i+j-r-1) = (-1)^r \sum_{k=\max\{0,i-p-1\}}^{\min\{r,i\}} (-1)^k \binom{r}{k} s(p+1, i-k).$$

Proof From the binomial theorem and the defining property of the Stirling numbers (see Section 4.4.3) we get

$$\begin{aligned}
\sum_{k=1}^{p+1} \left[\sum_{m=k}^{p+1} (-1)^{m-1} \ s(p, m-1) \binom{m-1}{k-1} \right] (-1)^{k-1} z^k
&= z \sum_{m=1}^{p+1} \left[\sum_{k=1}^{m} \binom{m-1}{k-1} (-z)^{k-1} \right] (-1)^{m-1} s(p, m-1) \\
&= z \sum_{m=1}^{p+1} (z-1)^{m-1} s(p, m-1) \\
&= z[z-1]_p \\
&= [z]_{p+1}
\end{aligned}$$

and therefore

$$\sum_{m=k}^{p+1} (-1)^{m-1} s(p, m-1) \binom{m-1}{k-1} = (-1)^{k+1} s(p+1, k). \tag{5.12}$$

Then we have

$$\begin{aligned}
&\sum_{j=\max\{0,r+1-i\}}^{p+r+1-i} (-1)^j \binom{i+j-1}{i-1} s(p, i+j-r-1) \\
&\text{(by (5.6))} = \sum_{k=\max\{0,i-p-1\}}^{\min\{r,i-1\}} \binom{r}{k} (-1)^{i-r-1} \left[\sum_{j=r-k}^{p+r+1-i} (-1)^{i+j-r-1} \cdot \right. \\
&\qquad \left. \cdot \binom{i+j-r-1}{i-k-1} \ s(p, i+j-r-1) \right] \\
&= (-1)^r \sum_{k=\max\{0,i-p-1\}}^{\min\{r,i\}} (-1)^k \binom{r}{k} s(p+1, i-k)
\end{aligned}$$

where in the last step we have used (5.12) with m replaced by $i+j-r$ and k replaced by $i-k$. □

The following lemma contains still another sum of products of two binomial coefficients. It will be used in the proof of Proposition 5.1.7.

Lemma 5.1.6 *For* $0 \le j \le [n/2]$ *we have:*

$$\sum_{h=j}^{[n/2]} \binom{n}{2h}\binom{h}{j} = \frac{n}{n-j}\binom{n-j}{j}2^{n-2j-1}.$$

Proof We have

$$\begin{aligned}
\sum_{h=j}^{[n/2]} \binom{n}{2h}\binom{h}{j} &= \sum_{h=j}^{[n/2]} \left[\sum_{k=h}^{n-h} \binom{n-1-k}{h-1}\binom{k}{h}\right]\binom{h}{j} \\
\text{(by (5.4))} \quad &= \sum_{k=j}^{n-j} \left[\sum_{h=j}^{\min\{k,n-k\}} \binom{n-k-1}{h-1}\binom{k-j}{k-h}\right]\binom{k}{j} \\
\text{(by (5.6))} \quad &= \sum_{k=j}^{n-j} \binom{n-j-1}{k-1}\binom{k}{j} \\
\text{(by (5.3))} \quad &= \sum_{k=j}^{n-j} \frac{k}{n-j}\binom{n-j}{k}\binom{k}{j} \\
\text{(by (5.4))} \quad &= \sum_{k=j}^{n-j} \frac{k}{n-j}\binom{n-j}{j}\binom{n-2j}{k-j} \\
\text{(by (5.3))} \quad &= \frac{1}{n-j}\binom{n-j}{j}\left[(n-2j)\sum_{k=j}^{n-j}\binom{n-2j-1}{k-j-1} + j\sum_{k=j}^{n-j}\binom{n-2j}{k-j}\right] \\
\text{(by (5.5))} \quad &= \frac{1}{n-j}\binom{n-j}{j}\left[(n-2j)2^{n-2j-1} + j2^{n-2j}\right] \\
&= \frac{n}{n-j}\binom{n-j}{j}2^{n-2j-1}
\end{aligned}$$

where the first equality follows from (5.9). □

Proposition 5.1.7 *We have the following power series development.*

$$\left(\frac{1+\sqrt{1+4z}}{2}\right)^n + \left(\frac{1-\sqrt{1+4z}}{2}\right)^n = \sum_{j=0}^{[n/2]} \frac{n}{n-j}\binom{n-j}{j}z^j$$

(which is actually an identity between polynomials).

Proof We have

$$\left(\frac{1+\sqrt{1+4z}}{2}\right)^n + \left(\frac{1-\sqrt{1+4z}}{2}\right)^n = \frac{1}{2^n}\sum_{k=0}^{n}[1+(-1)^k]\binom{n}{k}(1+4z)^{k/2}$$

$$\text{(setting } k=2h\text{)} \quad = \frac{1}{2^{n-1}}\sum_{h=0}^{[n/2]}\binom{n}{2h}(1+4z)^h$$

$$\text{(by the binomial theorem)} \quad = \frac{1}{2^{n-1}}\sum_{h=0}^{[n/2]}\binom{n}{2h}\sum_{j=0}^{h}\binom{h}{j}4^j z^j$$

$$= \frac{1}{2^{n-1}}\sum_{j=0}^{[n/2]}\left[\sum_{h=j}^{[n/2]}\binom{n}{2h}\binom{h}{j}\right]4^j z^j$$

$$\text{(by Lemma 5.1.6)} \quad = \sum_{j=0}^{[n/2]}\frac{n}{n-j}\binom{n-j}{j}z^j. \qquad \square$$

Proposition 5.1.8 *For $x, y \in \mathbb{C}$ and $a, b \in \mathbb{N}$ we have:*

$$\sum_{\substack{r,s,u,v\geq 0\\ r+s=a}}(-1)^{u+v}\binom{r}{u}\binom{s}{v}\binom{u+v}{a-b}(x+1)^{r-u}(y+1)^{s-v}$$
$$= (-1)^{a+b}\binom{a+1}{b+1}\frac{x^{b+1}-y^{b+1}}{x-y}.$$

Proof This is just a series of applications of elementary combinatorial identities.

$$\sum_{\substack{r,s,u,v\geq 0\\ r+s=a}}(-1)^{u+v}\binom{r}{u}\binom{s}{v}\binom{u+v}{a-b}(x+1)^{r-u}(y+1)^{s-v}$$

$$= \sum_{\substack{r,s,u,v\geq 0\\ r+s=a}}(-1)^{u+v}\binom{r}{u}\binom{s}{v}\left[\sum_{\substack{k+\ell=a-b\\ k,\ell\geq 0}}\binom{u}{k}\binom{v}{\ell}\right](x+1)^{r-u}(y+1)^{s-v}$$

$$\text{(by (5.4))} \quad = \sum_{\substack{k,\ell,r,s\geq 0\\ r+s=a\\ k+\ell=a-b}}(-1)^{r+s}\binom{r}{k}\binom{s}{\ell}\left[\sum_{u}\binom{r-k}{r-u}(-x-1)^{r-u}\right]\cdot$$
$$\cdot\left[\sum_{v}\binom{s-\ell}{s-v}(-y-1)^{s-v}\right]$$

$$\text{(by the binomial theorem)} = \sum_{\substack{k,\ell,r,s\geq 0\\ r+s=a\\ k+\ell=a-b}} \binom{r}{k}\binom{s}{\ell} x^{r-k}y^{s-\ell}(-1)^{a-b}$$

$$=_{(*)} \sum_{h=0}^{b}\left[\sum_k \binom{h+k}{h}\binom{a-h-k}{b-h}\right] x^h y^{b-h}(-1)^{a-b}$$
$$=_{(**)} (-1)^{a+b}\binom{a+1}{b+1}\frac{x^{b+1}-y^{b+1}}{x-y},$$

where the first equality follows from (5.6), in $=_{(*)}$ we used the equivalence

$$\begin{cases} r-k=h\\ s-\ell=b-h\\ r+s=a\end{cases} \iff \begin{cases} r=h+k\\ \ell=a-k-b\\ s=a-h-k\end{cases} \implies k+\ell=a-b$$

and in $=_{(**)}$ we have used (5.9) and the geometric identity $\frac{x^{b+1}-y^{b+1}}{x-y} = \sum_{h=0}^{b} x^h y^{b-h}$. □

5.1.3 Lassalle's coefficients

In this section, we introduce a family of positive integers, introduced by Lassalle in [79], that generalizes the binomial coefficients (see also [69], where a combinatorial interpretation of these coefficients is given).

Definition 5.1.9 For n, p, k nonnegative integers satisfying $0 \leq p \leq n$ and $k \geq 1$ we set:

$$\binom{n}{p}_k = \frac{n}{k}\sum_{r\geq 0}\binom{p}{r}\binom{n-p}{r}\binom{n-r-1}{k-r-1}.$$

We also set $\binom{n}{p}_0 = 0$.

It is clear that $\binom{n}{p}_1 = n$ and that $\binom{n}{p}_k = 0$ whenever $k > n$. We note that these coefficients generalize the binomial coefficients because for $p = 0$ we have, by (5.3),

$$\binom{n}{0}_k = \binom{n}{k}.$$

Exercise 5.1.10 Prove the following identities:

$$\binom{n}{p}_k = \binom{n}{n-p}_k,\quad \binom{n}{1}_k = k\binom{n}{k} \quad \text{and} \quad \binom{n}{p}_n = \binom{n}{p}.$$

We give now an alternative expression for $\binom{n}{p}_k$.

Proposition 5.1.11 *We have*

$$\binom{n}{p}_k = \sum_{i=0}^{n-k} (-1)^i \frac{n}{n-i} \binom{n-i}{i} \binom{n-i}{k} \binom{n-2i}{p-i}.$$

Proof Suppose first $k = 0$ (so that $\binom{n}{p}_k = 0$). We have

$$\begin{aligned}
\sum_{i=0}^{n} (-1)^i \frac{n}{n-i} \binom{n-i}{i} \binom{n-2i}{p-i} &= \sum_{i=0}^{n} (-1)^i \frac{n}{n-i} \binom{n-i}{p} \binom{p}{i} \\
\text{(by (5.3) and (5.1))} \quad &= \frac{n}{p} \sum_{i=0}^{n} (-1)^i \binom{n-i-1}{n-p-i} \binom{p}{i} \\
\text{(by (5.8))} \quad &= \frac{n}{p} \binom{n-p-1}{n-p} = 0,
\end{aligned}$$

where the first equality follows from (5.4). If $k \neq 0$ we have

$$\begin{aligned}
\binom{n}{p}_k &= \frac{n}{k} \sum_{r \geq 0} \binom{p}{r} \binom{n-p}{r} \binom{n-r-1}{k-r-1} \\
\text{(by (5.1))} \quad &= \frac{n}{k} \sum_{r \geq 0} \binom{p}{r} \binom{n-p}{r} \binom{n-r-1}{n-k} \\
\text{(by (5.8))} \quad &= \frac{n}{k} \sum_{r \geq 0} \binom{p}{r} \binom{n-p}{r} \sum_{i=0}^{n-k} (-1)^i \binom{r}{i} \binom{n-i-1}{n-k-i} \\
\text{(by (5.1))} \quad &= \sum_{i=0}^{n-k} (-1)^i \frac{n}{k} \binom{n-i-1}{k-1} \sum_{r \geq 0} \binom{n-p}{r} \left[\binom{p}{r} \binom{r}{i} \right] \\
\text{(by (5.4))} \quad &= \sum_{i=0}^{n-k} (-1)^i \frac{n}{k} \binom{n-i-1}{k-1} \binom{p}{i} \sum_{r \geq 0} \binom{n-p}{r} \binom{p-i}{r-i} \\
\text{(by (5.1))} \quad &= \sum_{i=0}^{n-k} (-1)^i \frac{n}{k} \binom{n-i-1}{k-1} \binom{p}{i} \sum_{r \geq 0} \binom{n-p}{r} \binom{p-i}{p-r} \\
\text{(by (5.6))} \quad &= \sum_{i=0}^{n-k} (-1)^i \frac{n}{k} \binom{n-i-1}{k-1} \binom{p}{i} \binom{n-i}{p} \\
\text{(by (5.4) and (5.3))} \quad &= \sum_{i=0}^{n-k} (-1)^i \frac{n}{n-i} \binom{n-i}{k} \binom{n-i}{i} \binom{n-2i}{p-i}.
\end{aligned}$$

□

Corollary 5.1.12 *For $1 \leq k \leq n$, the coefficient $\binom{n}{p}_k$ is a positive integer.*

Proof Positivity follows from the definition. It is an integer because we can express the result of Proposition 5.1.11 in the following form:

$$\binom{n}{p}_k = \sum_{i=0}^{n-k}(-1)^i \binom{n-i}{k}\binom{n-2i}{p-i}\left[\binom{n-i}{i} + \binom{n-i-1}{i-1}\right]. \qquad \square$$

We now present a two-variable generating function for the coefficients $\binom{n}{p}_k$.

Proposition 5.1.13 *We have*

$$\sum_{k=1}^{n}\sum_{p=0}^{n}\binom{n}{p}_k x^p y^k = G_n(x, y)$$

where

$$G_n(x, y) = \frac{1}{2^n}\left\{\left[(1+x)(1+y) + \sqrt{(1+x)^2(1+y)^2 - 4x(1+y)}\right]^n \right.$$
$$\left. + \left[(1+x)(1+y) - \sqrt{(1+x)^2(1+y)^2 - 4x(1+y)}\right]^n\right\} - 1 - x^n.$$

Proof From Proposition 5.1.11 and the binomial theorem we get

$$\begin{aligned}
\sum_{k=1}^{n}\sum_{p=0}^{n}\binom{n}{p}_k x^p y^k &= \sum_{k=1}^{n}\sum_{p=0}^{n} x^p y^k \sum_{i\geq 0}(-1)^i \frac{n}{n-i}\binom{n-i}{i}\binom{n-i}{k}\binom{n-2i}{p-i} \\
&= \sum_{i\geq 0}(-1)^i \frac{n}{n-i}\binom{n-i}{i}\left[(1+y)^{n-i} - 1\right]x^i(1+x)^{n-2i} \\
&= (1+x)^n(1+y)^n \sum_{i\geq 0}\frac{n}{n-i}\binom{n-i}{i}\left[-\frac{x}{(1+x)^2(1+y)}\right]^i \\
&\quad - (1+x)^n \sum_{i\geq 0}\frac{n}{n-i}\binom{n-i}{i}\left[-\frac{x}{(1+x)^2}\right]^i.
\end{aligned}$$

We can then apply Proposition 5.1.7 first with $z = -x/(1+x)^2(1+y)$ and then with $z = -x/(1+x)^2$. $\square$

We now present a recurrence relation for the coefficients $\binom{n}{p}_k$.

Lemma 5.1.14 *If* $k \neq 0$ *and* $1 \leq p \leq n$, *then:*

$$(n-p+1)\binom{n}{p-1}_k - p\binom{n}{p}_k = \frac{n}{n-1}(n-2p+1)\binom{n-1}{p-1}_k. \tag{5.13}$$

Proof First of all, note that

$$(n-p+1)\binom{p-1}{r}\binom{n-p+1}{r}-p\binom{p}{r}\binom{n-p}{r} = (n-2p+1)\left[\binom{p-1}{r}\binom{n-p}{r}-\binom{p-1}{r-1}\binom{n-p}{r-1}\right]. \tag{5.14}$$

Indeed, (5.14) is equivalent to the elementary algebraic identity

$$\frac{(n-p+1)^2}{r^2(n-p-r+1)}-\frac{p^2}{r^2(p-r)} = (n-2p+1)\left[\frac{1}{r^2}-\frac{1}{(p-r)(n-p-r+1)}\right]$$

(to verify it, multiply by $r^2(p-r)(n-p-r+1)$ and expand along the powers of r: both sides then equal the quantity $(n-2p+1)p(n-p+1)-r(n-2p+1)(n+1)$).

Then, the left-hand side of (5.13) is equal to:

$$\begin{aligned}
&\frac{n}{k}\sum_{r\geq 0}\binom{n-r-1}{k-r-1}\left[(n-p+1)\binom{p-1}{r}\binom{n-p+1}{r}-p\binom{p}{r}\binom{n-p}{r}\right]\\
\text{(by (5.14))} &= \frac{n}{k}(n-2p+1)\sum_{r\geq 0}\binom{n-r-1}{k-r-1}\left[\binom{p-1}{r}\binom{n-p}{r}-\binom{p-1}{r-1}\binom{n-p}{r-1}\right]\\
&= \frac{n}{k}(n-2p+1)\sum_{r\geq 0}\binom{p-1}{r}\binom{n-p}{r}\left[\binom{n-r-1}{k-r-1}-\binom{n-r-2}{k-r-2}\right]\\
\text{(by (5.7))} &= \frac{n}{k}(n-2p+1)\sum_{r\geq 0}\binom{p-1}{r}\binom{n-p}{r}\binom{n-r-2}{k-r-1}\\
&= \frac{n}{n-1}(n-2p+1)\binom{n-1}{p-1}_k.
\end{aligned}$$

□

We need another generating function for the the coefficients $\binom{n}{p}_k$ and to express it we introduce the following power series:

$$\varphi_{n,p}(z)=\sum_{h=0}^{\infty}\frac{(p+1)_h(n-p+1)_h}{(h+1)!h!}z^h, \tag{5.15}$$

$0\leq p\leq n$, $z\in\mathbb{C}$. It follows immediately from d'Alembert theorem that the radius of convergence of $\varphi_{n,p}$ is 1. Also note the particular case:

$$\varphi_{0,0}(z)=\sum_{h=0}^{\infty}\frac{(1)_h(1)_h}{(h+1)!h!}z^h=\sum_{h=0}^{\infty}\frac{z^h}{h+1}=\sum_{h=1}^{\infty}\frac{z^{h-1}}{h}=-\frac{1}{z}\log(1-z).$$

Proposition 5.1.15 *For* $0 \leq p \leq n$ *and* $|z| < 1$ *we have:*

$$\sum_{k=1}^{n} \binom{n}{p}_k \left(\frac{z}{1-z}\right)^k = nz\varphi_{n,p}(z).$$

Proof The proof is by induction on p. For $p = 0$ we have:

$$\begin{aligned}
\sum_{k=1}^{n} \binom{n}{0}_k \left(\frac{z}{1-z}\right)^k &= \sum_{k=1}^{n} \binom{n}{k} \left(\frac{z}{1-z}\right)^k \\
\text{(by (5.5))} \quad &= \frac{1}{(1-z)^n} - 1 \\
\text{(by (5.10))} \quad &= \sum_{h=1}^{\infty} \frac{(n)_h}{h!} z^h \\
&= \sum_{h=0}^{\infty} \frac{(n)_{h+1}}{(h+1)!} z^{h+1} \\
&= nz\varphi_{n,0}(z).
\end{aligned}$$

In view of Lemma 5.1.14, in order to prove the inductive step, it suffices to show that the function $\varphi_{n,p}(z)$ satisfies the identity

$$(n-p+1)\varphi_{n,p-1}(z) - p\varphi_{n,p}(z) = (n-2p+1)\varphi_{n-1,p-1}(z).$$

But this is elementary, since its left-hand side equals

$$\varphi_{n-1,p-1}(z) \cdot [(n-p+h+1) - (p+h)] = (n-2p+1)\varphi_{n-1,p-1}(z). \qquad \square$$

Remark 5.1.16 For p, q nonnegative integers, the hypergeometric function ${}_pF_q$ is defined by the power series

$${}_pF_q\left[\begin{matrix} a_1, a_2, \ldots, a_p \\ b_1, b_2, \ldots, b_q \end{matrix}; z\right] = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \cdots (a_p)_k z^k}{(b_1)_k (b_2)_k \cdots (b_q)_k k!} \tag{5.16}$$

Most of the identities in this chapter may be interpreted in terms of hypergeometric functions: for instance, $\binom{n}{k} = {}_2F_1\left[\begin{matrix} -k, k-n \\ 1 \end{matrix}; 1\right]$ and (5.6) is equivalent to the identity

$$\frac{(y-x)_k}{(y)_k} = {}_2F_1\left[\begin{matrix} -k, x \\ y \end{matrix}; 1\right].$$

We also have

$$\binom{n}{p}_k = \binom{n}{k} {}_3F_2\left[\begin{matrix} 1-k, -p, p-n \\ 1-n, 1 \end{matrix}; 1\right]$$

and

$$\varphi_{n,p}(z) = {}_2F_1\left[\begin{matrix} p+1, n-p+1 \\ 2 \end{matrix}; z\right].$$

In particular, the recurrence relation for $\varphi_{n,p}$ is a particular case of a contiguity relation for ${}_2F_1$. We refer to the books by Andrews, Askey and Roy [4], Bailey [6] and Rainville [105] for more details on hypergeometric series.

5.1.4 Binomial coefficients associated with partitions

In [80, 81] the following binomial coefficients were introduced: if λ is a partition of length $\ell = \ell(\lambda)$ and k is an integer ≥ 1, it is set

$$\left\langle \begin{matrix} \lambda \\ k \end{matrix} \right\rangle = \sum_{k_1, k_2, \ldots, k_\ell} \prod_{i=1}^{\ell(\lambda)} \binom{\lambda_i}{k_i},$$

where the sum is over all ℓ-parts compositions $(k_1, k_2, \ldots, k_\ell)$ of k such that $k_i \geq 1$ for $i = 1, 2, \ldots, \ell$. Clearly, $\left\langle \begin{matrix} \lambda \\ k \end{matrix} \right\rangle = 0$ if $k > |\lambda|$ or $k < \ell(\lambda)$. The coefficient $\left\langle \begin{matrix} \lambda \\ k \end{matrix} \right\rangle$ has an immediate combinatorial interpretation: it is the number of ways in which we can take k boxes from the diagram of λ in such a way that at least one box from each row is taken. Also the generating function for $\left\langle \begin{matrix} \lambda \\ k \end{matrix} \right\rangle$ is easy to derive (set $\ell = \ell(\lambda)$):

$$\begin{aligned}
\sum_{k=\ell(\lambda)}^{|\lambda|} \left\langle \begin{matrix} \lambda \\ k \end{matrix} \right\rangle x^k &= \sum_{\substack{k_1, k_2, \ldots, k_\ell \geq 1: \\ k_1 + k_2 + \cdots + k_\ell \leq |\lambda|}} \prod_{i=1}^{\ell(\lambda)} \binom{\lambda_i}{k_i} x^k \\
&= \prod_{i=1}^{\ell(\lambda)} \left[\sum_{k_i=1}^{\lambda_i} \binom{\lambda_i}{k_i} x_i^{k_i} \right] \\
&= \prod_{i=1}^{\ell(\lambda)} [(1+x)^{\lambda_i} - 1] \\
&= \prod_{i \geq 1} [(1+x)^i - 1]^{m_i(\lambda)}
\end{aligned}$$

where $m_i(\lambda)$ is the number of parts of λ that are equal to i (that is, $\lambda = (1^{m_1(\lambda)}, 2^{m_2(\lambda)}, \dots, n^{m_n(\lambda)})$, where $n = |\lambda|$). We can extend this definition to the coefficients $\binom{n}{p}_k$ by setting, for $|p| \le |\lambda|$, $k \ge 1$ and $\ell = \ell(\lambda)$

$$\left\langle \begin{matrix} \lambda \\ p \end{matrix} \right\rangle_k = \sum_{p_1, p_2, \dots, p_\ell} \sum_{k_1, k_2, \dots, k_\ell} \prod_{i=1}^{\ell(\lambda)} \binom{\lambda_i}{p_i}_{k_i}$$

where the sum is over all compositions $(p_1, p_2, \dots, p_\ell)$ of p and $(k_1, k_2, \dots, k_\ell)$ of k such that $0 \le p_i \le \lambda_i$ and $1 \le k_i \le \lambda_i$. Again we have

$$\left\langle \begin{matrix} \lambda \\ p \end{matrix} \right\rangle_k = 0 \text{ if } k > |\lambda| \text{ or if } k < \ell(\lambda). \tag{5.17}$$

Moreover, from Proposition 5.1.13 we get

$$\sum_{p=0}^{|\lambda|} \sum_{k=\ell(\lambda)}^{|\lambda|} \left\langle \begin{matrix} \lambda \\ p \end{matrix} \right\rangle_k x^p y^k = \prod_{i=1}^{\ell(\lambda)} G_{\lambda_i}(x, y) = \prod_{i \ge 1} [G_i(x, y)]^{m_i(\lambda)}. \tag{5.18}$$

Now we list some elementary properties of the coefficients $\left\langle \begin{matrix} \lambda \\ p \end{matrix} \right\rangle_k$.

$$\left\langle \begin{matrix} \lambda \\ p \end{matrix} \right\rangle_1 = \begin{cases} \binom{n}{p}_1 \equiv n & \text{if } \lambda = (n) \\ 0 & \text{otherwise} \end{cases} \tag{5.19}$$

$$\left\langle \begin{matrix} (n) \\ p \end{matrix} \right\rangle_k = \binom{n}{p}_k \tag{5.20}$$

$$\left\langle \begin{matrix} \lambda \\ 0 \end{matrix} \right\rangle_k = \left\langle \begin{matrix} \lambda \\ k \end{matrix} \right\rangle \qquad \left(\text{from } \binom{n}{0}_k = \binom{n}{k}\right) \tag{5.21}$$

$$\left\langle \begin{matrix} \lambda \\ 1 \end{matrix} \right\rangle_k = k \left\langle \begin{matrix} \lambda \\ k \end{matrix} \right\rangle = k \left\langle \begin{matrix} \lambda \\ 0 \end{matrix} \right\rangle_k \quad \left(\text{from } \binom{n}{1}_k = k\binom{n}{k} = k\binom{n}{0}_k\right) \tag{5.22}$$

$$\left\langle \begin{matrix} \lambda \\ p \end{matrix} \right\rangle_k = \left\langle \begin{matrix} \lambda \\ |\lambda| - p \end{matrix} \right\rangle_k \qquad \left(\text{from } \binom{n}{p}_k = \binom{n}{n-p}_k\right) \tag{5.23}$$

$$\left\langle \begin{matrix} \lambda \\ p \end{matrix} \right\rangle_{|\lambda|} = \binom{|\lambda|}{p} \tag{5.24}$$

from $\binom{n}{p}_n = \binom{n}{p}$ and $\sum_{p_1,p_2,\dots,p_\ell} \prod_{i=1}^{\ell} \binom{\lambda_i}{p_i} = \binom{|\lambda|}{p}$, where the sum is over all decompositions $(p_1, p_2, \dots, p_\ell)$ of p such that $0 \le p_i \le \lambda_i$ (this is an easy generalization of (5.6)).

5.1.5 Lassalle's symmetric function

We now introduce a family of symmetric functions that will play a fundamental role in the explicit formulae in the next section.

Definition 5.1.17 (Lassalle's symmetric functions) If n, k, p are integers satisfying $n \ge 1$, $1 \le k \le n$ and $0 \le p \le n$, we set

$$F_{npk} = \sum_{\mu \vdash n} \frac{\left\langle \begin{matrix} \mu \\ p \end{matrix} \right\rangle_k}{z_\mu} p_\mu, \tag{5.25}$$

where p_μ is the power sum symmetric function associated with μ (see Section 4.1.3) and $z_\mu = 1^{m_1} m_1! 2^{m_2} m_2! \cdots n^{m_n} m_n!$ if $\mu = (1^{m_1}, 2^{m_2}, \dots, n^{m_n})$ (see Proposition 4.1.1). We also set $F_{np0} = 0$ and $F_{000} = 1$.

Observe that in (5.25) the sum can be restricted to all $\mu \vdash n$ with $\ell(\mu) \le k$, because

$$\left\langle \begin{matrix} \mu \\ p \end{matrix} \right\rangle_k = 0 \text{ for } k \le \ell(\mu). \tag{5.26}$$

We now give some elementary properties of the functions F_{npk}.

For $k = 1$ we have $F_{np1} = p_n$ because $\left\langle \begin{matrix} (n) \\ p \end{matrix} \right\rangle_1 = n$ (by (5.19)) and $z_{(n)} = n$.

For $k = n$ we have $\left\langle \begin{matrix} \mu \\ p \end{matrix} \right\rangle_n = \binom{n}{p}$ (by (5.24)) and therefore

$$F_{npn} = \binom{n}{p} \sum_{\mu \vdash n} \frac{p_\mu}{z_\mu} = \binom{n}{p} h_n$$

(the last equality follows from Lemma 4.1.10). In particular, for $k = n$ and $p = 0$, $F_{n0n} = h_n$. For $p = 1$ we have $\left\langle \begin{matrix} \mu \\ 1 \end{matrix} \right\rangle_k = k \left\langle \begin{matrix} \mu \\ 0 \end{matrix} \right\rangle_k$ (by (5.22)) and therefore $F_{n1k} = k F_{n0k}$.

Finally, $F_{npk} = F_{n(n-p)k}$ since $\left\langle \begin{matrix} \mu \\ p \end{matrix} \right\rangle_k = \left\langle \begin{matrix} \mu \\ n-p \end{matrix} \right\rangle_k$ by (5.23).

We introduce one more notation. We recall (see Section 4.4.4) that if P is a symmetric polynomial in the variables $x_1, x_2, \ldots, x_n$ and $\lambda \vdash n$ then $P[C(\lambda)] = P(a_T(1), a_T(2), \ldots, a_T(n))$ denotes the polynomial P evaluated at the content of λ (T is any tableau in $\mathrm{Tab}(\lambda)$, but the value does not depend on the choice of the particular T). We set

$$d_r(\lambda) = \sum_{i=1}^{\ell(\lambda)} \sum_{j=1}^{\lambda_i} (j-i)^r \equiv p_r([C(\lambda)],$$

that is $d_r(\lambda)$ is the power sum symmetric function evaluated at the content of λ. In particular, $d_0(\lambda) = |\lambda|$. Finally, to simplify notation, we set

$$F_{nkp}(\lambda) := F_{nkp}[C(\lambda)] = \sum_{\mu=(\mu_1,\mu_2,\ldots,\mu_t)\vdash n} \frac{\left\langle \begin{matrix} \mu \\ p \end{matrix} \right\rangle_k}{z_\mu} d_{\mu_1}(\lambda) d_{\mu_2}(\lambda) \cdots d_{\mu_t}(\lambda). \tag{5.27}$$

Example 5.1.18 For instance,

$$F_{3q2} = \frac{\left\langle \begin{matrix} (3) \\ q \end{matrix} \right\rangle_2}{3} p_3 + \frac{\left\langle \begin{matrix} (2,1) \\ q \end{matrix} \right\rangle_2}{2} p_{2,1} + \frac{\left\langle \begin{matrix} (1,1,1) \\ q \end{matrix} \right\rangle_2}{1} p_{1,1,1}.$$

But, $\left\langle \begin{matrix} (1,1,1) \\ q \end{matrix} \right\rangle_2 = 0$,

$$\left\langle \begin{matrix} (3) \\ 3 \end{matrix} \right\rangle_2 = \left\langle \begin{matrix} (3) \\ 0 \end{matrix} \right\rangle_2 = \binom{3}{0}_2 = \binom{3}{2} = 3,$$

$$\left\langle \begin{matrix} (3) \\ 2 \end{matrix} \right\rangle_2 = \left\langle \begin{matrix} (3) \\ 1 \end{matrix} \right\rangle_2 = \binom{3}{1}_2 = 2\binom{3}{1} = 6,$$

$$\left\langle \begin{matrix} (2,1) \\ 3 \end{matrix} \right\rangle_2 = \left\langle \begin{matrix} (2,1) \\ 0 \end{matrix} \right\rangle_2 = \left\langle \begin{matrix} (2,1) \\ 2 \end{matrix} \right\rangle = \binom{2}{1}\binom{1}{1} = 2,$$

$$\left\langle \begin{matrix} (2,1) \\ 2 \end{matrix} \right\rangle_2 = \left\langle \begin{matrix} (2,1) \\ 1 \end{matrix} \right\rangle_2 = 2\left\langle \begin{matrix} (2,1) \\ 2 \end{matrix} \right\rangle = 4$$

and therefore

$$F_{332} = F_{302} = p_3 + p_{2,1} \text{ and } F_{322} = F_{312} = 2p_3 + 2p_{2,1}.$$

Remark 5.1.19 We have

$$
\begin{aligned}
d_r(\lambda) &= \sum_{i=1}^{\ell(\lambda)} \sum_{j=1}^{\lambda_i} (j-i)^r \\
\text{(by (4.65))} \quad &= \sum_{k=1}^{r} S(r,k) \sum_{i=1}^{\ell(\lambda)} \sum_{j=1}^{\lambda_i} [j-i]_k \\
\text{(as } [z+1]_{k+1} - [z]_{k+1} &= (k+1)[z]_k) \\
&= \sum_{k=1}^{r} \frac{S(r,k)}{k+1} \sum_{i=1}^{\ell(\lambda)} \sum_{j=1}^{\lambda_i} \{[j+1-i]_{k+1} - [j-i]_{k+1}\} \\
&= \sum_{k=1}^{r} \frac{S(r,k)}{k+1} \sum_{i=1}^{\ell(\lambda)} \{[\lambda_i - i + 1]_{k+1} - [-i+1]_{k+1}\}
\end{aligned}
$$

and therefore, if we set

$$
p_k^*(x_1, x_2, \dots, x_\ell) = \sum_{i=1}^{\ell} \{[x_i - i + 1]_k - [-i+1]_k\}
$$

we can write

$$
d_r(\lambda) = \sum_{k=1}^{r} \frac{S(r,k)}{k+1} p_{k+1}^*(\lambda_1, \lambda_2, \dots, \lambda_\ell).
$$

The polynomial p_k^* is a so-called *shifted symmetric function*. For instance, $d_0(\lambda) = p_1^*(\lambda)$, $d_1(\lambda) = \frac{1}{2} p_2^*(\lambda)$, $d_2(\lambda) = \frac{1}{3} p_3^*(\lambda) + \frac{1}{2} p_2^*(\lambda)$, $d_3(\lambda) = \frac{1}{4} p_4^*(\lambda) + p_3^*(\lambda) + \frac{1}{2} p_2^*(\lambda)$ and $d_4(\lambda) = \frac{1}{5} p_5^*(\lambda) + \frac{7}{4} p_4^*(\lambda) + 2 p_3^*(\lambda) + \frac{1}{2} p_2^*(\lambda)$. This means that it is symmetric in the "shifted" variables $x_i - i$, for $i = 1, 2, \dots, \ell$. Therefore, both $d_r(\lambda)$ and $F_{nkp}(\lambda)$ are symmetric polynomials in the content $\{(j-i) : 1 \le i \le \ell, 1 \le j \le \lambda_i\}$ but are shifted symmetric polynomials in the variables $\lambda_1, \lambda_2, \dots, \lambda_\ell$. We do not treat here the theory of shifted symmetric functions; it was developed in several papers by Olshanski and his collaborators, see for instance [98]. We will use two elementary facts on shifted symmetric polynomials. First of all, a polynomial $p(x_1, x_2, \dots, x_\ell)$ is shift-symmetric if and only if it satisfies the identity

$$
\begin{aligned}
&p(x_1, x_2, \dots, x_{i-1}, x_i, x_{i+1}, x_{i+2}, \dots, x_\ell) \\
&\qquad = p(x_1, x_2, \dots, x_{i-1}, x_{i+1} - 1, x_i + 1, x_{i+2}, \dots, x_\ell) \qquad (5.28)
\end{aligned}
$$

for $i = 1, 2, \dots, \ell$. In particular, $d_r(\lambda)$ and F_{nkp} satisfy this identity:

$$
d_r(\lambda_1, \lambda_2, \dots, \lambda_i, \lambda_{i+1}, \dots, \lambda_\ell) = d_r(\lambda_1, \lambda_2, \dots, \lambda_{i+1} - 1, \lambda_i + 1, \dots, \lambda_\ell).
$$

The second fact is the following: suppose that we have an identity satisfied by the functions $F_{nkp}(\lambda)$ for all partitions $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$ of length ℓ: for instance $\sum_{n,k,p} a_{nkp} F_{nkp}(\lambda) = 0$, where the sum is over a finite set and the coefficients a_{nkp} are real (or complex) numbers. Then the identity holds true for all real (or complex) values of $\lambda_1, \lambda_2, \dots, \lambda_\ell$. We refer to this fact as to the *principle of analytic continuation*.

5.2 Taylor series for the Frobenius quotient

This section is devoted to some technical tools that lead to a far-reaching generalization of Proposition 4.2.11. It is based on Lassalle papers [80] and [81] and Lascoux [78].

5.2.1 The Frobenius function

Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$ be a partition. The *Frobenius function* associated with λ is the rational function

$$F(z;\lambda) = \prod_{i=1}^{\ell(\lambda)} \frac{z - \lambda_i + i}{z + i}.$$

We also introduce the following terminology: for a fixed positive integer m, the rational function

$$[z]_m \frac{F(z-m;\lambda)}{F(z;\lambda)} = [z]_m \prod_{i=1}^{\ell(\lambda)} \frac{z - \lambda_i + i - m}{z + i - m} \frac{z + i}{z - \lambda_i + i}$$

(where $[z]_m$ is the falling factorial) will be called the *Frobenius quotient*.

Remark 5.2.1 In the notation of Proposition 4.2.11, it is easy to see that

$$[z]_m \frac{F(z-m;\lambda)}{F(z;\lambda)} = [x]_m \frac{\phi(x-m)}{\phi(x)}$$

where $x = \ell + z$ and $\phi(x) = \prod_{i=1}^{\ell}(x - \lambda_i - \ell + i)$.

Actually, Frobenius used a slightly different notation to express a partition and the corresponding Frobenius function. Let r be the number of boxes in the main diagonal of the diagram of λ. Then we set $\alpha_i = \lambda_i - i$, $i = 1, 2, \dots, r$ and $\beta_j = \lambda'_j - j$, $j = 1, 2, \dots, r$. In other words, α_i is the number of boxes in the ith row that are on the right of the box of coordinates (i, i), while β_j is the number of boxes in column j that are below the box (j, j). The numbers $(\alpha_1, \alpha_2, \dots, \alpha_r | \beta_1, \beta_2, \dots, \beta_r)$ are the *Frobenius coordinates* (or the *characteristics*) of the partition λ.

For instance, the Frobenius coordinates of the partition $\lambda = (6, 4, 3, 3, 2)$ are $(5, 2, 0|4, 3, 1)$ (Figure 5.1).

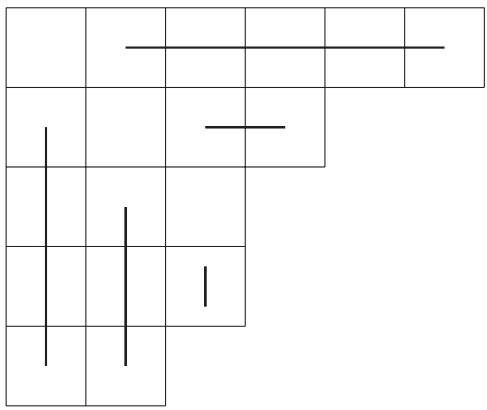


Figure 5.1

In order to express $F(z; \lambda)$ in terms of the Frobenius coordinates, we need a combinatorial lemma.

Lemma 5.2.2 *Let* $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$ *be a partition with Frobenius coordinates* $(\alpha_1, \alpha_2, \dots, \alpha_r | \beta_1, \beta_2, \dots, \beta_r)$. *We then have*

$$\{0, 1, 2, \dots, \ell - 1\} \coprod \{\alpha_1 + \ell, \alpha_2 + \ell, \dots, \alpha_r + \ell\}$$
$$= \{\lambda_1 + \ell - 1, \lambda_2 + \ell - 2, \dots, \lambda_\ell\} \coprod \{\ell - 1 - \beta_1, \ell - 1 - \beta_2, \dots, \ell - 1 - \beta_r\}$$

and both sides of the above equality represent two sets of $\ell + r$ *distinct numbers.*

Proof It is clear that

$$\lambda_i + \ell - i = \alpha_i + \ell \quad i = 1, 2, \dots, r.$$

It remains to show that

$$\{0, 1, 2, \dots, \ell - 1\} = \{\lambda_{r+1} + \ell - r - 1, \lambda_{r+2} + \ell - r - 2, \dots, \lambda_\ell\} \coprod$$
$$\coprod \{\ell - 1 - \beta_1, \ell - 1 - \beta_2, \dots, \ell - 1 - \beta_r\}. \quad (5.29)$$

To prove this identity, consider the segments at the end of the rows and at the end of the columns. Consecutively label all these segments with the numbers $0, 1, 2, \dots, \ell - 1$, starting from the bottom as in the examples in Figure 5.2.

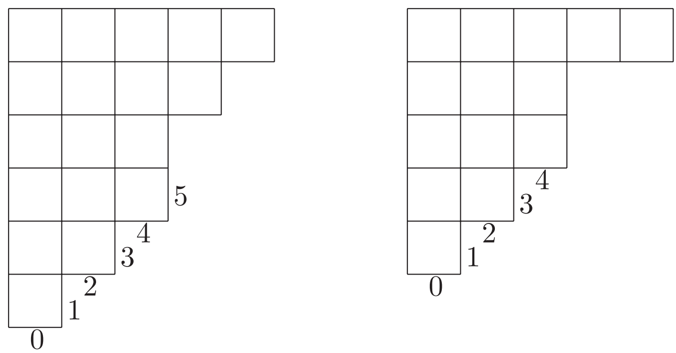


Figure 5.2

Note that the end of row r is the first unlabelled end. This may be proved in the following way: each number at the end of a row is moved to the first box of the same row; the number at the end of column j is first moved to the box (j, j) of the main diagonal and then to the box $(j, 1)$ (Figures 5.3(a)(b)).

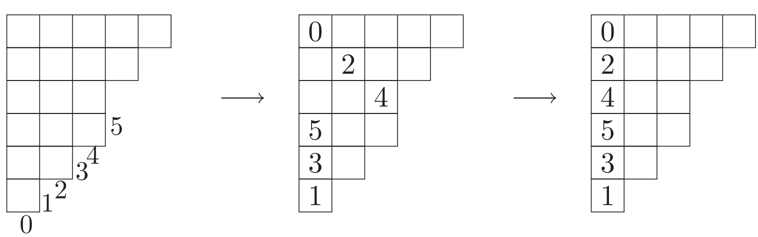


Figure 5.3(a)

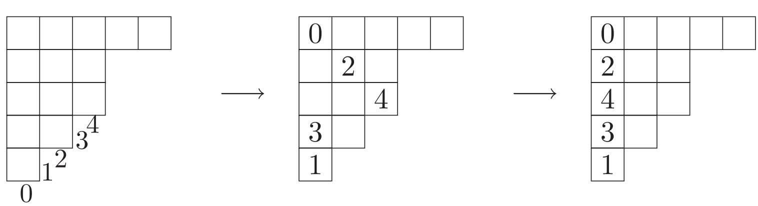


Figure 5.3(b)

Then, to prove (5.29) it suffices to note that the labels attached to the rows are exactly

$$\lambda_j + \ell - j, \quad j = r+1, r+2, \ldots, \ell,$$

while the labels attached at the end of the columns are

$$\ell - 1 - \beta_1, \ell - 1 - \beta_2, \ldots, \ell - 1 - \beta_r.$$

Indeed, arguing by induction we have the following. First of all, the label attached to the first column is $\ell - 1 - \beta_1 = \ell - 1 - (\lambda'_1 - 1) = \ell - \lambda'_1 = 0$. Suppose the ith column has attached the label $\ell - 1 - \beta_i$. Set $j = \lambda'_i - \lambda'_{i+1}$. Note that possibly $j = 0$. Then the numbers $(\ell - 1 - \beta_i) + 1, (\ell - 1 - \beta_i) + 2, \dots, (\ell - 1 - \beta_i) + j$ are attached to the j rows above and the label that we attach to the $(i+1)$st column is

$$\begin{aligned}(\ell - 1 - \beta_i) + (j+1) &= \ell - 1 - ((\lambda'_i - i) - (j+1)) \\ &= \ell - 1 - ((\lambda'_i - j) - (i+1)) \\ &= \ell - 1 - (\lambda'_{i+1} - (i+1)) \\ &= \ell - 1 - \beta_{i+1}.\end{aligned}$$

On the other hand, the label attached to the last (i.e. the ℓth) row is $\lambda_\ell + \ell - \ell = \lambda_\ell$ since the numbers $0, 1, 2, \dots, \lambda_\ell - 1$ are already attached to the first λ_ℓ columns. Suppose the kth row has the label $\lambda_k + \ell - k$. Set $j = \lambda_{k-1} - \lambda_k$. Then the numbers $(\lambda_k + \ell - k) + 1, (\lambda_k + \ell - k) + 2, \dots, (\lambda_k + \ell - k) + j$ are attached to the j columns on the right and the label that we attach to the $(k-1)$st row is

$$\lambda_k + \ell - k + j + 1 = (\lambda_k + j) + \ell - (k-1) = \lambda_{k-1} + \ell - (k-1). \qquad \square$$

We can now give Frobenius' original axpression for the Frobenius function.

Corollary 5.2.3 *Let λ be a partition and let $(\alpha_1, \alpha_2, \dots, \alpha_r | \beta_1, \beta_2, \dots, \beta_r)$ be its Frobenius coordinates. Then*

$$F(z; \lambda) = \prod_{i=1}^{r} \frac{z - \alpha_i}{z + \beta_i + 1}.$$

Proof The equality

$$\prod_{i=1}^{r}(z - \alpha_i) \cdot \prod_{i=1}^{\ell(\lambda)}(z + i) = \prod_{i=1}^{\ell(\lambda)}(z - \lambda_i + i) \cdot \prod_{i=1}^{r}(z + \beta_i + 1)$$

can be deduced from the set equality

$$\begin{aligned}&\{-\ell, -\ell+1, -\ell+2, \dots, -1\} \coprod \{\alpha_1, \alpha_2, \dots, \alpha_r\} \\ &\quad = \{\lambda_1 - 1, \lambda_2 - 2, \dots, \lambda_\ell - \ell\} \coprod \{-1 - \beta_1, -1 - \beta_2, \dots, -1 - \beta_r\}\end{aligned}$$

which follows by subtracting "ℓ" from each element in the equality in the statement of the previus lemma. $\square$

5.2.2 Lagrange interpolation formula

Fix two sets of variables, $\{x_1, x_2, \ldots, x_n\}$ and $\{y_1, y_2, \ldots, y_m\}$. The symmetric polynomials in the *sum* of the two alphabets is just the symmetric polynomials in the set of variables $\{x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_m\}$. Set (cf. Lemma 4.1.7)

$$E_x(t) = \prod_{i=1}^{n}(1 + x_i t) \quad E_y(t) = \prod_{j=1}^{m}(1 + y_j t)$$
$$H_x(t) = \frac{1}{\prod_{i=1}^{n}(1 - x_i t)} \quad H_y(t) = \frac{1}{\prod_{j=1}^{m}(1 - y_j t)}.$$

Then the elementary and complete symmetric functions in the sum of the two alphabets may be obtained by means of the generating series:

$$E_{x,y}(t) \equiv E_x(t)E_y(t) = \sum_{k=0}^{n} e_k(x, y)t^k$$

and

$$H_{x,y}(t) \equiv H_x(t)H_y(t) = \sum_{k=0}^{n} h_k(x, y)t^k.$$

We define the symmetric polynomials in the *difference* x/y on the two alphabets (it is not a set-theoretic difference nor an algebraic difference) as follows: the complete symmetric functions $h_k(x_1, x_2, \ldots, x_n | y_1, y_2, \ldots, y_m)$ are defined by means of the following generating function:

$$\begin{aligned} H_{x/y}(t) &:= \frac{H_x(t)}{H_y(t)} \\ &\equiv H_x(t)E_y(-t) \\ &\equiv \frac{\prod_{j=1}^{m}(1 - y_j t)}{\prod_{i=1}^{n}(1 - x_i t)} \end{aligned}$$

and

$$H_{x/y}(t) = \sum_{k=0}^{\infty} h_k(x_1, x_2, \ldots, x_n | y_1, y_2, \ldots, y_m)t^k. \tag{5.30}$$

Now $h_k(x/y)$ is a polynomial in the variables $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_m$ which is symmetric in each set of variables, separately. See [83] p. 58, and [77]. Now we give a basic interpolation formula attributed to Lagrange.

Lemma 5.2.4 (Lagrange) *Let* $\{x_1, x_2, \ldots, x_n\}$ *and* $\{y_1, y_2, \ldots, y_m\}$ *be two sets of variables and let* r *be a nonnegative integer. Then*

$$\sum_{i=1}^{n} x_i^r \cdot \frac{\prod_{j=1}^{m}(x_i - y_j)}{\prod_{\substack{k=1\\k\neq i}}^{n}(x_i - x_k)} = h_{r-n+m+1}(x/y). \tag{5.31}$$

Proof First of all, we prove the following interpolation formula: if $x_1, x_2, \ldots, x_n$ are distinct real (or complex) numbers

$$t^{n-m-1} \cdot \prod_{j=1}^{m}(1 - ty_j) = \sum_{i=1}^{n}\left[\left(\prod_{\substack{k=1\\k\neq i}}^{n}\frac{1 - tx_k}{x_i - x_k}\right)\cdot\prod_{j=1}^{m}(x_i - y_j)\right]. \tag{5.32}$$

Both sides of (5.32) are polynomials of degree $n-1$ in the variable t, so it suffices to show that they assume the same values at n distinct points. Setting $t = 1/x_h$ in the left-hand side of (5.32) we immediately get the value $\left(\frac{1}{x_h}\right)^{n-1} \cdot \prod_{j=1}^{m}(x_h - y_j)$, while in the right-hand side, for $t = 1/x_h$, only the term with $i = h$ is nonzero and therefore we get again

$$\prod_{\substack{k=1\\k\neq h}}^{n}\frac{1 - \frac{x_k}{x_h}}{x_h - x_k}\cdot\prod_{j=1}^{m}(x_h - y_j) = \left(\frac{1}{x_h}\right)^{n-1}\cdot\prod_{j=1}^{m}(x_h - y_j).$$

Since (5.32) is verified for $t = 1/x_h$, $h = 1, 2, \ldots, n$, it is true for all other values of t. If we multiply both sides of (5.32) by $H_x(t) = 1/\prod_{i=1}^{n}(1 - x_i t)$ and we apply (5.30) we get

$$\begin{aligned}\sum_{i=1}^{n}\frac{\prod_{j=1}^{m}(x_i - y_j)}{(1 - x_i t)\prod_{\substack{k=1\\k\neq i}}^{n}(x_i - x_k)} &= t^{n-m-1}\cdot\frac{\prod_{j=1}^{m}(1 - ty_j)}{\prod_{i=1}^{n}(1 - tx_i)}\\ &= \sum_{k=0}^{\infty} h_k(x/y)t^{k+n-m-1}.\end{aligned}$$

On the other hand, a simple application of the geometric series expansion $\frac{1}{1-x_i t} = \sum_{r=0}^{\infty} t^r x_i^r$ yields

$$\sum_{i=1}^{n}\frac{\prod_{j=1}^{m}(x_i - y_j)}{(1 - x_i t)\prod_{\substack{k=1\\k\neq i}}^{n}(x_i - x_k)} = \sum_{r=0}^{\infty} t^r\left[\sum_{i=1}^{n} x_i^r \frac{\prod_{j=1}^{m}(x_i - y_j)}{\prod_{\substack{k=1\\k\neq i}}^{n}(x_i - x_k)}\right].$$

Then (5.31) follows by equating the coefficients of t^r in the corresponding power series expansions. □

For a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell) \vdash n$ and two integers k, m with $1 \leq k \leq \ell$, we set $\lambda - \epsilon_k m = (\lambda_1, \lambda_2, \ldots, \lambda_{k-1}, \lambda_k - m, \lambda_{k+1}, \ldots, \lambda_\ell)$. We can also extend the formula in Proposition 4.2.10 by setting

$$d_{\lambda-\epsilon_k m} = \frac{(n-m)!a_\delta(\lambda_1+\ell-1, \lambda_2+\ell-2, \ldots, \lambda_k+\ell+k-m, \ldots, \lambda_\ell)}{(\lambda_1+\ell-1)!(\lambda_2+\ell-2)!\cdots(\lambda_k+\ell-k-m)!\cdots\lambda_\ell!}. \tag{5.33}$$

We introduce the following notation:

$$\begin{aligned} d_\lambda(k,m) &= \frac{n!}{(n-m)!}\cdot\frac{d_{\lambda-\epsilon_k m}}{d_\lambda} \\ &= \frac{(\lambda_k+\ell-k)!}{(\lambda_k+\ell-k-m)!}\cdot\prod_{\substack{i=1\\ i\neq k}}^{\ell}\frac{\lambda_k-\lambda_i+i-k-m}{\lambda_k-\lambda_i+i-k}. \end{aligned} \tag{5.34}$$

Theorem 5.2.5 *In the Taylor series at infinity*

$$[z]_m\frac{F(z-m;\lambda)}{F(z;\lambda)} = \sum_{r=-m}^{\infty} C_r(\lambda;m)z^{-r}$$

we have

$$C_{r+1}(\lambda;m) = -m\sum_{k=1}^{\ell(\lambda)} d_\lambda(k,m)(\lambda_k-k)^r$$

for $r = 0, 1, 2, \ldots$

Proof Consider the alphabets $\{a_i = \lambda_i - i,\ i = 1, 2, \ldots, \ell\}$ and $\{b_i = \lambda_i - i + m,\ i = 1, 2, \ldots, \ell+m\}$ (where $\lambda_{\ell+1} = \lambda_{\ell+2} = \cdots = \lambda_{\ell+m} = 0$). Note that

$$\begin{aligned} z^m[1/z]_m\frac{F(1/z-m;\lambda)}{F(1/z;\lambda)} &= \prod_{i=1}^{m}[1+z(i-m)]\prod_{i=1}^{\ell}\frac{1+zi}{1+z(i-m)}\prod_{i=1}^{\ell}\frac{1-z(\lambda_i-i+m)}{1-z(\lambda_i-i)} \\ &=_{(*)} \prod_{i=1}^{\ell}\frac{1-z(\lambda_i-i+m)}{1-z(\lambda_i-i)}\cdot\prod_{i=1}^{m}[1-z(m-\ell-i)] \\ &= H_{a/b}(z), \end{aligned}$$

where $=_{(*)}$ follows from the elementary identity

$$\begin{aligned} \frac{\prod_{i=1}^{m}[1+z(i-m)]\prod_{i=1}^{\ell}(1+zi)}{\prod_{i=1}^{\ell}[1+z(i-m)]} &= \frac{\prod_{i=-m+1}^{\ell}(1+zi)}{\prod_{i=-m+1}^{\ell-m}(1+zi)} \\ &= \prod_{i=1}^{m}[1+z(i+\ell-m)]. \end{aligned}$$

It follows that

$$
\begin{aligned}
[z]_m \frac{F(z-m;\lambda)}{F(z;\lambda)} &= z^m H_{a/b}(1/z) \\
&= \sum_{k=0}^{\infty} h_k(a/b) z^{m-k} \\
&= \sum_{r=-m}^{\infty} h_{r+m}(a/b) z^{-r}.
\end{aligned}
\tag{5.35}
$$

We now apply the Lagrange lemma: we have, for $1 \le i \le \ell$,

$$
\begin{aligned}
\frac{\prod_{i=1}^{\ell+m}(a_k - b_i)}{\prod_{\substack{i=1,\\ i\neq k}}^{\ell}(a_k - a_i)} &= \frac{\prod_{i=1}^{\ell+m}(\lambda_k - k - \lambda_i + i - m)}{\prod_{\substack{i=1,\\ i\neq k}}^{\ell}(\lambda_k - k - \lambda_i + i)} \\
&= -m \prod_{\substack{i=1,\\ i\neq k}}^{\ell} \frac{\lambda_k - k - \lambda_i + i - m}{\lambda_k - k - \lambda_i + i} \cdot \prod_{i=1}^{m}(\lambda_k - k + i + \ell - m) \\
&= -m \left(\prod_{\substack{i=1,\\ i\neq k}}^{\ell} \frac{\lambda_k - k - \lambda_i + i - m}{\lambda_k - k - \lambda_i + i} \right) \cdot \frac{(\lambda_k + \ell - k)!}{(\lambda_k + \ell - k - m)!} \\
&= -m d_\lambda(k, m).
\end{aligned}
$$

Therefore, the Lagrange lemma yields, for $r \ge 0$,

$$
-m \sum_{k=1}^{\ell(\lambda)} d_\lambda(k, m)(\lambda_k - k)^r = h_{r+m+1}(a/b)
$$

and this, combined with (5.35), ends the proof. □

5.2.3 The Taylor series at infinity for the Frobenius quotient

We now present the main result of this section: the Taylor series at infinity for the Frobenius quotient. We still need another definition: for a partition λ, r a nonnegative integer and m an integer (but it may be any complex number) we set

$$c_r^\lambda(m) = \sum_{\substack{n,q,h\geq 0\\ q+h+2n\leq r}} (-m)^{n-1}(m+1)^h \binom{n+h+q-1}{h} \cdot \left[\sum_{k=0}^{\min\{n,r-2n-h\}} \binom{n+|\lambda|-1}{n-k} F_{r-2n-h,q,k}(\lambda)\right], \quad (5.36)$$

where $F_{r-2n-h,q,k}(\lambda)$ is the Lassalle symmetric function evaluated at the content of λ (see (5.27): we recall that it is a symmetric function in the content of λ but a shifted symmetric function in the variables $\lambda_1, \lambda_2, \ldots, \lambda_\ell$; see Remark 5.1.19). In the following theorem, we expand the quotient $\frac{F(z-m;\lambda)}{F(z;\lambda)}$ in descending powers of $(z+1)$.

Theorem 5.2.6 *We have*

$$\frac{F(z-m;\lambda)}{F(z;\lambda)} = -m\sum_{r\geq 0} c_r^\lambda(m)\frac{1}{(z+1)^r}.$$

Proof It is a long series of implications where we have merged the arguments in the proofs of Theorem 3.1. and Theorem 5.1 in [80].

$$\begin{aligned}
\frac{F(z-m;\lambda)}{F(z;\lambda)} &= \prod_{i=1}^{\ell(\lambda)} \frac{z-\lambda_i+i-m}{z+i-m}\cdot\frac{z+i}{z-\lambda_i+i}\\
&= \prod_{i=1}^{\ell(\lambda)}\prod_{j=1}^{\lambda_i} \frac{z-j+i-m}{z-j+i-m+1}\cdot\frac{z-j+i+1}{z-j+i}\\
&= \prod_{i=1}^{\ell(\lambda)}\prod_{j=1}^{\lambda_i}\left[1-\frac{-m}{(z-j+i-m)(z-j+i+1)}\right]^{-1}\\
&= \prod_{i=1}^{\ell(\lambda)}\prod_{j=1}^{\lambda_i}\exp\left\{\sum_{k=1}^{\infty}\frac{(-m)^k}{k}[(z-j+i-m)(z-j+i+1)]^{-k}\right\}
\end{aligned}$$

(since $\frac{1}{1-t} = \exp[-\log(1-t)] = \exp[\sum_{k=1}^\infty \frac{t^k}{k}]$)

$$= \prod_{i=1}^{\ell(\lambda)}\prod_{j=1}^{\lambda_i}\exp\left\{\sum_{k=1}^{\infty}\frac{(-muv)^k}{k}[1+v(j-i)]^{-k}[1+u(j-i)]^{-k}\right\}$$

(by setting $u = -\frac{1}{z+1}$ and $v = -\frac{1}{z-m}$)

$$= \prod_{i=1}^{\ell(\lambda)}\prod_{j=1}^{\lambda_i}\exp\left\{\sum_{k=1}^{\infty}\frac{(-muv)^k}{k}\sum_{r,s\geq 0}\frac{(k)_r(k)_s}{r!s!}(-u)^r(-v)^s(j-i)^{r+s}\right\}$$

(since $(1-t)^{-k} = \sum_{r\geq 0} \frac{(k)_r}{r!} t^r$)

$$= \exp\left\{\sum_{k=1}^{\infty} \frac{(-muv)^k}{k}\left[\sum_{r,s\geq 0} \frac{(k)_r (k)_s}{r!s!}(-u)^r(-v)^s d_{r+s}(\lambda)\right]\right\}$$

$$= \exp\left\{\sum_{k=1}^{\infty} \frac{(-muv)^k}{k}\left[\sum_{t=0}^{\infty} d_t(\lambda)\sum_{s=0}^{t}(-u)^{t-s}(-v)^s \frac{(k)_{t-s}(k)_s}{(t-s)!s!}\right]\right\}$$

$$= \exp\left[\sum_{t=0}^{\infty} d_t(\lambda)(-u)^t \sum_{s=0}^{t}\left(\frac{v}{u}\right)^s \sum_{k=1}^{\infty}\frac{(-muv)^k (k)_{t-s}(k)_s}{k(t-s)!s!}\right]$$

$$= (1+muv)^{-|\lambda|}\exp\left[\sum_{t=1}^{\infty} d_t(\lambda)(-u)^t \sum_{s=0}^{t}\left(\frac{v}{u}\right)^s(-muv)\varphi_{t,s}(-muv)\right]$$

(since, by (5.15), $\varphi_{t,s}(-muv) = \sum_{h=0}^{\infty}\frac{(s+1)_h(t-s+1)_h}{h!(h+1)!}(-muv)^h$ equals $-\frac{1}{muv}\sum_{k=1}^{\infty}\frac{(-muv)^k(k)_{t-s}(k)_s}{k(t-s)!s!}$, $d_0(\lambda) = |\lambda|$ and, for $t=0$, $\sum_{k=1}^{\infty}\frac{(-muv)^k}{k}$ $= -\log(1+muv)$)

$$= (1+muv)^{-|\lambda|}\exp\left[\sum_{t=1}^{\infty} d_t(\lambda)(-u)^t \frac{1}{t} G_t\left(\frac{v}{u}, \frac{-muv}{1+muv}\right)\right]$$

(by Proposition 5.1.15 and Proposition 5.1.13)

$$= (1+muv)^{-|\lambda|}\prod_{t=1}^{\infty}\exp\left[d_t(\lambda)(-u)^t \frac{1}{t} G_t\left(\frac{v}{u}, \frac{-muv}{1+muv}\right)\right]$$

$$= (1+muv)^{-|\lambda|}\sum_{h_1,h_2,\ldots\geq 0}\prod_{t=1}^{\infty}\frac{1}{h_t!}\left[(-u)^t\frac{d_t(\lambda)}{t} G_t\left(\frac{v}{u}, \frac{-muv}{1+muv}\right)\right]^{h_t}$$

$$= (1+muv)^{-|\lambda|}\sum_{\mu}(-u)^{|\mu|}\frac{1}{z_\mu}\prod_{t=1}^{\infty}\left[d_t(\lambda) G_t\left(\frac{v}{u}, \frac{-muv}{1+muv}\right)\right]^{m_t(\mu)}$$

(if $\sum_\mu$ is the sum over all partitions $\mu = (1^{h_1}, 2^{h_2}, \ldots)$, including $\mu = (0)$, and $z_\mu = h_1!1^{h_1}h_2!2^{h_2}\cdots$)

$$= (1+muv)^{-|\lambda|}\sum_{\mu}(-u)^{|\mu|}\frac{1}{z_\mu}\left[\sum_{p=0}^{|\mu|}\sum_{k=\ell(\mu)}^{|\mu|}\left\langle \begin{matrix}\mu\\ p\end{matrix}\right\rangle_k \left(\frac{v}{u}\right)^p\left(\frac{-muv}{1+muv}\right)^k\right]\cdot$$

$$\cdot\prod_{t=1}^{\infty}[d_t(\lambda)]^{m_t(\mu)}$$

(by (5.18))

$$= (1+muv)^{-|\lambda|}\sum_{p,q\geq 0}(-u)^q(-v)^p\sum_{k=0}^{p+q}\sum_{\mu\vdash p+q}\frac{\left\langle \begin{matrix}\mu\\ p\end{matrix}\right\rangle_k}{z_\mu}\prod_{t\geq 1}[d_t(\lambda)]^{m_t(\mu)}\left(\frac{-muv}{1+muv}\right)^k$$

(by setting $q=|\mu|-p$)

$$= \sum_{p,q\geq 0}(-u)^q(-v)^p\sum_{k=0}^{p+q}\left[F_{p+q,q,k}(\lambda)\frac{(-muv)^k}{(1+muv)^{k+|\lambda|}}\right]$$

(by Definition 5.1.17, by (5.27) and the convention $F_{000}=1$)

$$= \sum_{p,q\geq 0}(-u)^q(-v)^p\sum_{k=0}^{p+q}F_{p+q,q,k}(\lambda)\sum_{n=k}^{\infty}\binom{|\lambda|+n-1}{n-k}(-muv)^n$$

(by (5.11))

$$= \sum_{n=0}^{\infty}(-muv)^n\sum_{p,q\geq 0}(-u)^q(-v)^p\sum_{k=0}^{\min\{n,p+q\}}\binom{|\lambda|+n-1}{n-k}F_{p+q,q,k}(\lambda)$$

$$= \sum_{n=0}^{\infty}(-m)^n\sum_{p,q\geq 0}\frac{1}{(z+1)^{p+q+2n}}\left(1-\frac{m+1}{z+1}\right)^{-q-n}\cdot$$

$$\cdot\left[\sum_{k=0}^{\min\{n,p+q\}}\binom{|\lambda|+n-1}{n-k}F_{p+q,q,k}(\lambda)\right]$$

(we have set $-u=\frac{1}{z+1}$ and $-v=\frac{1}{z+1}(1-\frac{m+1}{z+1})^{-1}$)

$$= \sum_{n,p,q,h\geq 0}(-m)^n(m+1)^h\binom{h+q+n-1}{h}\cdot$$

$$\cdot\left[\sum_{k=0}^{\min\{n,p+q\}}\binom{|\lambda|+n-1}{n-k}F_{p+q,q,k}(\lambda)\right]\frac{1}{(z+1)^{p+q+2n+h}}$$

(by (5.11))

$$= (-m)\sum_{r\geq 0}c_r^{\lambda}(m)\frac{1}{(z+1)^r}$$

(setting $r=p+q+2n+h$). □

We can now give the Taylor series at infinity of the Frobenius quotient.

Corollary 5.2.7 *We have*

$$[z]_m \frac{F(z-m;\lambda)}{F(z;\lambda)} = -m \sum_{h \geq -m} \left[\sum_{\substack{q,r \geq 0 \\ q+r-h \leq m}} c_r^\lambda(m)(-1)^q \binom{r+q-1}{q} s(m,q+r-h) \right] z^{-h}.$$

Proof From (5.10) we get

$$\frac{1}{(1+z)^r} = z^{-r} \cdot \frac{1}{(1+z^{-1})^r} = \sum_{q=0}^{\infty} (-1)^q \binom{r+q-1}{q} z^{-q-r}$$

and therefore (recalling the definition of the Stirling numbers of the first kind, see Section 4.4.3),

$$[z]_m \frac{F(z-m;\lambda)}{F(z;\lambda)} = -m \sum_{t=0}^{m} s(m,t) z^t \sum_{q,r \geq 0} c_r^\lambda(m)(-1)^q \binom{r+q-1}{q} z^{-q-r}$$

(if $t = q + r - h$)

$$= -m \sum_{h \geq -m} \left[\sum_{\substack{q,r \geq 0 \\ q+r-h \leq m}} c_r^\lambda(m)(-1)^q \binom{r+q-1}{q} s(m,q+r-h) \right] z^{-h}. \qquad \square$$

We now combine the results in Theorem 5.2.5 and Corollary 5.2.7.

Theorem 5.2.8 *For $r \geq 0$, we have:*

$$\sum_{k=1}^{\ell(\lambda)} d_\lambda(k,m)(\lambda_k - k)^r = (-1)^r \sum_{k=0}^{r} \sum_{h=k}^{m+1+k} c_h^\lambda(m)(-1)^k \binom{r}{k} s(m+1,h-k).$$

Proof The left-hand side is exactly $1/-m$ times the coefficients of $1/z^{r+1}$ in Theorem 5.2.5, while in Corollary 5.2.7 $1/z^{r+1}$ has the coefficient (without $(-m)$)

$$\sum_{\substack{q,h \geq 0 \\ q+h-(r+1) \leq m}} c_h^\lambda(m)(-1)^q \binom{h+q-1}{q} s(m,q+h-r-1)$$

$$= (-1)^r \sum_{k=0}^{r} \sum_{h=k}^{m+1+k} c_h^\lambda(m)(-1)^k \binom{r}{k} s(m+1,h-k),$$

where equality follows from Proposition 5.1.5. $\square$

5.2.4 Some explicit formulas for the coefficients $c_r^\lambda(m)$

In this section, we give some explicit formulas for the coefficients $c_r^\lambda(m)$. We recall that $d_r(\lambda) = \sum_{i=1}^{\ell(\lambda)} \sum_{j=1}^{\lambda_i} (j-i)^r$ (and $d_0(\lambda) = |\lambda|$).

Proposition 5.2.9 *Let $r \geq 1$. Consider the terms in (5.36) with $k = 0$. Then necessarily $q = 0$, $1 \leq n \leq r/2$ and $h = r - 2n$. Moreover, these terms are equal to*

$$(m)^{n-1}(m+1)^{r-2n}\binom{r-n-1}{r-2n}\binom{|\lambda|+n-1}{n}$$

for $1 \leq n \leq r/2$.

Proof It suffices to note that $F_{np0} \neq 0$ if and only if $n = p = 0$ and, in this case, one has $F_{000} = 1$. Therefore we have $q = 0$ and $h = r - 2n$. Moreover, under these conditions we have $\binom{n+h+q-1}{h} = \binom{r-n-1}{r-2n}$, which is equal to 0 if $r \geq 1$ and $n = 0$. □

Proposition 5.2.10 *Fix r, n, k such that $r \geq 2$, $1 \leq n \leq r/2$ and $k = 1$. Then the corresponding terms in (5.36) are*

$$\sum_{h=0}^{r-2n-1} (-m)^{n-1}(m+1)^h \binom{n+|\lambda|-1}{n-1} \times \left[\binom{r-n}{h+1} - \binom{n+h-1}{h+1}\right] d_{r-2n-h}(\lambda).$$

Proof First of all, we have $F_{r-2n-h,q,1} = p_{r-2n-h}$ (see Section 5.1.5). Moreover, the conditions in the sums in (5.36) yields the bounds $0 \leq h \leq r - 2n - 1$ and $0 \leq q \leq r - 2n - h$. Finally,

$$\begin{aligned}\sum_{q=0}^{r-2n-h} \binom{n+h+q-1}{h} &= \sum_{q=0}^{r-2n-h} \binom{n+q-1+h}{n+q-1} \\ \text{(setting } p = n+q-1\text{)} \quad &= \sum_{p=0}^{r-n-h-1} \binom{p+h}{p} - \sum_{p=0}^{n-2} \binom{p+h}{p} \\ \text{(by Exercise 5.1.3)} \quad &= \binom{r-n}{h+1} - \binom{n+h-1}{h+1}.\end{aligned}$$

□

We now give the formulas for $c_r^\lambda(m)$ for $0 \le r \le 7$.

$$c_0^\lambda(m) = -\frac{1}{m}$$

$$c_1^\lambda(m) = 0$$

$$c_2^\lambda(m) = d_0(\lambda) = |\lambda|$$

$$c_3^\lambda(m) = (m+1)d_0(\lambda) + 2d_1(\lambda)$$

$$c_4^\lambda(m) = (m+1)^2 d_0(\lambda) + 3(m+1)d_1(\lambda) + 3d_2(\lambda) - \frac{m}{2}d_0(\lambda)(d_0(\lambda)+1)$$

$$\begin{aligned} c_5^\lambda(m) = {} & (m+1)^3 d_0(\lambda) + \sum_{h=0}^{2}\binom{4}{3-h}(m+1)^h d_{3-h}(\lambda) \\ & - m(m+1)d_0(\lambda)(d_0(\lambda)+1) - 2md_1(\lambda)(d_0(\lambda)+1) \end{aligned}$$

$$\begin{aligned} c_6^\lambda(m) = {} & (m+1)^4 d_0(\lambda) + \sum_{h=0}^{3}\binom{5}{h+1}(m+1)^h d_{4-h}(\lambda) \\ & - \frac{3}{2}m(m+1)^2 d_0(\lambda)(d_0(\lambda)+1) \\ & - m\sum_{h=0}^{1}(m+1)^4(d_0(\lambda)+1)\left[\binom{4}{h+1} - 1\right]d_{2-h}(\lambda) \\ & + \frac{1}{6}m^2 d_0(\lambda)(d_0(\lambda)+1)(d_0(\lambda)+2) - 2md_1(\lambda) - 2md_2(\lambda) \end{aligned}$$

$$\begin{aligned} c_7^\lambda(m) = {} & (m+1)^5 d_0(\lambda) + \sum_{h=0}^{4}(m+1)^h\binom{6}{h+1}d_{5-h}(\lambda) \\ & - 2m(m+1)^3 d_0(\lambda)(d_0(\lambda)+1) \\ & - m\sum_{h=0}^{2}(m+1)^h(d_0(\lambda)+1)\left[\binom{5}{h+1} - 1\right]d_{3-h}(\lambda) \\ & + m^2(d_0(\lambda)+2)(d_0(\lambda)+1)\left[\frac{m+1}{2}d_0(\lambda) + d_1(\lambda)\right] \\ & - 3m\left[2(m+1)d_1(\lambda)^2 + 2(m+1)d_2(\lambda) + d_3(\lambda) + d_1(\lambda)d_2(\lambda)\right]. \end{aligned}$$

Exercise 5.2.11 Check the above formulas.

Hint: For $r = 0$ we necessarily have $n = q = h = k = 0$, and since $F_{000} = 1$ we get $c_0^\lambda(m) = -1/m$. For $r = 1$, we necessarily have $n = k = 0$; for $h = 1$ and $q = 0$ we have $\binom{n+h+q-1}{h} = \binom{0}{1} = 0$, while for $h = 0$ and $q = 1$ we have $F_{110} = 0$. In conclusion, $c_1^\lambda(m) = 0$. For $2 \le r \le 5$ the formulae in Proposition 5.2.9 and Proposition 5.2.10 suffice to compute $c_r^\lambda(m)$. For $r = 6$

we also need the formulas $F_{2q2} = \binom{2}{q} h_2$ and $h_2 = \frac{1}{2} p_{1,1} + \frac{1}{2} p_2$. Finally, for $r = 7$ we also need the formulas in Example 5.1.18.

5.3 Lassalle's explicit formulas for the characters of the symmetric group

In this section, we complete the exposition of the results in [81] by providing Lassalle's explicit formula for the characters of the symmetric group. Actually, these are formulas for the characters of the symmetric group normalized by the dimension of the corresponding induced representation, namely χ^λ / d_λ. That is, they are spherical functions of the Gelfand pair $(\mathfrak{S}_n \times \mathfrak{S}_n, \widetilde{\mathfrak{S}}_n)$ (see Example 1.5.26) and the eigenvalues of the convolution operators with kernel $\frac{1}{|\mathcal{C}|} 1_{\mathcal{C}}$, with $\mathcal{C}$ a conjugacy class of $\mathfrak{S}_n$ (see Corollary 1.5.12). Following [81] we introduce the notation

$$\hat{\chi}^\lambda_\mu = \frac{1}{d_\lambda} \chi^\lambda_\mu.$$

We extend this definition to virtual characters, that is, to any $\alpha \in \mathbb{Z}^\ell$ (see (4.35) and (5.33)), by setting $\hat{\chi}^\alpha_\mu = \chi^\alpha_\mu / \chi^\alpha_{(1^n)}$.

5.3.1 Conjugacy classes with one nontrivial cycle

We first present the value of $\hat{\chi}^\lambda_\mu$ when $\mu = (m, 1^{n-m})$, that is, the value of $\hat{\chi}^\lambda$ on a conjugacy class with exactly one nontrivial cycle. To start, we give a basic recursion identity that follows immediately from the version of Murnaghan's rule in Theorem 4.2.15. The coefficient $d_\lambda(k, m)$ is as in (5.34).

Proposition 5.3.1 (Basic recursion formula) *Let $\lambda, \mu \vdash n$ and m a part of μ. We have*

$$\hat{\chi}^\lambda_\mu = \frac{1}{[n]_m} \sum_{k=1}^{\ell(\lambda)} d_\lambda(k, m) \hat{\chi}^{\lambda - \epsilon_k m}_{\mu \setminus m}.$$

Proof From Theorem 4.2.15 we deduce

$$\hat{\chi}^\lambda_\mu = \frac{1}{d_\lambda} \sum_{k=1}^{\ell(\lambda)} \chi^{\lambda - \epsilon_k m}_{\mu \setminus m} = \frac{1}{[n]_m} \sum_{k=1}^{\ell(\lambda)} d_\lambda(k, m) \hat{\chi}^{\lambda - \epsilon_k m}_{\mu \setminus m}.$$

□

Theorem 5.3.2 *Let $\lambda \vdash n$, $2 \le m \le n$ and $\mu = (m, 1^{n-m})$. Then*

$$\hat{\chi}^\lambda_\mu = \frac{1}{[n]_m} \sum_{h=2}^{m+1} s(m+1, h) c^\lambda_h(m).$$

Proof From Proposition 5.3.1 we get

$$\hat{\chi}^\lambda_\mu = \frac{1}{[n]_m}\sum_{k=1}^{\ell(\lambda)} d_\lambda(k,m).$$

Then we may apply Theorem 5.2.8 with $r = 0$, keeping into account that $s(m+1,0) = 0$ and $c_1^\lambda(m) = 0$. □

Example 5.3.3 We now give some examples of applications of Theorem 5.3.2. We compute $\hat{\chi}^\lambda_{m,1^{n-m}}$ for $2 \le m \le 6$. We shall use the results in Section 5.2.4 and the table of the Stirling numbers of the first kind (see Section 4.4.3). Recalling that $d_0(\lambda) = |\lambda| = n$, we have

$$\begin{aligned}
\hat{\chi}^\lambda_{2,1^{n-2}} &= \frac{1}{[n]_2}\left[-3c_2^\lambda(2) + c_3^\lambda(2)\right] = \frac{2}{n(n-1)}d_1(\lambda)\\
\hat{\chi}^\lambda_{3,1^{n-3}} &= \frac{1}{[n]_3}\left[11c_2^\lambda(3) - 6c_3^\lambda(3) + c_4^\lambda(3)\right] = \frac{1}{[n]_3}\left[3d_2(\lambda) - \frac{3}{2}n(n-1)\right]\\
\hat{\chi}^\lambda_{4,1^{n-4}} &= \frac{1}{[n]_4}\left[-50c_2^\lambda(4) + 35c_3^\lambda(4) - 10c_4^\lambda(4) + c_5^\lambda(4)\right]\\
&= \frac{1}{[n]_4}\left[4d_3(\lambda) - 4(2n-3)d_1(\lambda)\right]\\
\hat{\chi}^\lambda_{5,1^{n-5}} &= \frac{1}{[n]_5}\left[274c_2^\lambda(5) - 225c_3^\lambda(5) + 85c_4^\lambda(5) - 15c_5^\lambda(5) + c_6^\lambda(5)\right]\\
&= \frac{1}{[n]_5}\left[5d_4(\lambda) - 5(3n-10)d_2(\lambda) - 10d_1(\lambda)^2 + 25\binom{n}{3} - 15\binom{n}{2}\right]\\
\hat{\chi}^\lambda_{6,1^{n-6}} &= \frac{1}{[n]_6}\left[-1764c_2^\lambda(6) + 1624c_3^\lambda(6) - 735c_4^\lambda(6) + 175c_5^\lambda(6) - 21c_6^\lambda(6) + c_7^\lambda(6)\right]\\
&= \frac{1}{[n]_6}[6d_5(\lambda) + 24(7-n)d_3(\lambda) + 12(3n-4)(n-5)d_1(\lambda) - 18d_1(\lambda)d_2(\lambda)].
\end{aligned}$$

Remark 5.3.4 We can extend the formula in Theorem 5.3.2 in the following way:

$$\hat{\chi}^\alpha_\mu = \frac{1}{[n]_m}\sum_{h=2}^{m+1} s(m+1,h)c_h^\alpha(m) \tag{5.37}$$

for all $\alpha \in \mathbb{Z}^\lambda$. Now, if $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_i, \alpha_{i+1}, \ldots, \alpha_\ell)$ and $\alpha' = (\alpha_1, \alpha_2, \ldots, \ldots, \alpha_{i+1} - 1, \alpha_i + 1, \ldots, \alpha_\ell)$ then $\hat{\chi}^\alpha = \hat{\chi}^{\alpha'}$ (by (4.36)). Since the right-hand side is a shift symmetric polynomial (see Remark 5.1.19) by (5.28) the identity (5.37) holds true for every $\alpha \in \mathbb{Z}^\ell$. For instance, from

$d_1(\lambda) = \frac{1}{2} p_2^*(\lambda)$ we get

$$\hat{\chi}^{\lambda}_{2,1^{n-2}} = \frac{1}{n(n-1)} p_2^*(\lambda) = \frac{1}{n(n-1)} \sum_{i=1}^{\ell} \{[\lambda_i - i + 1]_2 - [-i+1]_2\}.$$

Often, this formula is written in the following way:

$$\hat{\chi}^{\lambda}_{2,1^{n-2}} = \frac{1}{n(n-1)} \sum_{i=1}^{\ell} \lambda_i(\lambda_i - 2i + 1),$$

see [20, Theorem 10.6.2] and [26].

5.3.2 Conjugacy classes with two nontrivial cycles

We now give the explicit formula for $\hat{\chi}^{\lambda}_{\mu}$ when $\mu = (m, q, 1^{n-m-q})$. First we need two technical lemmas.

Lemma 5.3.5 *Let λ be a partition of n and h, m, k, q positive integers with $k \leq \ell(\lambda)$. Then*

$$c_h^{\lambda - m\epsilon_k}(q) = c_h^{\lambda}(q) + mq \sum_{\substack{r,s,t \geq 0 \\ r+s+t+2=h}} c_t^{\lambda}(q)(\lambda_k - k + q + 1)^r (\lambda_k - k - m + 1)^s.$$

Proof We first note that both sides may be seen as polynomials in the variables $\lambda_1, \lambda_2, \ldots, \lambda_\ell$, so that, by the principle of analytic continuation (see Remark 5.1.19), it suffices to prove the identity when $\lambda - m\epsilon_k$ is a partition. Note also that $c_h^{\lambda - m\epsilon_k}(q)$ is a shifted symmetric polynomial in $\lambda_1, \lambda_2, \ldots, \lambda_k - m, \ldots, \lambda_\ell$, so that also the second member must be shifted symmetric in the same variable. By elementary manipulations we get

$$\begin{aligned}
\frac{F(z-q-1; \lambda - m\epsilon_k)}{F(z-1; \lambda - m\epsilon_k)} \frac{F(z-1;\lambda)}{F(z-q-1;\lambda)} & \\
&= \frac{z - \lambda_k + k - q + m - 1}{z - \lambda_k + k + m - 1} \frac{z - \lambda_k + k - 1}{z - \lambda_k + k - q - 1} \\
&= 1 + \frac{mq}{(z - \lambda_k + k + m - 1)(z - \lambda_k + k - q - 1)} \\
&= 1 + \frac{mq}{z^2} \sum_{r,s=0}^{\infty} (\lambda_k - k - m + 1)^s (\lambda_k - k + q + 1)^r z^{-r-s},
\end{aligned}$$

that is

$$\frac{F(z-q-1;\lambda-m\epsilon_k)}{F(z-1;\lambda-m\epsilon_k)} = \frac{F(z-q-1;\lambda)}{F(z-1;\lambda)}\left[1+\frac{mq}{z^2}\sum_{r,s=0}^{\infty}(\lambda_k-k-m+1)^s(\lambda_k-k+q+1)^r z^{-r-s}\right].$$

Then it suffices to apply Theorem 5.2.6 (with z replaced by $z-1$). □

Lemma 5.3.6 *Let λ be a partition of n and h, m, k, q positive integers with $k \le \ell(\lambda)$. Then*

$$\sum_{k=1}^{\ell(\lambda)} d_\lambda(k,m) c_h^{\lambda-m\epsilon_k}(q) = \sum_{j=2}^{m-1} c_j^\lambda(m) c_h^\lambda(q) s(m+1, j) + mq \sum_{\substack{a,t\ge 0\\ a+t=h-2}} \sum_{j=0}^{a+m+1} c_t^\lambda(q) c_j^\lambda(m) s(m+1, j-a+b) \cdot \sum_{b=\max\{0,a-j\}}^{\min\{a,m+1+a-j\}} \binom{a+1}{b+1} \frac{q^{b+1}+m(-m)^b}{q+m}.$$

Proof By Lemma 5.3.5 and the binomial theorem (5.5) we have

$$\sum_{k=1}^{\ell(\lambda)} d_\lambda(k,m) c_h^{\lambda-m\epsilon_k}(q) = \sum_{k=1}^{\ell(\lambda)} d_\lambda(k,m) c_h^\lambda(q) + mq \sum_{\substack{r,s,t\ge 0\\ r+s+t+2=h}} c_t^\lambda(q) \sum_{u=0}^{r}\sum_{v=0}^{s}\binom{r}{u}\binom{s}{v} \cdot \left[\sum_{k=1}^{\ell(\lambda)} d_\lambda(k,m)(\lambda_k-k)^{u+v}\right](q+1)^{r-u}(1-m)^{s-v}$$

(by applying Theorem 5.2.8 to both summands)

$$= c_h^\lambda(q)\sum_{j=2}^{m+1} c_j^\lambda(m) s(m+1;j) + mq \sum_{\substack{r,s,t\ge 0\\ r+s+t+2=h}} c_t^\lambda(q) \sum_{u,v\ge 0}\binom{r}{u}\binom{s}{v}(q+1)^{r-u}(1-m)^{s-v} \cdot (-1)^{u+v}\sum_{i=0}^{u+v}\sum_{j=i}^{m+i+1}(-1)^i c_j^\lambda(m)\binom{u+v}{i} s(m+1, j-i). \tag{5.38}$$

By setting $a = r + s$ and $b = a - i$, the order of the sums in the second summand may be manipulated as follows:

$$\sum_{\substack{r,s,t\geq 0\\ r+s+t+2=h}} \sum_{u=0}^{r}\sum_{v=0}^{s}\sum_{i=0}^{u+v}\sum_{j=i}^{m+i+1} =_{(a=r+s)} \sum_{\substack{a,t\geq 0\\ a+t=h-2}} \sum_{i=0}^{a} \sum_{j=i}^{m+i+1} \sum_{\substack{r,s\geq 0\\ r+s=a}} \sum_{\substack{0\leq u\leq r\\ 0\leq v\leq s\\ u+v\geq i}}$$

$$=_{(i=a-b)} \sum_{\substack{a,t\geq 0\\ a+t=h-2}} \sum_{b=0}^{a} \sum_{j=a-b}^{m+1+a-b} \sum_{\substack{r,s\geq 0\\ r+s=a}} \sum_{\substack{0\leq u\leq r\\ 0\leq v\leq s\\ u+v\geq a-b}}$$

$$= \sum_{\substack{a,t\geq 0\\ a+t=h-2}} \sum_{j=0}^{m+1+a} \sum_{b=\max\{0,a-j\}}^{\min\{a,m+1+a-j\}} \sum_{\substack{r,s\geq 0\\ r+s=a}} \sum_{\substack{0\leq u\leq r\\ 0\leq v\leq s\\ u+v\geq a-b}} .$$

Now, the product of all the factors depending on r, s, u, v is:

$$(-1)^{u+v}\binom{u+v}{i}\binom{r}{u}\binom{s}{v}(q+1)^{r-u}(1-m)^{s-v}$$
$$= (-1)^{u+v}\binom{u+v}{a-b}\binom{r}{u}\binom{s}{v}(q+1)^{r-u}(1-m)^{s-v}.$$

Therefore, we may apply Proposition 5.1.8 with $x = q$ and $y = -m$, getting the desired expression for the second summand in (5.38). □

Theorem 5.3.7 *Let λ be a partition of n and let m, q be positive integer with $m + q \leq n$. Then we have*

$$\hat{\chi}^{\lambda}_{m,q,1^{n-m-q}} = \frac{1}{[n]_{m+q}} \sum_{h=2}^{q+1}\sum_{j=2}^{m+1} c^{\lambda}_h(q)c^{\lambda}_j(m)s(q+1,h)s(m+1,j)$$
$$+ \frac{1}{[n]_{m+q}} mq \sum_{h=0}^{q-1} \sum_{j=0}^{q+m-h} c^{\lambda}_h(q)c^{\lambda}_j(m)\cdot$$
$$\cdot \sum_{a,b\geq 0} \binom{a+1}{b+1} s(q+1, a+h+2)s(m+1, j-a+b)\frac{q^{b+1}+m(-m)^b}{m+q}.$$

Proof From the basic recursion formula (Proposition 5.3.1) and Theorem 5.3.2 (see, in particular (5.37)) we get

$$\begin{aligned}
\chi^{\lambda}_{m,q,1^{n-m-q}} &= \frac{1}{[n]_m}\sum_{k=1}^{\ell(\lambda)} d_\lambda(k,m)\chi^{\lambda-\epsilon_k m}_{q,1^{n-m-q}} \\
&= \frac{1}{[n]_m}\frac{1}{[n-m]_q}\sum_{k=1}^{\ell(\lambda)} d_\lambda(k,m)\sum_{h=2}^{q+1} s(q+1,h)c_h^{\lambda-\epsilon_k m}(q) \\
&= \frac{1}{[n]_{m+q}}\sum_{h=2}^{q+1} s(q+1,h)\sum_{k=1}^{\ell(\lambda)} d_\lambda(k,m)c_h^{\lambda-\epsilon_k m}(q).
\end{aligned}$$

We can now apply Lemma 5.3.6 and the following manipulation of sums:

$$\begin{aligned}
\sum_{h=2}^{q+1}\sum_{\substack{a,t\geq 0\\ a+t=h-2}}\sum_{j=0}^{a+m+1} &=_{(h=a+t+2)} \sum_{t=0}^{q-1}\sum_{a=0}^{q-1-t}\sum_{j=0}^{a+m+1} \\
&= \sum_{t=0}^{q-1}\sum_{j=0}^{m+q-t}\sum_{a=\max\{0,j-m-1\}}^{q-1-t}.
\end{aligned}$$

Finally, we also replace t with h to uniform notation of both terms. □

Example 5.3.8 It is easy to check that for $q = 1$ the formula in Theorem 5.3.7 becomes the formula in Theorem 5.3.2. For $q = 2$, the formula in Theorem 5.3.7 becomes

$$\begin{aligned}
[n]_{m+2}\hat{\chi}^{\lambda}_{m,2,1^{n-m-2}} &= [c_3^\lambda(2) - 3c_2^\lambda(2)]\sum_{j=2}^{m+1} s(m+1,j)c_j^\lambda(m) \\
&\quad - m\sum_{j=2}^{m+2} c_j^\lambda(m)[2s(m+1,j-1) - (m+1)s(m+1,j)].
\end{aligned}$$

In particular, for $m = q = 2$ we get

$$\begin{aligned}
[n]_{m+2}\hat{\chi}^{\lambda}_{2,2,1^{n-4}} &= [c_3^\lambda(2) - 3c_2^\lambda(2)]^2 - 26c_2^\lambda(2) + 18c_3^\lambda(2) - 4c_4^\lambda(2) \\
&= 4d_1(\lambda)^2 - 12d_2(\lambda) + 4n(n-1).
\end{aligned}$$

For $q = 3$, the expression in Theorem 5.3.7 becomes

$$[n]_{m+3}\hat{\chi}^\lambda_{m,3,1^{n-m-3}} = [c^\lambda_4(3)-6c^\lambda_3(3) + (11+3m)c^\lambda_2(3)]\sum_{j=2}^{m+1} s(m+1, j)c^\lambda_j(m)$$

$$- m\sum_{j=2}^{m+3} c^\lambda_j(m)[(6m-7)s(m+1, j)+\frac{27+m^3}{3+m}s(m+1, j)$$

$$- 3(m+1)s(m+1, j-1) + 3s(m+1, j-2)].$$

For instance, for $q = 3 = m$ we get

$$[n]_6\hat{\chi}^\lambda_{3,3,1^{n-6}} = [c^\lambda_4(3) - 6c^\lambda_3(3) + 20c^\lambda_2(3)]^2$$

$$- 876c^\lambda_2(3) + 810c^\lambda_3(3) - 375c^\lambda_4(3) + 90c^\lambda_5(3) - 9c^\lambda_6(3).$$

5.3.3 The explicit formula for an arbitrary conjugacy class

In this section we complete our exposition of the result in Lassalle's paper [81] and we give the explicit formula for $\hat{\chi}^\lambda_\mu$, when μ is an arbitrary partition.

We introduce the following notation. For h a positive integer, $\mathsf{M}^{(h)}$ is the set of upper triangular $h \times h$ matrices whose entries are nonnegative integers and whose diagonal entries are equal to zero. If $a \in \mathsf{M}^{(h)}$ (so that $a = (a_{i,j})$ with $a_{i,j}$ nonnegative integer and $a_{i,j} = 0$ unless $i < j$) we denote by a' the $(h-1) \times (h-1)$ submatrix obtained by deleting the first column and the first row. We denote by $\mathsf{E}^{(h)}$ the subset of $\mathsf{M}^{(h)}$ formed by those matrices whose entries belong to {0, 2} (the nonzero entries are equal to 2).

Let $\rho = (\rho_1, \rho_2, \dots, \rho_h)$ be a partition of n (in h parts). With $\epsilon \in \mathsf{E}^{(h)}$ we associate the matrix $\theta \in \mathsf{M}^{(h)}$ defined by setting, for $1 \le i \le j \le h$

$$\theta_{i,j} = \begin{cases} 1 & \text{if } \epsilon_{i,j} = 0 \\ \rho_i\rho_j & \text{if } \epsilon_{i,j} = 2. \end{cases}$$

Whit this notation, if $i_1, i_2, \dots, i_h$ are nonnegative integers, we define the coefficients $A^{(\epsilon)}_{i_1,i_2,\dots,i_h}(\rho_1, \rho_2, \dots, \rho_h)$ by setting

$$A^{(\epsilon)}_{i_1,i_2,\dots,i_h}(\rho_1,\rho_2, \dots, \rho_h)$$

$$= \sum_{a,b\in\mathsf{M}^{(h)}} \left\{ \prod_{1\le i<j\le h} \left[\theta_{i,j}\binom{a_{i,j}+1}{b_{i,j}+1} \frac{\rho_i(-\rho_i)^{b_{i,j}} + \rho_j^{b_{i,j}+1}}{\rho_i+\rho_j} \right] \cdot \right.$$

$$\left. \cdot \prod_{k=1}^{h} s(\rho_k + 1, i_k + \sum_{\ell=1}^{k-1}(a_{\ell,k} + \epsilon_{\ell,k}) - \sum_{\ell=k+1}^{h} (a_{k,\ell} - b_{k,\ell})) \right\}$$

where the sum over $a_{i,j}$ and $b_{i,j}$ is restricted to $a_{i,j} = b_{i,j} = 0$ when $\epsilon_{i,j} = 0$. Moreover, for $h = 1$ the product over $\prod_{1\le i<j\le h}$ is set equal to 1.

Lemma 5.3.9

(i) *The sum* $\sum_{a,b\in \mathsf{M}^{(h)}}$ *appearing in the definition of* $A^{(\epsilon)}$ *is always finite (and therefore* $A^{(\epsilon)}$ *is well defined).*

(ii) *The coefficients* $i_1, i_2, \ldots, i_h$ *are subject to the following conditions:* $1 \le i_k \le \rho_k + 1$, $k = 1, 2, \ldots, h$ *if* ϵ *is identically zero,* $\sum_{k=1}^{h} i_k \le |\rho| + h - \sum_{k=1}^{h}\sum_{\ell=1}^{k-1} \epsilon_{\ell,k}$ *in the generic case.*

(iii) *We have the recursive property:*

$$\begin{aligned} A^{(\epsilon)}_{i_1,i_2,\ldots,i_h}&(\rho_1, \rho_2, \ldots, \rho_h)\\ &= \sum_{\substack{a_{1,2}\cdots a_{1,h}\\ b_{1,2}\cdots b_{1,h}}} \prod_{k=2}^{h} \theta_{1,k} \binom{a_{1,k}+1}{b_{1,k}+1} \frac{\rho_1(-\rho_1)^{b_{1,k}} + \rho_k^{b_{1,k}+1}}{\rho_1+\rho_k}\cdot\\ &\quad\cdot s(\rho_1 + 1, i_1 - \sum_{\ell=2}^{h}(a_{1,\ell} - b_{1,\ell}))\cdot\\ &\quad\cdot A^{(\epsilon')}_{i_2+a_{1,2}+\epsilon_{1,2},\ldots,i_h+a_{1,h}+\epsilon_{1,h}}(\rho_2, \ldots, \rho_h). \end{aligned}$$

Proof (i) The coefficients $a_{k,\ell}$ and $b_{k,\ell}$ are subject to the conditions:

(1) $i_k + \sum_{\ell=1}^{k-1}(a_{\ell,k}+\epsilon_{\ell,k}) - \sum_{\ell=k+1}^{h}(a_{k,\ell} - b_{k,\ell}) \le \rho_k+1$ for $k = 1, 2, \ldots, h$
(2) $b_{\ell,k} \le a_{\ell,k}$ for $1 \le \ell < k \le h$

(1) for the existence of the Stirling numbers and (2) for the existence of the binomial coefficients. If we take $1 \le s \le h$ and we sum (1) from $k = s$ to $k = h$, keeping into account that

$$\sum_{k=s}^{h}\sum_{\ell=1}^{k-1} a_{\ell,k} = \sum_{k=s}^{h}\sum_{\ell=1}^{s-1} a_{\ell,k} + \sum_{k=s+1}^{h}\sum_{\ell=s}^{k-1} a_{\ell,k} = \sum_{k=s}^{h}\sum_{\ell=1}^{s-1} a_{\ell,k} + \sum_{\ell=s}^{h-1}\sum_{k=\ell+1}^{h} a_{\ell,k},$$

we get the conditions

$$\sum_{k=s}^{h}(i_k + \sum_{\ell=1}^{k-1}\epsilon_{\ell,k} + \sum_{\ell=1}^{s-1} a_{\ell,k} + \sum_{\ell=k+1}^{h} b_{k,\ell}) \le \sum_{k=s}^{h}\rho_k + (h - s + 1) \tag{5.39}$$

for $s = 1, 2, \ldots, h$. In particular, for $1 \le s \le h$ and $1 \le t \le s - 1$, we get

$$a_{t,s} \le \sum_{k=s}^{h}\sum_{\ell=1}^{s-1} a_{\ell,k} \le \sum_{k=s}^{h}\rho_k + (h - s + 1) \le |\rho| + h,$$

that is, $a_{\ell,s}$, which is the generic nonzero elements in a in $\mathsf{M}^{(h)}$, remains bounded.

(2) implies that also the coefficients in b remain bounded.

(ii) This follows from (5.39) written for $s = 1$ and the conditions for the existence of the Stirling numbers.

(iii) This factorization property is trivial: we limit ourselves to observe that the second argument in the kth Stirling number may be written in the form

$$(i_k + a_{1,k} + \epsilon_{1,k}) + \left[\sum_{\ell=2}^{k-1} (a_{\ell,k} + \epsilon_{\ell,k}) - \sum_{\ell=k+1}^{h} (a_{k,\ell} - b_{k,\ell}) \right]$$

for $k = 2, 3, \ldots, h$ and that the second term contains terms depending only on the coefficients in a'. □

The following is a generalization of Lemma 5.3.6.

Lemma 5.3.10 *Let $\lambda \vdash n$ and let $\rho = (\rho_1, \rho_2, \ldots, \rho_h)$ be an h-parts partition with $|\rho| \leq n$ and $i_2, i_3, \ldots, i_h$ be nonnegative integers. Then we have:*

$$\sum_{\ell=1}^{\ell(\lambda)} d_\lambda(\ell, \rho_1) \prod_{k=2}^{h} c_{i_k}^{\lambda - \rho_1 \epsilon_\ell}(\rho_k)$$

$$= \sum_{t_1 \geq 0} c_{t_1}^{\lambda}(\rho_1) \sum_{\substack{a_{1,2} \cdots a_{1,h} \\ b_{1,2} \cdots b_{1,h}}} \sum_{\epsilon_{1,2} \cdots \epsilon_{1,h} \in \{0,2\}} s(\rho_1 + 1, t_1 - \sum_{\ell=2}^{h} (a_{1,\ell} - b_{1,\ell})) \cdot$$

$$\cdot \prod_{k=2}^{h} \left[c_{t_k}^{\lambda}(\rho_k) \theta_{1,k} \binom{a_{1,k} + 1}{b_{1,k} + 1} \frac{\rho_1 (-\rho_1)^{b_{1,k}} + \rho_k^{b_{1,k}+1}}{\rho_1 + \rho_k} \right]$$

where $t_k = i_k - a_{1,k} - \epsilon_{1,k}$ for $k = 2, 3, \ldots, h$.

Proof Set

$$B_k(r, s, u, v) = (-1)^{u+v} \binom{r}{u} \binom{s}{v} (1 + \rho_k)^{r-u} (1 - \rho_1)^{s-v}.$$

From Lemma 5.3.5 and the binomial theorem, we get

$$c_{i_k}^{\lambda - \rho_1 \epsilon_\ell}(\rho_k) = c_{i_k}^{\lambda}(\rho_k) + \rho_1 \rho_k \sum_{\substack{r,s,t \geq 0 \\ r+s+t+2 = i_k}} c_t^{\lambda}(\rho_k)(\lambda_\ell - \ell + \rho_k + 1)^r (\lambda_\ell - \ell - \rho_1 + 1)^s$$

$$= c_{i_k}^{\lambda}(\rho_k) + \rho_1 \rho_k \sum_{\substack{r,s,t \geq 0 \\ r+s+t+2 = i_k}} c_t^{\lambda}(\rho_k) \sum_{u,v \geq 0} B_k(r, s, u, v)(\ell - \lambda_\ell)^{u+v}.$$

Therefore, recalling the convention on the relationship between $\epsilon_{i,j}$ and $\theta_{i,j}$ and introducing the convention that the sum over r_k, s_k is restricted to $r_k = s_k = 0$

when $\epsilon_{i,k} = 0$, we can write:

$$\sum_{\ell=1}^{\ell(\lambda)} d_\lambda(\ell,\rho_1)\prod_{k=2}^{h} c_{i_k}^{\lambda-\rho_1\epsilon_\ell} = \sum_{\ell=1}^{\ell(\lambda)} d_\lambda(\ell,\rho_1)\prod_{k=2}^{h}\left[\sum_{\epsilon_{i,k}\in\{0,2\}}\theta_{1,k}\sum_{\substack{r_k,s_k,t_k\geq 0\\ r_k+s_k+t_k+\epsilon_{1,k}=i_k}} c_{t_k}^{\lambda}(\rho_k)\cdot\right.$$

$$\left.\cdot\sum_{u_k,v_k\geq 0} B_k(r_k,s_k,u_k,v_k)(\ell-\lambda_\ell)^{u_k+v_k}\right]$$

$$= \sum_{\epsilon_{1,2}\cdots\epsilon_{1,h}\in\{0,2\}}\ \sum_{\substack{r_2\cdots r_h\geq 0\\ s_2\cdots s_h\geq 0\\ t_2\cdots t_h\geq 0\\ r_k+s_k+t_k+\epsilon_{1,k}=i_k\\ k=2,\dots,h}}\ \sum_{\substack{u_2\cdots u_h\geq 0\\ v_2\cdots v_h\geq 0}}\ \prod_{k=2}^{h}\left[\theta_{1,k}c_{t_k}^{\lambda}(\rho_k)B_k(r_k,s_k,u_k,v_k)\right]\cdot$$

$$\cdot(-1)^{\sum_{k=2}^{h}(u_k+v_k)}\sum_{\ell=1}^{\ell(\lambda)} d_\lambda(\ell,\rho_1)(\ell-\lambda_\ell)^{\sum_{k=2}^{h}(u_k+v_k)}. \tag{5.40}$$

This cumbersome expression is obtained simply by developing the product $\prod_{k=2}^{h}$ and grouping into the sum $\sum_{\ell=1}^{\ell(\lambda)}$ all the terms containing ℓ. From Theorem 5.2.8 we get:

$$\sum_{\ell=1}^{\ell(\lambda)} d_\lambda(\ell,\rho_1)(\ell-\lambda_\ell)^{\sum_{k=2}^{h}(u_k+v_k)}$$

$$= \sum_{t_1,j\geq 0}(-1)^j\binom{\sum_{\ell=2}^{h}(u_\ell+v_\ell)}{j} s(\rho_1+1,t_1-j)c_{t_1}^{\lambda}(\rho_1), \tag{5.41}$$

while from (5.6) we get

$$\binom{\sum_{\ell=2}^{h}(u_\ell+v_\ell)}{j} = \sum_{\substack{j_2\cdots j_h\geq 0\\ j_2+\cdots+j_h=j}}\binom{u_2+v_2}{j_2}\cdots\binom{u_h+v_h}{j_h}. \tag{5.42}$$

We also need to reformulate Proposition 5.1.8 in the following form:

$$\sum_{\substack{r_k,s_k,u_k,v_k\geq 0\\ r_k+s_k=a_{1,k}}} B_k(r_k,s_k,u_k,v_k)\binom{u_k+v_k}{a_{1,k}-b_{1,k}}$$

$$= (-1)^{a_{1,k}-b_{1,k}}\binom{a_{1,k}+1}{b_{1,k}+1}\frac{\rho_1(-\rho_1)^{b_{1,k}}+\rho_k^{b_{1,k}+1}}{\rho_1+\rho_k}. \tag{5.43}$$

Then the formula in the statement may be obtained by inserting (5.41) and (5.42) in (5.40) and using (5.43) to simplify the expression. □

Theorem 5.3.11 (Lassalle's explicit formula) *Let λ be a partition of n, let $\rho = (\rho_1, \rho_2, \ldots, \rho_h)$ be a partition with $|\rho| \le n$ and set $\sigma = (\rho, 1^{n-|\rho|})$. Then we have*

$$\hat{\chi}^\lambda_\sigma = \frac{1}{[n]_{|\rho|}} \sum_{\epsilon \in \mathsf{E}^{(h)}} \sum_{i_1, i_2, \ldots, i_h \ge 0} A^{(\epsilon)}_{i_1, i_2, \ldots, i_h}(\rho_1, \rho_2, \ldots, \rho_h) \prod_{k=1}^{h} c^\lambda_{i_k}(\rho_k). \tag{5.44}$$

Proof The proof is by induction on h. We have already obtained the case $h = 1$ (in Theorem 5.3.2) and $h = 2$ (in Theorem 5.3.7). Indeed, for $h = 1$ we have $\epsilon = (\epsilon_{1,1}) = a = (a_{1,1}) = b = (b_{1,1}) = 0$ and therefore $A^{(\epsilon)}_{i_1}(\rho_1) = s(\rho_1 + 1, i_1)$, and (5.44) reduces to the formula in Theorem 5.3.2. Even though this case is not necessary for the induction, we check the formula for $h = 2$: we can write $A^{(\epsilon)}$ with a, b, ϵ, θ in place of $a_{1,2}, b_{1,2}, \epsilon_{1,2}, \theta_{1,2}$ so that

$$A^{(\epsilon)}_{i_1, i_2}(\rho_1, \rho_2) = \sum_{a, b \ge 0} \theta \binom{a+1}{b+1} \frac{\rho_1(-\rho_1)^b + \rho_2^{b+1}}{\rho_1 + \rho_2} \cdot$$
$$\cdot\, s(\rho_1 + 1, i_1 - a + b) s(\rho_2 + 1, i_2 + a + \epsilon),$$

with the convention that if $\epsilon = 0$ then $\theta = 1$ and the sum over a, b is restricted to $a = b = 0$, while if $\epsilon = 2$, $\theta = \rho_1 \rho_2$ (and the sum over a, b is for all possible values). Then the formula (5.44) coincides with that in Theorem 5.3.7, since the latter may be written in the form

$$\hat{\chi}^\lambda_{\rho_1, \rho_2, 1^{n-\rho_1-\rho_2}} = \frac{1}{[n]_{\rho_1+\rho_2}} \sum_{\epsilon \in \{0,2\}} \sum_{i_1, i_2 \ge 0} c^\lambda_{i_1}(\rho_1) c^\lambda_{i_2}(\rho_2) A^{(\epsilon)}_{i_1, i_2}(\rho_1, \rho_2) c^\lambda_{i_1}(\rho_1) c^\lambda_{i_2}(\rho_2).$$

Note also that by the principle of analytic continuation (Remark 5.1.19) we may assume that the inductive hypothesis holds true also when λ is replaced by an arbitrary sequence of integers (that is, χ^λ is replaced by a virtual character). Now we can apply the inductive hypothesis to the basic recurrence formula. We have, by Proposition 5.3.1 and induction,

$$[n]_{|\rho|} \hat{\chi}^\lambda_\sigma = \sum_{\ell=1}^{\ell(\lambda)} d_\lambda(\ell, \rho_1) [n - \rho_1]_{|\rho| - \rho_1} \hat{\chi}^{\lambda - \rho_1 \epsilon_\ell}_{\sigma \setminus \rho_1}$$
$$= \sum_{\ell=1}^{\ell(\lambda)} d_\lambda(\ell, \rho_1) \sum_{\epsilon' \in \mathsf{E}^{(h-1)}} \sum_{i_2 \cdots i_h \ge 0} A^{(\epsilon')}_{i_2, \ldots, i_h}(\rho_2, \ldots, \rho_h) \prod_{k=2}^{h} c^{\lambda - \rho_1 \epsilon_\ell}_{i,k}(\rho_k)$$

which equals, by Lemma 5.3.10,

$$\sum_{\epsilon'\in \mathsf{E}^{(h-1)}}\sum_{i_2\cdots i_h\geq 0} A^{(\epsilon')}_{i_2,\dots,i_h}(\rho_2,\dots,\rho_h)\sum_{t_1\geq 0} c^\lambda_{t_1}(\rho_1)\sum_{\substack{a_{1,2}\cdots a_{1,h}\geq 0\\ b_{1,2}\cdots b_{1,h}\geq 0}}$$

$$\sum_{\epsilon_{1,2}\cdots\epsilon_{1,h}\in\{0,2\}}\prod_{k=2}^{h}\left[c^\lambda_{t_k}(\rho_k)\theta_{1,k}\binom{a_{1,k}+1}{b_{1,k}+1}\frac{\rho_1(-\rho_1)^{b_{1,k}}+\rho_k^{b_{1,k}+1}}{\rho_1+\rho_k}\right]\cdot$$

$$\cdot\, s(\rho_1+1, t_1-\sum_{\ell=2}^{h}(a_{1,\ell}-b_{1,\ell}))$$

$$=\sum_{\epsilon\in\mathsf{E}^{(h)}}\sum_{t_1\cdots t_h\geq 0} A^{(\epsilon)}_{t_1,\dots,t_h}(\rho_1,\dots,\rho_h)\prod_{k=1}^{h} c^\lambda_{t_k}(\rho_k),$$

where the last equality follows from Lemma 5.3.9.(iii). □

Exercise 5.3.12 Show that setting $\rho_h = 1$ one gets exactly the formula for $h-1$.

Exercise 5.3.13 Check the following formulas from [81] in two ways:

- by applying the formula in Theorem 5.3.11;
- by using the basic recursion formula, Lemma 5.3.6, Lemma 5.3.10 and the formula for $\hat{\chi}^\lambda_{2,2,1^{n-4}}$ in Example 5.3.8.

$$\begin{aligned}[n]_6\hat{\chi}^\lambda_{2,2,2,1^{n-6}} &= (c^\lambda_3(2)-3c^\lambda_2(2))^3+2(c^\lambda_3(2)-3c^\lambda_2(2))\cdot\\ &\quad\cdot(27c^\lambda_3(2)-47c^\lambda_2(2)-6c^\lambda_4(2))\\ &\quad+40c^\lambda_5(2)-240c^\lambda_4(2)+560c^\lambda_3(2)-600c^\lambda_2(2)\end{aligned}$$

and

$$\begin{aligned}[n]_7\hat{\chi}^\lambda_{3,2,2,1^{n-7}} &= (c^\lambda_3(2)-3c^\lambda_2(2))^2-4c^\lambda_4(2)+18c^\lambda_3(2)-50c^\lambda_2(2))\cdot\\ &\quad\cdot(c^\lambda_4(3)-6c^\lambda_3(3)+11c^\lambda_2(3))\\ &\quad-12(c^\lambda_3(2)-3c^\lambda_2(2))(c^\lambda_5(3)-8c^\lambda_4(3)+23c^\lambda_3(3)-28c^\lambda_2(3))\\ &\quad+72(c^\lambda_6(3)-10c^\lambda_5(3)+40c^\lambda_4(3)-80c^\lambda_3(3)+79c^\lambda_2(3)).\end{aligned}$$

5.4 Central characters and class symmetric functions

This section is an exposition of the results of Corteel, Goupil and Schaeffer in [24]. Their work originated from some conjectures of Katriel (see [72] and [50]). We derive most of the results of that paper from the explicit formulas in Section 5.3. We also give a version of the classical asymptotic estimate of Kerov–Vershik (see [121]).

5.4.1 Central characters

In Section 5.3 we have obtained for $\hat{\chi}^\lambda_\sigma$ an expression in terms of symmetric functions evaluated on the content $c(\lambda)$ of λ. If $\sigma = (\rho, 1, \dots, 1)$, then such symmetric function depends only on ρ. Following [24], for $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_k)$ partition of length k, (with $\sigma_k > 0$), the *reduced partition* $\tilde{\sigma}$ associated with σ is defined by setting

$$\tilde{\sigma} = (\sigma_1 - 1, \sigma_2 - 1, \dots, \sigma_k - 1).$$

Given a permutation π of cycle type σ, its *reduced cycle type* is just $\tilde{\sigma}$. If μ is partition, with $|\mu| + \ell(\mu) \le n$, we denote by $\mathcal{C}_\mu(n)$ the conjugacy class of $\mathfrak{S}_n$ formed by those partitions of reduced cycle type μ. Note that this established a bijection between

$$\Xi_n := \{\text{partition } \mu : |\mu| + \ell(\mu) \le n\}$$

and the set of all conjugacy classes of $\mathfrak{S}_n$. For instance, $\mathcal{C}_1(n)$ is the set of all transpositions, $\mathcal{C}_2(n)$ the set of all 3-cycles, and so on.

There is an obvious algebraic interpretation of the correspondence $\Xi_n \ni \mu \mapsto \mathcal{C}_\mu(n)$: In view of Theorem 4.4.4, $|\mu|$ is the minimal number of transpositions needed to write $\pi \in \mathcal{C}_\mu(n)$ as a product of transpositions, and this minimal number is achieved by writing the first cycle into the product of μ_1 transpositions, the second cycle into the product of μ_2 transpositions and so on. We introduce the following notation: if μ and η are partitions, $\mu \cup \theta$ is the partition obtained taking the parts of μ and η and rearranging them in decreasing order. For instance, $(5, 3, 1, 1) \cup (4, 2) = (5, 4, 3, 2, 1, 1)$. We write $\mu <_a \eta$ when the parts of η are obtained by adding some parts of μ. For instance, $(2, 2, 2, 1, 1, 1) <_a (4, 2, 2, 1)$. Clearly, $\mu <_a \eta$ implies $\mu \trianglelefteq \eta$ and $\ell(\mu) \ge \ell(\eta)$, with equalities if and only if $\mu = \eta$. For $\mu, \eta \in \Xi_n$, we denote by $\mathcal{C}_\mu(n)\mathcal{C}_\eta(n)$ the convolution of the characteristic functions of those conjugacy classes. Clearly, there exist coefficients $C^\theta_{\mu,\eta}(n)$ such that

$$\mathcal{C}_\mu(n)\mathcal{C}_\eta(n) = \sum_{\theta \in \Xi_n} C^\theta_{\mu,\eta}(n)\mathcal{C}_\theta(n).$$

We give some basic properties of those coefficients.

Proposition 5.4.1 *The following conditions are necessary in order to have* $C^\theta_{\mu,\eta}(n) \neq 0$.

(i) $|\mu| + |\eta|$ *and* $|\theta|$ *must have the same parity;*
(ii) *we must have:* $|\theta| \le |\mu| + |\eta|$*;*
(iii) *if* $|\theta| = |\mu| + |\eta|$ *then* $(\mu \cup \eta) <_a \theta$ *and* $C^\theta_{\mu,\eta}$ *does not depend on* n.

Proof It is an easy consequence of Theorem 4.4.4 and the details are left as an exercise. We limit ourselves to prove that if $|\theta| = |\mu| + |\eta|$ then $C^{\theta}_{\mu,\eta}$ does not depend on n. Suppose that π is a permutation of reduced type θ. Then $C^{\theta}_{\mu,\eta}$ is equal to the number of ways we may rewrite $\pi = \pi_1\pi_2$, with π_1 of reduced type μ and π_2 of reduced type η. If we write π_1 and π_2 as in Theorem 4.4.4, then in the product $\pi_1\pi_2 = \pi$ there is not simplification between transpositions. This implies that if $A \subseteq \{1, 2, \dots, n\}$ is the subset of numbers moved by π (i.e., $\{1, 2, \dots, n\} \setminus A$ are fixed points of π) then π_1 and π_2 belong to the symmetric group $\mathfrak{S}_A$ over A. Therefore the number of possible factorizations $\pi = \pi_1\pi_2$ depends only of the size of A (and on θ, μ, η obviously), but it does not depend on n. □

Remark 5.4.2 In order to clarify the case $|\theta| = |\mu| + |\eta|$ in the last proposition, consider the following construction in Theorem 4.4.4. Suppose that π is a permutation of reduced type θ and $\pi = t_1 t_2 \cdots t_{\theta_1} \cdots t_{\theta_1+\cdots+\theta_{h-1}} \cdots t_{|\theta|}$ is a decomposition as in Theorem 4.4.4. We can construct a graph whose vertices are the numbers moved by $t_1, t_2, \dots, t_{|\theta|}$ (and therefore by π) and whose edges are the pairs $\{i_k, j_k\}$ if $t_k = (i_k \to j_k \to i_k)$, $k = 1, 2, \dots, |\theta|$. Clearly, the connected components of this graph correspond to the cycle of π. Suppose that π_1 is of reduced type μ, π_2 of reduced type η, π of reduced type θ and $|\theta| = |\mu| + |\eta|$. If we multiply a decomposition of π_1 with a decomposition of π_2 then we get a decompositon of π and the connected components of the graph of π may be obtained by gluing the connected components of π_1 with those of π_2 that have common vertices. For instance, if $\mu = (3, 2)$, $\eta = (2, 1)$, $\theta = (5, 3)$, $\pi_1 = (1 \to 2 \to 3 \to 4 \to 1)(5 \to 6 \to 7 \to 5)$, $\pi_2 = (1 \to 8 \to 9 \to 1)(5 \to 10 \to 5)$, $\pi = (1 \to 8 \to 9 \to 2 \to 3 \to 4 \to 1)(5 \to 10 \to 6 \to 7 \to 5)$ we have the permutations shown in Figure 5.4(a) and the composition in Figure 5.4(b).

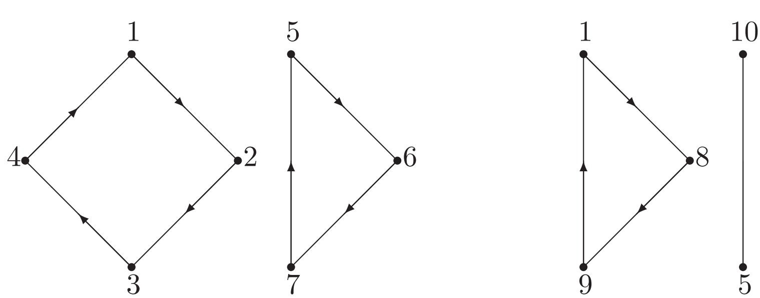


Figure 5.4(a) The two permutations π_1 (left) and π_2 (right)

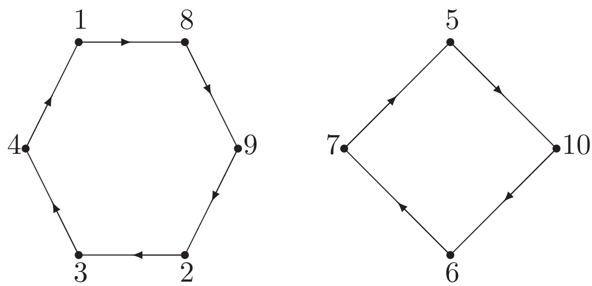


Figure 5.4(b) The composition $\pi = \pi_1\pi_2$

Exercise 5.4.3 Prove that $C^{\theta}_{\mu,\eta}$ is a polynomial in n and that its degree is less than equal to $\frac{1}{2}(|\mu| + |\eta| - |\theta|) + \ell(\mu) + \ell(\eta) - \ell(\theta)$.

Hint: prove the bound for $C^{\theta}_{1,\eta}(n)$ and then examine $\underbrace{\mathcal{C}_1(n)\cdots\mathcal{C}_1(n)}_{|\mu|\text{-times}}\mathcal{C}_\eta(n)$; see [24, Proposition 2.1] for more details.

In what follows, we introduce the order $\sqsubset$ in Ξ_n by setting $\mu \sqsubset \eta$ when $|\mu| > |\eta|$ or $|\mu| = |\eta|$ and $\mu <_a \eta$. In other words,

$$\mu \sqsubset \eta \Leftrightarrow \{|\mu| > |\eta| \text{ or } [|\mu| = |\eta| \text{ and } \mu <_a \eta]\}.$$

Proposition 5.4.4 *Let $\mu = (1^{m_1}, 2^{m_2}, \ldots, k^{m_k})$. Then we have the following expansion:*

$$\mathcal{C}_\mu(n) = \frac{1}{m_1!m_2!\cdots m_k!}\mathcal{C}_{\mu_1}(n)\mathcal{C}_{\mu_2}(n)\cdots\mathcal{C}_{\mu_k}(n) + \sum_{\substack{\nu\in\Xi_n:\\ \nu\sqsupset\mu}} d^\nu_\mu(n)\mathcal{C}_{\nu_1}(n)\mathcal{C}_{\nu_2}(n)\cdots\mathcal{C}_{\nu_t}(n)$$

for some coefficients $d^\nu_\mu(n)$. For $|\nu| = |\mu|$ the coefficients d^ν_μ does not depend on n.

Proof A repeated application of Proposition 5.4.1 yields

$$\mathcal{C}_{\mu_1}(n)\mathcal{C}_{\mu_2}(n)\cdots\mathcal{C}_{\mu_k}(n) = (m_1!m_2!\cdots m_k!)\mathcal{C}_\mu(n) + \sum_{\substack{\nu\in\Xi_n:\\ \nu\sqsupset\mu}} b^\nu_\mu(n)\mathcal{C}_\nu(n)$$

for some coefficients $b^\nu_\mu(n)$ (and for $|\nu| = |\mu|$, $b^\nu_\mu(n)$ does not depend on n). This is a triangular relation (compare with Lemma 3.6.6) that may be easily solved, giving the relation in the statement. □

If $\lambda \vdash n$ and μ is a partition with $|\mu| + \ell(\mu) \leq n$, the *central character* ω_μ evaluated at λ is the coefficient

$$\omega_\mu^\lambda = \frac{|\mathcal{C}_\mu(n)|}{d_\lambda}\chi_\sigma^\lambda \equiv |\mathcal{C}_\mu(n)|\hat{\chi}_\sigma^\lambda$$

if $\mu = \tilde{\sigma}$, $\sigma \vdash n$.

5.4.2 Class symmetric functions

From Theorem 5.3.11, we know that there exists a symmetric polynomial, that we denote by $\omega_\mu^{(n)}$, such that

$$\omega_\mu^\lambda = \omega_\mu^{(n)}[C(\lambda)] \tag{5.45}$$

where $\omega_\mu^{(n)}[C(\lambda)]$ denotes the evaluation of $\omega_\mu^{(n)}$ on the content of $\lambda \vdash n$. The symmetric polynomials $\omega_\mu^{(n)}$ are called *class symmetric functions*. Observe that they are not homogeneous. Note that in Theorem 5.3.11 σ does not denote a reduced partition, so that the formula for ω_μ^λ is obtained from those for $|\mathcal{C}_\mu(n)|\hat{\chi}_\sigma^\lambda$, $\sigma \vdash n$ such that $\tilde{\sigma} = \mu$.

Now we give a basic relation between the polynomial $\omega_\mu^{(n)}$ and the conjugacy classes.

Proposition 5.4.5 *We have*

$$\omega_\mu^{(n)}\omega_\eta^{(n)} = \sum_{\theta \in \Xi_n} C_{\mu,\eta}^\theta \omega_\theta^{(n)}$$

with the same coefficients in Proposition 5.4.4.

Proof From Corollary 1.5.12 we get $\mathcal{C}_\mu(n)\chi^\lambda = \omega_\mu^{(n)}[C(\lambda)]\chi^\lambda$ and therefore

$$\begin{aligned}\omega_\mu^{(n)}[C(\lambda)]\omega_\eta^{(n)}[C(\lambda)]\chi^\lambda &= \mathcal{C}_\mu(n)\mathcal{C}_\eta(n)\chi^\lambda \\ &= \sum_{\theta \in \Xi_n} C_{\mu,\eta}^\theta(n)\mathcal{C}_\theta(n)\chi^\lambda \\ &= \left\{\sum_{\theta \in \Xi_n} C_{\mu,\eta}^\theta \omega_\theta^{(n)}[C(\lambda)]\right\}\chi^\lambda. \end{aligned}$$

□

Corollary 5.4.6 *We have*

$$\omega_\mu^{(n)} = \frac{1}{m_1!m_2!\cdots m_k!}\omega_{\mu_1}^{(n)}\omega_{\mu_2}^{(n)}\cdots\omega_{\mu_k}^{(n)} + \sum_{\substack{\nu \in \Xi_n: \\ \nu \sqsupset \mu}} d_\mu^\nu(n)\omega_{\nu_1}^{(n)}\omega_{\nu_2}^{(n)}\cdots\omega_{\nu_t}^{(n)}$$

with the same coefficients of Proposition 5.4.4.

From Definition 5.1.17 (and (5.27)), the definition of $c_r^\lambda(m)$ (see (5.36)) and Theorem 5.3.11, it follows that there exist coefficients $\Omega_\mu^\nu(n)$, which are polynomials in n, such that

$$\omega_\mu^{(n)} = \sum_\nu \Omega_\mu^\nu(n) p_\nu, \tag{5.46}$$

where p_ν are the usual power sum symmetric functions. Note also that if again $\tilde{\sigma} = \mu$ and $\sigma = (\rho, 1^{n-|\rho|})$ then

$$\frac{|\mathcal{C}_\mu(n)|}{[n]_{|\rho|}} = \frac{n!}{z_\sigma [n]_{|\rho|}} = \frac{1}{z_\rho}. \tag{5.47}$$

For instance, from Example 5.3.3 we know that

$$\omega_{(4)}^{(n)} = p_4 - 2p_{1,1} - (3n-10)p_2 + 5\binom{n}{3} - 3\binom{n}{2},$$

while from Example 5.3.8 we know that $\omega_{1,1}^{(n)} = \frac{1}{2}p_{1,1} - \frac{3}{2}p_2 + \binom{n}{2}$. Clearly, the expansion (5.46) is unique: take n large and apply (iv) in Theorem 4.1.12 to each homogeneous component of $\omega_\mu^{(n)}$. The assertions above are proved in detail together with some other properties of the coefficients $\Omega_\mu^\nu(n)$ in the following theorem.

Theorem 5.4.7

(i) *$\Omega_\mu^\nu(n)$ is a polynomial in n of degree at most $\frac{|\mu|-|\nu|}{2} + \ell(\mu) - \ell(\nu)$ whose coefficients depend only on ν and μ.*

(ii) *$\Omega_\mu^\mu(n) = \frac{1}{m_1!m_2!\cdots m_k!}$ if $\mu = (1^{m_1}, 2^{m_2}, \dots, k^{m_k})$.*

(iii) *If $|\nu| = |\mu|$ and $\nu \neq \mu$ then $\Omega_\mu^\nu(n)$ does not depend on n. Moreover, it is equal to zero unless $\mu <_a \nu$. If $|\nu| > |\mu|$ then p_ν does not appear in* (5.46).

(iv) *$\Omega_\mu^\nu(n)$ is equal to zero if $|\mu|$ and $|\nu|$ do not have the same parity.*

Proof First note that, for the polynomials $c_r^\lambda(m)$, we have the following expansion:

$$c_r^\lambda(m) = (r-1)p_{r-2}[C(\lambda)] + \sum_{|\nu|<r-2} A_r^\nu(n) p_\nu[C(\lambda)] \tag{5.48}$$

where $A_r^\nu(n)$ is a polynomial in n of degree at most

$$\frac{r-|\nu|}{2} - \ell(\nu). \tag{5.49}$$

Indeed, the term with p_{r-2} is obtained by setting in (5.36) $n = k = 1, h = 0$ and summing from $q = 0$ to $q = r-2$, taking into account that $F_{r-2,q,1} = p_{r-2}$. Note also that the n in (5.36) is just a summation index, while the n in the present theorem coincides with $|\lambda|$ and that the polynomial in $|\lambda|$ is given by the binomial coefficient $\binom{n+|\lambda|-1}{n-k}$, which is a polynomial in $|\lambda|$ of degree $n-k$. But p_ν appears in $c_r^\lambda(m)$ only in the terms with $r-2n-h=|\nu|$ and $k \geq \ell(\nu)$ (see (5.26)), and these conditions yield

$$n-k \leq \frac{r-|\nu|}{2} - \ell(\nu).$$

If we examine the formula (5.44), the power sum symmetric function p_ν comes from the product $\prod_{k=1}^h c_{i_k}^\lambda(\rho_k)$, selecting a $p_{\nu^{(k)}}$ appearing in $c_{i_k}^\lambda(\rho_k)$ in such a way that $(\nu^{(1)}, \nu^{(2)}, \dots, \nu^{(h)})$ is a composition from which ν may be obtained rearranging the parts in decreasing order. Therefore, (5.48) ensures that the coefficients of p_ν is the sum of products of the form

$$A_{i_1}^{\nu^{(1)}}(n) A_{i_2}^{\nu^{(2)}}(n) \cdots A_{i_h}^{\nu^{(h)}}(n).$$

Assume that $\rho_h \geq 2$. Since $i_1, i_2, \dots, i_h$ must satisfy the condition $\sum_{k=1}^h i_h \leq |\rho| + \ell(\rho)$ (see (ii) in Lemma 5.3.9) from (5.49) we deduce that the coefficient of p_ν is a polynomial in n of degree at most

$$\begin{aligned}\frac{i_1+i_2+\cdots+i_h-|\nu^{(1)}|-|\nu^{(2)}|-\cdots-|\nu^{(h)}|}{2} - \ell(\nu^{(1)}) - \ell(\nu^{(2)}) - \cdots - \ell(\nu^{(h)}) \\ \leq \frac{|\rho|+\ell(\rho)-|\nu|}{2} - \ell(\nu) = \frac{|\mu|-|\nu|}{2} + \ell(\mu) - \ell(\nu)\end{aligned}$$

(because $\mu = (\rho_1 - 1, \rho_2 - 1, \dots, \rho_h - 1)$, $h = \ell(\rho) = \ell(\mu)$ and therefore $|\mu| = |\rho| + \ell(\rho)$). This proves (i).

From (5.48) and (ii) in Lemma 5.3.9 it also follows that p_μ appears in $\prod_{k=1}^h c_{i_k}^\lambda(\rho_k)$ only when $\epsilon \equiv 0$ and $i_k = \rho_k + 1$, $k = 1, 2, \dots, h$. Moreover, in ω_μ^λ its coefficients are

$$\frac{|\mathcal{C}_\mu(n)|}{[n]_{|\rho|}} \rho_1 \rho_2 \cdots \rho_h = \frac{\rho_1 \rho_2 \cdots \rho_h}{z_\rho} = \frac{1}{m_1! m_2! \cdots m_k!}$$

where $\mu = (1^{m_1}, 2^{m_2}, \dots, k^{m_k})$ and the first equality follows from (5.47). This proves (ii).

Now we prove (iii). First note that from (5.48), (i) and (iv) in the present theorem and Theorem 5.3.2 we get, for $\mu = (r)$,

$$\omega_r^{(n)} = p_r + \sum_{|\nu| \leq r-2} \Omega_r^\nu(n) p_\nu. \tag{5.50}$$

Then all the assertions in (iii) are an immediate consequence of (5.50) and Corollary 5.4.6. Note that we get also an alternative proof of (ii). To prove (iv), first note that from Lemma 3.6.10 and (vii) in Proposition 1.3.4 it follows that $\chi^\lambda(\pi)\epsilon(\pi) = \chi^{\lambda'}(\pi)$, for $\lambda \vdash n$, $\pi \in \mathfrak{S}_n$, and therefore (for the central characters) $\omega^\lambda_\mu = (-1)^{|\mu|}\omega^{\lambda'}_\mu$ which in terms of $\omega^{(n)}_\mu[C(\lambda)]$ yields

$$\omega^{(n)}_\mu[C(\lambda')] = (-1)^{|\mu|}\omega^{(n)}_\mu[C(\lambda)] \tag{5.51}$$

because if π has reduced type μ then clearly $\epsilon(\pi) = (-1)^{|\mu|}$ (π may be written as a product of $\mu_1 + \mu_2 + \cdots + \mu_k$ transpositions; this is the easy part of Theorem 4.4.4). On the other hand, we have $C(\lambda') = -C(\lambda)$ and clearly

$$p_\nu[C(\lambda')] = p_\nu[-C(\lambda)] = (-1)^{|\nu|}p_\nu[C(\lambda)]. \tag{5.52}$$

Now write $\omega^{(n)}_\mu$ in the following form:

$$\omega^{(n)}_\mu = \sum_{|\nu|\text{even}} \Omega^\nu_\mu(n)p_\nu + \sum_{|\nu|\text{odd}} \Omega^\nu_\mu(n)p_\nu.$$

If $|\mu|$ is even, (5.51) and (5.52) force

$$\sum_{|\nu|\text{odd}} \Omega^\nu_\mu(n)p_\nu[C(\lambda)] = 0 \tag{5.53}$$

for all $n \geq 1$ and for all $\lambda \vdash n$. But we can express the right-hand side of (5.53) as a shifted symmetric polynomial in λ (see Remark 5.1.19 and recall that $n \equiv d_0(\lambda) = p^*_1(\lambda)$ is also a shifted symmetric polynomial). Then the principle of analytic continuation ensures that $\Omega^\nu_\mu = 0$ when $|\nu|$ is odd.

The case $|\mu|$ odd is similar and this ends the proof of (iv). □

Exercise 5.4.8 ([50], [7]) Show that if $\lambda^{(n)}$ is a sequence of partitions such that $\lambda^{(n)} \vdash n$, $\lambda^{(n)}_1 \leq A\sqrt{n}$ and $\ell(\lambda^{(n)}) \leq A\sqrt{n}$ for some positive constant A, then

$$\omega^{(n)}_\mu[C(\lambda^{(n)})] = \mathcal{O}\left(n^{|\mu|/2+\ell(\mu)}\right).$$

Hint: first prove that $p_r[C(\lambda)] = \mathcal{O}(n^{r/2+1})$ and then use Theorem 5.4.7.

Exercise 5.4.9 ([50])

(i) Show that if $x_1, x_2, \ldots, x_n$ are the YJM elements then

$$\mathcal{C}_\mu(n) = \omega^{(n)}_\mu(x_1, x_2, \ldots, x_n).$$

Hint: $\mathcal{C}_\mu(n)\chi^\lambda = \omega^{(n)}_\mu[C(\lambda)]\chi^\lambda = \omega^{(n)}_\mu(x_1, x_2, \ldots, x_n)\chi^\lambda$; see Proposition 4.4.12.

(ii) Recalling that e_k is the k-th elementary symmetric function, show that

$$e_k = \sum_{\mu\vdash k} \omega^{(n)}_\mu.$$

Hint: recall Theorem 4.4.18.

From (ii) and (iii) in Theorem 5.4.7 we get the following expression for $\omega_\mu^{(n)}$: if $\mu = (1^{m_1}, 2^{m_2}, \dots, k^{m_k})$ then

$$\omega_\mu^{(n)} = \frac{1}{m_1! m_2! \cdots m_k!} p_\mu + \sum_{\substack{|\nu|=|\mu| \\ \mu <_a \nu}} \Omega_\mu^\nu(n) p_\nu + \sum_{|\nu| \le |\mu|-2} \Omega_\mu^\nu(n) p_\nu.$$

5.4.3 Kerov–Vershik asymptotics

Now we go back to the normalized characters $\hat{\chi}_\sigma^\lambda$. Clearly, if $\sigma = (\rho, 1^{n-|\rho|})$ with $\rho = (\rho_1, \rho_2, \dots, \rho_h)$, $\rho_h \ge 2$, and $\mu = \tilde{\sigma}$, then

$$\begin{aligned}
\hat{\chi}_\sigma^\lambda &= \frac{1}{|\mathcal{C}_\mu(n)|} \omega_\mu^{(n)}[C(\lambda)] \\
&= \frac{z_\rho (n-|\rho|)!}{n!} \omega_\mu^{(n)}[C(\lambda)] \\
&= \frac{\rho_1 \rho_2 \cdots \rho_h}{[n]_{|\rho|}} p_\mu[C(\lambda)] + \sum_{\substack{|\nu|=|\mu| \\ \mu <_a \nu}} z_\rho \frac{\Omega_\mu^\nu(n)}{[n]_{|\rho|}} p_\nu[C(\lambda)] \\
&\quad + \sum_{|\nu| \le |\mu|-2} z_\rho \frac{\Omega_\mu^\nu(n)}{[n]_{|\rho|}} p_\nu[C(\lambda)].
\end{aligned} \tag{5.54}$$

From this expression, it is easy to deduce the following form of the celebrated *Kerov–Vershik asymptotic formula for the characters of* $\mathfrak{S}_n$ [121]. It was rediscovered by Wasserman [122], whose proof was reproduced in [39]. In [7] these asymptotics are generalized and extended using methods from the free probability theory.

Proposition 5.4.10 *In the same hypothesis of* (5.54)*, we have the asymptotic formula*

$$\hat{\chi}_\sigma^\lambda = \frac{\rho_1 \rho_2 \cdots \rho_h}{[n]_{|\rho|}} p_\mu[C(\lambda)] + \mathcal{O}\left(\frac{1}{|\lambda|}\right)$$

where the constant in $\mathcal{O}\left(\frac{1}{|\lambda|}\right)$ *depends only on* ρ.

Proof Clearly, we have

$$|d_r(\lambda)| \le \sum_{i=1}^{\ell(\lambda)} \sum_{j=1}^{\lambda_i} |j-i|^r \le= |\lambda| \cdot |\lambda|^r = |\lambda|^{r+1}$$

and therefore

$$|p_\nu[C(\lambda)]| \le |\lambda|^{|\nu|+\ell(\nu)}. \tag{5.55}$$

From (i) in Theorem 5.4.7 and (5.55) we deduce the following asymptotic estimate: for $|\nu| < |\mu|$ (so that $|\nu| \leq |\mu| - 2$ by (iv) in Theorem 5.4.7):

$$\left| \frac{\Omega^{\nu}_{\mu}(n)}{[n]_{|\rho|}} p_{\nu}[C(\lambda)] \right| = \mathcal{O}\left(\frac{1}{|\lambda|^{\beta}} \right), \tag{5.56}$$

where $\beta = -\left[\frac{|\mu|-|\nu|}{2} + \ell(\mu) - \ell(\nu)\right] + [|\mu| + \ell(\mu)] - [|\nu| + \ell(\nu)] = \frac{|\mu|-|\nu|}{2} \geq 1$. On the other hand, if $|\nu| = |\mu|$, $\nu \neq \mu$, by (iv) in Theorem 5.4.7 we know that Ω^{ν}_{μ} does not depend on n and that $\mu <_a \nu$, so that $\ell(\nu) < \ell(\mu)$. Therefore (5.55) gives

$$\left| \frac{\Omega^{\nu}_{\mu}(n)}{[n]_{|\rho|}} p_{\nu}[C(\lambda)] \right| = \mathcal{O}\left(\frac{1}{|\lambda|^{\gamma}} \right), \tag{5.57}$$

where $\gamma = [|\mu| + \ell(\mu)] - [|\nu| + \ell(\nu)] = \ell(\mu) - \ell(\nu) \geq 1$. Using the estimate (5.56) and (5.57) in (5.54), one gets the result. $\square$

6

Radon transforms, Specht modules and the Littlewood–Richardson rule

The aim of this chapter is to introduce the reader to the combinatorics of pairs of partitions, a powerful method discovered by James in his fundamental paper [64], also described in his lecture notes [65]. In particular, this leads to a proof of the Littlewood–Richardson rule (which gives the decomposition of $\mathrm{Ind}_{\mathfrak{S}_{n-m}\times\mathfrak{S}_m}^{\mathfrak{S}_n}(S^\mu \boxtimes S^\lambda)$, where $\lambda \vdash m$ and $\nu \vdash n-m$).

The reader may limit himself to the first section devoted to the proof of the Littlewood–Richardson rule. Other proofs of this deep result may be found in the monographs by Sagan [108], Macdonald [83], James and Kerber [66], Fulton [42] and in the references therein. The proof we present is surely not the quickest (those in [66] and in [83] are slightly simpler); however, the combinatorics that we develop is quite powerful and we shall use it in the second section.

This chapter should be thought of as a sequel to Chapter 3, but the reader may also consider another possibility. The present chapter indeed, does not require the knowledge of the whole of Chapter 3: once Sections 3.1–3.3 and Sections 3.6.1, 3.6.2, 3.6.4 and 3.7.1 (that is, the basic facts on Young modules) have been read, the reader will possess all the necessary prerequisites to read Sections 6.1.1–6.1.3, the whole of Section 6.2, and then he may go back to Sections 3.7.2 and 3.7.3. After that, he may complete Section 6.1 up to the Littlewood–Richardson rule. Our treatment of the Young rule by means of Radon transforms, altough entirely based on James' work, is also inspired by Appendix C in Sternberg's monograph [115] and by several indications in Diaconis' book [26]. However, the emphasis on orthogonal decompositions of the Young modules and the connection with the Gelfand–Tsetlin bases for the $\mathfrak{S}_a$-invariant vectors in M^a, are new. We also point out that James' original motivation was to develop a characteristic-free approach to the representation theory of $\mathfrak{S}_n$: he was interested in linear representations of $\mathfrak{S}_n$ on vector spaces over arbitrary fields (possibly with positive characteristic). On the other hand, his approach led to most interesting results and applications also for ordinary representations (i.e. on vector spaces over the complex field $\mathbb{C}$), as previously indicated in the books by Diaconis and Sternberg cited above.

6.1 The combinatorics of pairs of partitions and the Littlewood–Richardson rule

6.1.1 Words and lattice permutations

Let m be a positive integer. The elements of the *alphabet* $\{1, 2, \dots, m\}$ are called *letters*. A *word* of *length n* over the alphabet $\{1, 2, \dots, m\}$ is a *sequence* $w = x_1x_2\cdots x_n$ where each symbol x_i is a letter in $\{1, 2, \dots, m\}$, $i = 1, 2, \dots, n$. Suppose that $x_i = j \in \{1, 2, \dots, m\}$. We then say that x_i is a *j-symbol* in w. The *type* of w is the m-tuple $a = (a_1, a_2, \dots, a_m)$ where $a_j = |\{1 \leq i \leq n : x_i = j\}|$ for $j = 1, 2, \dots, m$. In other words, a_j is the number of j-symbols in the word w. Clearly, $a_1 + a_2 + \cdots + a_m = n$, that is, a is a composition of n.

Definition 6.1.1 Let $w = x_1x_2\cdots x_n$ be a word. A symbol in w is said to be *good* or *bad* according to the following:

(i) Each 1-symbol is good.
(ii) The $(j+1)$-symbol x_i is good if in the subword $x_1x_2\cdots x_{i-1}$ the number of good j-symbols is strictly larger than the number of good $(j+1)$-symbols; otherwise it is bad.

Example 6.1.2 Consider the word $w = 1221133223$ of length 10 in the alphabet $\{1, 2, 3\}$. In the table below we label each good (resp. bad) symbol with a g (resp. b).

1	2	2	1	1	3	3	2	2	3
g	g	b	g	g	g	b	g	g	g

Definition 6.1.3 Let $w = x_1x_2\cdots x_n$ be a word of length n over the alphabet $\{1, 2, \dots, m\}$. Let $j \in \{2, \dots, m\}$. We say that w has the *lattice permutation property* with respect to the letter j if, for $i = 1, 2, \dots, n$, the number of j's in the subword $x_1x_2\cdots x_i$ is less than or equal to the number of $j-1$'s therein. We say that w is a *lattice permutation* (or a *ballot sequence*, or *Yamamouchi word*) if it has the lattice permutation property with respect to all letters $j = 2, \dots, m$.

Example 6.1.4 Consider the word $w_1 = 112323233124$ of length 12 over the alphabet $\{1, 2, 3, 4\}$. Then w_1 has the lattice permutation property with respect to the letter 4 (but not with respect to 2 and 3). On the other hand, the word

$w_2 = 112123231124$ has the lattice permutation property with respect to 2, 3, 4, that is, it is a lattice permutation.

Lemma 6.1.5 *Let $w = x_1 x_2 \cdots x_n$ be a word.*

(i) *If every j-symbol of w is good, then w has the lattice permutation property with respect to the letter j.*
(ii) *w is a lattice permutation if and only if every symbol in w is good.*

Proof (i) For every $k \leq n$ denote by $w_k = x_1 x_2 \cdots x_k$ the initial subword of length k. Then we have

$$\begin{aligned} |\{(j-1)\text{-symbols in } w_k\}| &\geq |\{\text{good } (j-1)\text{-symbols in } w_k\}| \\ &\geq |\{\text{good } j\text{-symbols in } w_k\}| \\ &= |\{j\text{-symbols in } w_k\}|. \end{aligned}$$

(ii) Suppose that w is a lattice permutation and let $j_0 \in \{2, 3, \ldots, m\}$. Then, if all j-symbols with $j \leq j_0 - 1$ are good, we have that all the j_0-symbols are also good by Definition 6.1.3. The converse follows from (i). □

Note that the word in Example 6.1.2 has the lattice permutation property with respect to the letter 3, but the 3-symbol x_7 is bad.

Definition 6.1.6 Let $w = x_1 x_2 \cdots x_n$ be a word of length n over the alphabet $\{1, 2, \ldots, m\}$. Let $2 \leq j \leq m$.

(i) Suppose that there exists $i \in \{1, 2, \ldots, n\}$ such that $x_{i-1} = j - 1$ and $x_i = j$; then we say that (the $(j-1)$-symbol) x_{i-1} is 1-*step* j-*paired* to (the j-symbol) x_i and, reciprocally, x_i is 1-*step* j-*paired* to x_{i-1}.
(ii) Suppose there exist $1 \leq i < k \leq n$ such that $x_i = j - 1$ and $x_k = j$ and every $(j-1)$-symbol x_ℓ with $i < \ell < k$ (resp. every j-symbol x_h with $i < h < k$) is $(\ell' - \ell)$-step (resp. $h - h'$-step) j-paired to some j-symbol $x_{\ell'}$, where $\ell < \ell' < k$ (resp. to a $(j-1)$-symbol $x_{h'}$, where $i < h' < h$). Then we say that (the $(j-1)$-symbol) x_i is $(k-i)$-*step* j-*paired* to (the j-symbol) x_k.

If a $(j-1)$-symbol x_i is $(k-i)$-*step* j-paired to a j-symbol x_k, for some $1 \leq i < k \leq n$, then we simply say that x_i and x_k are j-*paired*. A j-symbol x_k (resp. a $(j-1)$-symbol x_ℓ) which is not j-paired is said to be j-*unpaired*.

Clearly, in order to determine which couples are j-paired, one must start by looking for, in order, the 1-step, the 2-step pairings, and so on. It is also obvious that the j-pairing is unique: if the $(j-1)$-symbol x_i is j-paired to a j-symbol

x_k, then for each j-symbol (resp. $(j-1)$-symbol) x_h with $j < h < k$, x_i and x_h (resp. x_h and x_k) cannot be j-paired.

There is a simple way to describe j-pairing in a word w. Replace each letter $j-1$ occuring in w by a left-hand bracket and each letter j by a right-hand bracket. The resulting expression is a parenthesis system and the symbols are paired following the usual rules for brackets.

Example 6.1.7 Consider the alphabet $\{1, 2, 3\}$ and let $j = 3$. In the picture below, we write a word over the alphabet $\{1, 2, 3\}$ and we label every 1-symbol with the letter 1 and every 2-symbol (resp. every 3-symbol) with a left-hand (resp. right-hand) bracket. In addition, paired brackets are in turn labelled with the same letter in $\{a, b, c, d, e, f\}$, while unpaired brackets are unlabelled.

$$\begin{array}{cccccccccccccccccccccc}
1 & 2 & 3 & 1 & 3 & 2 & 2 & 3 & 1 & 1 & 3 & 3 & 2 & 1 & 2 & 3 & 1 & 1 & 2 & 3 & 3 & 3 \\
1 & (_a &)_a & 1 &) & (_e & (_b &)_b & 1 & 1 &)_e &) & (_f & 1 & (_c &)_c & 1 & 1 & (_d &)_d &)_f &)
\end{array}$$

Lemma 6.1.8 *A word w is a lattice permutation with respect to a letter $j \geq 2$ if and only if each j-symbol is j-paired.*

Proof The "if " part is obvious. Conversely, suppose that $w = x_1x_2 \cdots x_n$ is a lattice permutation with respect to j. Suppose that $x_k = j$ for some $1 \leq k \leq n$. Then the number of $(j-1)$-symbols in the subword $x_1x_2 \cdots x_{k-1}$ is strictly larger than the number of j-symbols therein. Consider the largest position $i \leq k-1$ of a $(j-1)$-symbol which is not j-paired to a j-symbol in $x_{i+1}x_{i+2} \cdots x_{k-1}$. Note that in the subword $x_{i+1}x_{i+2} \cdots x_{k-1}$ every j-symbol must be j-paired with a $(j-1)$-symbol: otherwise, taking the least $i+1 \leq t \leq k-1$ such that $x_t = j$ and x_t is not j-paired with a $(j-1)$-symbol in $x_{i+1}x_{i+2} \cdots x_{t-1}$, then x_t would be j-paired with x_i. It follows that the $(j-1)$-symbol x_i is j-paired to x_k. This shows that every j-symbol is j-paired to some $(j-1)$-symbol. □

In terms of parentheses, a word as in Lemma 6.1.8 has an expression of the form

$$P_0 \; (\; P_1 \; (\; P_2 \; (\cdots (\; P_{h-1} \; (\; P_h$$

where $P_0, P_1, \ldots, P_h$ are j-closed, possible empty, parenthesis systems.

Lemma 6.1.9 *Let $w = x_1x_2 \ldots x_n$ be a word and denote by $w_i = x_1x_2 \ldots x_i$ the initial subword of length i for $i = 1, 2, \ldots, n$. Suppose that in w every j-symbol is good. Then,*

$$|\{j\text{-paired } (j-1)\text{-symbols in } w_i\}| \leq |\{\text{good } (j-1)\text{-symbols in } w_i\}| \quad (6.1)$$

for every $i = 1, 2, \ldots, n$.

Proof Suppose that x_{i+1} is a $(j-1)$-symbol and that it is j-unpaired. Then,

$$\begin{aligned}|\{j\text{-paired } (j-1)\text{-symbols in } w_i\}| &= |\{j\text{-symbols in } w_i\}| \\ &\leq |\{\text{good } (j-1)\text{-symbols in } w_i\}|\end{aligned} \tag{6.2}$$

where the equality follows from the fact that the $(j-1)$-symbol x_{i+1} is j-unpaired (by (i) of Lemma 6.1.5 w is a lattice permutation with respect to j, and by Lemma 6.1.8 every j-symbol is j-paired to a $(j-1)$-symbol), while the inequality follows from the fact that every j-symbol is good. Consider the word $w' = x_1 x_2 \dots x_n (j-1) = w(j-1)$ of length $n+1$. Then, the $(j-1)$-symbol x_{n+1} is clearly j-unpaired in w' and the j-symbols in w' (which are the same of those in w) are still good therein. Applying (6.2) to w' we obtain (6.1) for $i = n$. We can now prove (6.1) by reverse induction, starting from $i = n$ and proving it for $i = n-1, n-2, \dots, 2, 1$. Suppose that (6.1) holds for $i = k+1$ and let us prove it for $i = k$. Suppose first that $x_{k+1} = j-1$. If this symbol is j-paired then (6.1) trivially holds for $i = k$ (indeed $|\{j\text{-paired } (j-1)\text{-symbols in } w_k\}| = |\{j\text{-paired } (j-1)\text{-symbols in } w_{k+1}\}| - 1$, while the number of good $(j-1)$-symbols present in w_k may only decrease by at most one). If the $(j-1)$-symbol x_{k+1} is j-unpaired then (6.1) follows from (6.2). Finally, if $x_{k+1} \neq j-1$, then the $(j-1)$-symbols in w_k and in w_{k+1} are the same and induction applies. □

6.1.2 Pairs of partitions

In this section, we allow the parts of compositions and partitions to be equal to zero. That is, we consider a composition $a = (a_1, a_2, \dots, a_h)$ of n as a sequence of nonnegative integers such that $a_1 + a_2 + \cdots + a_h = n$. Similarly, a partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_h)$ of n is a composition such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_h \geq 0$. Also, 0 will denote the trivial partition (of 0).

Definition 6.1.10

(i) A *pair of partitions for* n is a couple (λ, a), where $a = (a_1, a_2, \dots, a_m)$ is a composition of n, $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_m)$ is a partition of $k \leq n$ and $\lambda_i \leq a_i$ for $i = 1, 2, \dots, m$.

(ii) If (λ, a) is a pair of partitions for n, we denote by $W(\lambda, a)$ the set of all words of type a in which the number of good j-symbols is at least λ_j for $j = 1, 2, \dots, n$.

Note that a partition λ of length h can be seen as a partition of length $m > h$ by setting $\lambda_{h+1} = \lambda_{h+2} = \cdots = \lambda_m = 0$.

Remark 6.1.11

(i) $W(0, a)$ is just the set of all words of type a;

(ii) $W(\lambda, \lambda)$ is the set of all lattice permutations of type λ;

(iii) if $\lambda_1 < a_1$ and we set $\overline{\lambda} = (a_1, \lambda_2, \ldots, \lambda_m)$, then $W(\lambda, a) \equiv W(\overline{\lambda}, a)$, because every 1-symbol is good;

(iv) in every word w, the number of good j-symbols is greater than or equal to the number of good $(j+1)$-symbols. Therefore, in the definition of $W(\lambda, a)$ it is not restrictive to suppose that λ is a partition.

From now on, unless otherwise specified, we assume that $\lambda_1 = a_1$ (see Remark 6.1.11.(iii)). We also adopt the following graphical representation for a pair of partitions (λ, a). We draw a Young frame of shape a and we put a $\bullet$ inside the first (from left to right) λ_j boxes in row j, for $j = 1, 2, \ldots, m$. For instance, the pair of partitions of 23 given by ((7, 3, 3, 1), (7, 5, 8, 3)) is represented by the diagram in Figure 6.1.

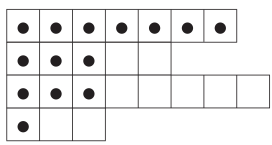


Figure 6.1

We now define two operations A_j and R_j on the set of pairs of partitions (λ, a) that satisfy the following conditions: $\lambda_{j-1} = a_{j-1}$ and $\lambda_j < a_j$. Roughly speaking, A_j "adds" 1 to λ_j, while R_j "raises" $a_j - \lambda_j$ boxes (without $\bullet$) from row j to row $j-1$.

Definition 6.1.12 Let (λ, a) be a pair of partitions and suppose that for an index j one has $\lambda_{j-1} = a_{j-1}$ and $\lambda_j < a_j$.

(a) The pair of partitions $A_j(\lambda, a)$ is defined in the following way:
– if $\lambda_{j-1} > \lambda_j$ then $A_j(\lambda, a) = (\mu, a)$, where the partition μ is given by $\mu = (\lambda_1, \lambda_2, \ldots, \lambda_{j-1}, \lambda_j + 1, \lambda_{j+1}, \ldots, \lambda_m)$;
– if $\lambda_{j-1} = \lambda_j$ then $A_j(\lambda, a) = (0, 0)$.

(b) $R_j(\lambda, a)$ is the pair of partitions (λ, b), where $b = (a_1, a_2, \ldots, a_{j-2}, a_{j-1} + a_j - \lambda_j, \lambda_j, a_{j+1}, \ldots, a_m)$; if $j = 2$ we also replace $\lambda_1 (\equiv a_1)$ with $\lambda_1 + a_2 - \lambda_2$.

Example 6.1.13 For instance,

$$A_3((5, 4, 2, 1), (5, 4, 4, 2)) = ((5, 4, 3, 1), (5, 4, 4, 2))$$

and

$$R_3((5, 4, 2, 1), (5, 4, 4, 2)) = ((5, 4, 2, 1), (5, 6, 2, 2)),$$

as in Figure 6.2.

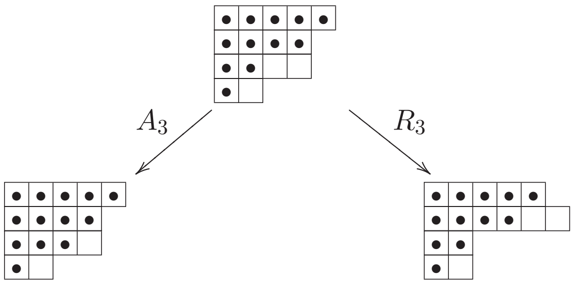


Figure 6.2

Also, $A_3((5, 4, 4, 1), (5, 4, 5, 2)) = (0, 0)$ and $R_2((4), (4, 3, 1)) = ((7), (7, 0, 1))$ (Figure 6.3).

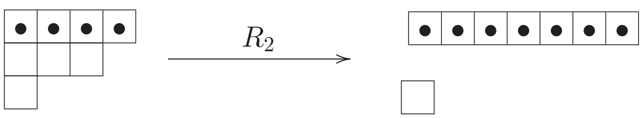


Figure 6.3

The following proposition is obvious (note that in the definition of R_2 we replace λ_1 with $\lambda_1 + a_2 - \lambda_2$, that is, under repeated applications of the operators of type A and R, we always have $\lambda_1 = a_1$).

Proposition 6.1.14 *Let (λ, a) be a pair of partitions for n. Consider a sequence of operations*

$$(\lambda, a) \to R_{j_\ell}^{k_\ell} \cdots R_{j_1}^{k_1}(\lambda, a) \to A_{i_q}^{h_q} \cdots A_{i_1}^{h_1} R_{j_\ell}^{k_\ell} \cdots R_{j_1}^{k_1}(\lambda, a) \to$$
$$\to R_{s_r}^{t_r} \cdots R_{s_1}^{t_1} A_{i_q}^{h_q} \cdots A_{i_1}^{h_1} R_{j_\ell}^{k_\ell} \cdots R_{j_1}^{k_1}(\lambda, a) \to \cdots$$

where $k_1, \ldots, k_\ell, h_1, \ldots, h_q, t_1, \ldots, t_r, \ldots$ are positive integers. Then, eventually one reaches a pair of partitions of n of the form (μ, μ).

Example 6.1.15 We have the representations shown in Figure 6.4 and 6.5.

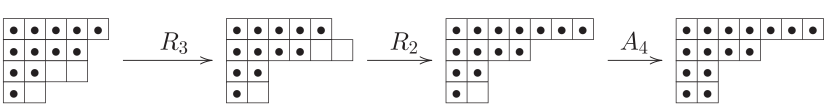


Figure 6.4

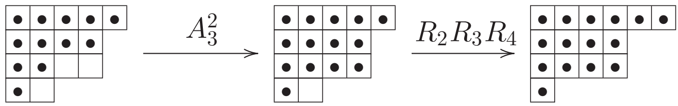


Figure 6.5

Let (λ, a) be a pair of partitions for n with $\lambda \neq a$. We construct a rooted, binary, oriented tree as follows. The root is (λ, a). Choose a j such that one may apply to (λ, a) the operators A_j and R_j (it is always possible to apply A_j, R_j where j is the first index such that $\lambda_{j-1} = a_{j-1}$ and $\lambda_j < a_j$). Then, the first level of the tree is given in Figure 6.6.

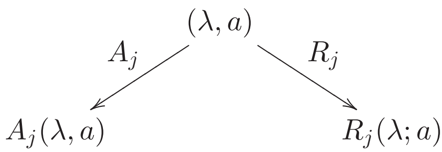


Figure 6.6

Then, we apply the previous step to the pair of partitions $A_j(\lambda, a)$ and $R_j(\lambda, a)$, and so on. When we get a pair of partitions of the form (μ, μ), the procedure at that vertex stops. Eventually, we end up with a tree whose terminal vertices, called *leaves*, are of the form (μ, μ).

An example is given in Figure 6.7.

Example 6.1.16

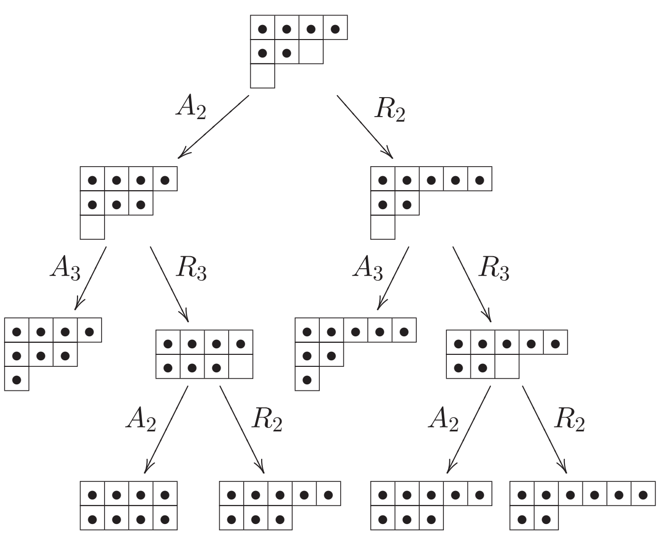


Figure 6.7

Clearly, given (λ, a), in most cases, we may construct more than one tree (depending on the j we start at each step), but we shall prove (see Theorem 6.1.30) that the number of terminal vertices of the form (ν, ν) only depends on (λ, a) and ν.

Proposition 6.1.17 *Let (λ, a) be a pair of partitions for n. Then there exist a composition $b = (b_1, b_2, \ldots, b_h)$ of n and a sequence of operations of type A and R that lead from $((b_1), b)$ to (λ, a).*

Proof Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$. Take $b = (a_1, \lambda_2, \ldots, \lambda_m, a_2 - \lambda_2, \ldots, a_m - \lambda_m)$ (recall that $a_1 = \lambda_1$). Then we have

$$A_m^{\lambda_m} A_{m-1}^{\lambda_{m-1}} \cdots A_2^{\lambda_2}((a_1), b) = (\lambda, b)$$

and

$$[(R_{m+1}R_{m+2} \cdots R_{2m-1})(R_m R_{m+1} \cdots R_{2m-2}) \cdots \cdots (R_3 R_4 \cdots R_{m+1})](\lambda, b) = (\lambda, a).$$

□

Example 6.1.18 For $(\lambda, a) = ((5, 2, 2, 1), (5, 4, 4, 2))$, see Figure 6.8.

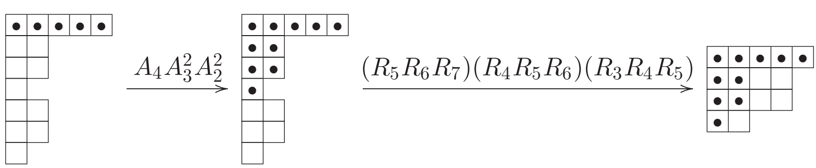


Figure 6.8

6.1.3 James' combinatorial theorem

Let (λ, a) be a pair of partitions for n. Suppose that $\lambda_{j-1} = a_{j-1}$ and $\lambda_j < a_j$ (so that we may apply A_j and R_j). Consider a word $w \in W(\lambda, a) \setminus W(A_j(\lambda, a))$. The number of $(j-1)$-symbols in w is equal to λ_{j-1} and all of them are good. On the other hand, the number of j-symbols is equal to a_j: λ_j of them are good, the remaining $a_j - \lambda_j$ are bad.

Definition 6.1.19 For $w \in W(\lambda, a) \setminus W(A_j(\lambda, a))$ we denote by $\Phi(w)$ the word obtained from w by replacing every bad j-symbol with a $(j-1)$-symbol.

For instance, for $(\lambda, a) = ((5, 4, 2, 1), (5, 4, 4, 2))$ and $j = 3$ (so that $A_3(\lambda, a) = ((5, 4, 3, 1), (5, 4, 4, 2)))$ and $w = 113214332314212$ of length 15 in the alphabet {1, 2, 3, 4} we have:

$$w = \underset{g}{1}\ \underset{g}{1}\ \underset{b}{3}\ \underset{g}{2}\ \underset{g}{1}\ \underset{b}{4}\ \underset{g}{3}\ \underset{b}{3}\ \underset{g}{2}\ \underset{g}{3}\ \underset{g}{1}\ \underset{g}{4}\ \underset{g}{2}\ \underset{g}{1}\ \underset{g}{2} \tag{6.3}$$

$w \in W(\lambda, a) \setminus W(A_3(\lambda, a))$ and

$$v = \Phi(w) = \underset{g}{1}\ \underset{g}{1}\ \underset{g}{2}\ \underset{g}{2}\ \underset{g}{1}\ \underset{b}{4}\ \underset{g}{3}\ \underset{g}{2}\ \underset{b}{2}\ \underset{g}{3}\ \underset{g}{1}\ \underset{g}{4}\ \underset{g}{2}\ \underset{g}{1}\ \underset{g}{2} \tag{6.4}$$

Lemma 6.1.20 $\Phi(w) \in W(R_j(\lambda, a))$ *for all* $w \in W(\lambda, a) \setminus W(A_j(\lambda, a))$.

Proof Let $w = x_1 x_2 \cdots x_n$ and $\Phi(w) = y_1 y_2 \cdots y_n$. As usual, for $k = 1, 2, \ldots, n$ we set $w_k = x_1 x_2 \cdots x_k$ and $\Phi(w)_k = y_1 y_2 \cdots y_k$. We first show that for $i = 1, 2, \ldots, n$,

$$|\{\text{good } (j-1)\text{-symbols in } w_i\}| \ \leq\ |\{\text{good } (j-1)\text{-symbols in } \Phi(w)_i\}|. \tag{6.5}$$

Clearly, (6.5) is trivial for $j = 2$ so that we may assume $j \geq 3$. We proceed by induction on i. First note that the statement is trivial for $i = 1$ (for any word v, x_1 is good if and only if $x_1 = 1$). Now we examine the worst case in the passage from step i to step $i + 1$. Suppose that (6.5) is true and that $x_{i+1} = j - 1 = y_{i+1}$, that x_{i+1} is good but y_{i+1} is bad. Since the number of $(j-2)$-symbols in w_i and in $\Phi(w)_i$ is the same (as the quality good/bad), this implies that the inequality in (6.5) is strict: the left-hand side must be strictly smaller than the number of good $(j-2)$-symbols in w_i, while the right-hand side must be equal to such a quantity. Therefore, (6.5) also holds for w_{i+1} and $\Phi(w)_{i+1}$. In all the other cases, the passage from step i to step $i + 1$ is trivial and thus (6.5) is proved.

Set $(\lambda, b) := R_j(\lambda, a)$. Clearly, $\Phi(w)$ is of type b. But (6.5) implies that $\Phi(w)$ contains at least λ_{j-1} good $(j-1)$-symbols and that all the j-symbols in $\Phi(w)$ are good (they are the same that were good in w). Another consequence is that, for $k > j$, the quality of each k-symbol does not change in the passage from w to $\Phi(w)$ (this is trivial for $k < j - 1$). In conclusion, $\Phi(w) \in W(R_j(\lambda, a))$. □

The main result of this section consists in showing that the map $w \mapsto \Phi(w)$ is indeed a bijection between $W(\lambda, a) \setminus W(A_j(\lambda, a))$ and $W(R_j(\lambda, a))$. This will be achieved by constructing an explicit inverse map.

Suppose that $v \in W(R_j(\lambda, a))$. Then the number of j-symbols in v is equal to λ_j, and all of them are good. On the other hand, the number of $(j-1)$-symbols is equal to $a_{j-1} + a_j - \lambda_j$, and at least λ_{j-1} of them are good. In particular, v is a lattice permutation with respect to j and therefore, by

Lemma 6.1.8, each j-symbol in v is j-paired to a $(j-1)$-symbol. If we replace the $(j-1)$-symbols and the j-symbols by left-hand brackets and right-hand brackets, respectively (see Example 6.1.7) we get an expression of the form

$$P_0(P_1(P_2 \cdots (P_{a_{j-1}+a_j-2\lambda_j-1}(P_{a_{j-1}+a_j-2\lambda_j} \tag{6.6}$$

where, for $k = 0, 1, \ldots, a_{j-1} + a_j - 2\lambda_j$, each P_k is a closed parenthesis system. Indeed, the j-unpaired $(j-1)$-symbols are $(a_{j-1} + a_j - \lambda_j) - \lambda_j = a_{j-1} + a_j - 2\lambda_j$ many.

Now, if $v = \Phi(w)$ for some $w \in W(\lambda, a) \setminus W(A_j(\lambda, a))$, then necessarily w is obtained from v by changing the first $a_j - \lambda_j$ j-unpaired $(j-1)$-symbols into j-symbols. Otherwise, assuming that all the $(j-1)$-symbols in w are good, the number of good j-symbols in w would be greater than λ_j. In other words, in the expression (6.6), we must reverse the first $a_j - \lambda_j$ unpaired left-hand brackets, which become unpaired right-hand brackets:

$$P_0)P_1)P_2 \cdots)P_{a_j-\lambda_j}(P_{a_j-\lambda_j+1}(\cdots (P_{a_{j-1}+a_j-2\lambda_j}. \tag{6.7}$$

Definition 6.1.21 For $v \in W(R_j(\lambda, a))$, we define $\Psi(v)$ as the word w obtained by replacing back in (6.7) the left-hand brackets and right-hand brackets with $(j-1)$-symbols and the j-symbols, respectively.

For instance, if v is as in (6.4), then $w = \Psi(v)$ is as in (6.3). The most difficult point is now to prove that $\Psi(v) \in W(\lambda, a) \setminus W(A_j(\lambda, a))$. We need another technical lemma.

Lemma 6.1.22 *Let $v = y_1 y_2 \cdots y_n \in W(R_j(\lambda, a))$ and $w = x_1 x_2 \cdots x_n$. Suppose that $w = \Psi(v)$. Then, for all i such that $x_{i+1} = j-1$ we have*

$$|\{\text{good } (j-2)\text{-symbols in } w_i\}| \;>\; |\{(j-1)\text{-symbols in } w_i\}|. \tag{6.8}$$

Proof First of all, note that if $x_{i+1} = j-1$ then also $y_{i+1} = j-1$. We divide the proof into two cases. In the first one we assume that v_i contains the first $a_j - \lambda_j$ j-unpaired $(j-1)$-symbols in v; compare with (6.6), (6.7) and the definition of Ψ (these j-unpaired $(j-1)$-symbols in v become j-symbols in w). In the second case, we assume that the $(a_j - \lambda_j)$th j-unpaired $(j-1)$-symbol in v occurs in position $> i+1$ (also note that the $(j-1)$-symbol y_{i+1} cannot be one of the first $(a_j - \lambda_j)$ j-unpaired $(j-1)$-symbols in v as these become j-symbols in w).

First case. Since $v \in W(R_j(\lambda, a))$, it contains at most $(a_j + a_{j-1} - \lambda_j) - \lambda_{j-1} = a_j - \lambda_j$ bad $(j-1)$-symbols. Therefore

$$\begin{aligned}|\{\text{good } (j-2)\text{-symbols in } w_i\}| &= |\{\text{good } (j-2)\text{-symbols in } v_i\}| \\ &> |\{(j-1)\text{-symbols in } v_i\}| - (a_j - \lambda_j) \\ &= |\{(j-1)\text{-symbols in } w_i\}|\end{aligned}$$

and (6.8) holds in this case.

Second case. Now, the $(j-1)$-symbol y_{i+1} is j-paired. Moreover, every j-symbol in v is good and Lemma 6.1.9 applies:

$$\begin{aligned}|\{(j-1)\text{-symbols in } w_i\}| &= |\{j\text{-paired } (j-1)\text{-symbols in } v_i\}| \\ (\text{by Lemma 6.1.9}) &\leq |\{\text{good } (j-1)\text{-symbols in } v_i\}| \\ &\leq |\{\text{good } (j-2)\text{-symbols in } v_i\}| \\ &= |\{\text{good } (j-2)\text{-symbols in } w_i\}|,\end{aligned}$$

where the first inequality is strict when the $(j-1)$-symbol y_{i+1} is bad (apply Lemma 6.1.9 to v_{i+1}), while the second inequality is strict when the $(j-1)$-symbol y_{i+1} is good. This proves (6.8) in the second case. □

Lemma 6.1.23 *Let $v \in W(R_j(\lambda, a))$. Then $\Psi(v) \in W(\lambda, a) \setminus W(A_j(\lambda, a))$.*

Proof From Lemma 6.1.22 it follows that every $(j-1)$-symbol in $\Psi(v)$ is good, while from the discussion preceding the definition of Ψ (see also (6.6) and (6.7)), it follows that $\Psi(v)$ contains exactly λ_j good j-symbols. □

Summarizing we have:

Theorem 6.1.24 (James combinatorial theorem for words) *Let (λ, a) be a pair of partitions for n. Suppose that $\lambda_{j-1} = a_{j-1}$ and $\lambda_j < a_j$. Then the map*

$$\Phi : W(\lambda, a) \setminus W(A_j(\lambda, a)) \to W(R_j(\lambda, a))$$

is a bijection and its inverse is Ψ. In particular,

$$|W(\lambda, a)| = |W(A_j(\lambda, a))| + |W(R_j(\lambda, a))|.$$

Proof In the discussion preceding Definition 6.1.21 we have shown that if $v \in W(R_j(\lambda, a))$ and $v = \Phi(w)$ for some $w \in W(\lambda, a) \setminus W(A_j(\lambda, a))$, then necessarily $w = \Psi(v)$. Then the theorem follows from Lemma 6.1.23. □

6.1.4 Littlewood–Richardson tableaux

Let ν be a partition of n and μ be a partition of $n - m$ such that $\mu \preceq \nu$ (cf. Section 3.1). Let ν/μ be the corresponding skew diagram and T a skew tableau of shape ν/μ (cf. Section 3.5.1). We say that T is *semistandard* if the numbers in T are strictly increasing from top to bottom along each column and weakly increasing from left to right along each row. The *weight* of T is the composition $a = (a_1, a_2, \dots, a_k)$, where a_i denotes the number of boxes in T occupied by the number i, for $i = 1, 2, \dots, k$. The *word* $w(T)$ associated with T is the sequence consisting of the numbers in the first row of T in reverse order followed (on the right) by the numbers in the second row still in reverse order, and so on. For instance, if T is as shown in Figure 6.9,

$$T = \begin{array}{cccccccc} & & 1 & 1 & 2 & 3 & 5 & 6 \\ & 1 & 2 & 3 & 3 & & & \\ 2 & 4 & 4 & & & & & \end{array}$$

Figure 6.9

then

$$w(T) = 6532113321442.$$

A box in T containing the number j is called a j-box. We say that a j-box is good/bad if the corresponding j-symbol in $w(T)$ is good/bad. Similarly a $j/(j-1)$-box is j-paired (resp. j-unpaired) if the corresponding $j/(j-1)$-symbol in $w(T)$ is j-paired (resp. j-unpaired).

Definition 6.1.25 Let (λ, a) be a pair of partitions for m, μ a partition of $n - m$ and ν a partition of n such that $\mu \preceq \nu$, $1 \le m \le n$.

(1) A *Littlewood–Richardson* tableau of *shape* ν/μ and *type* (λ, a) is a semistandard tableau T of shape ν/μ and weight a such that $w(T)$ belongs to $W(\lambda, a)$. If $a = \lambda$, we say that T is of *type* λ.

(2) We denote by $C^{\nu}_{(\lambda,a),\mu}$ the set of all Littlewood–Richardson tableaux of shape ν/μ and type (λ, a). If $\lambda = a$ we simply write $C^{\nu}_{\lambda,\mu}$.

(3) We denote by $c^{\nu}_{(\lambda,a),\mu}$ (resp. $c^{\nu}_{\lambda,\mu}$) the cardinality of $C^{\nu}_{(\lambda,a),\mu}$ (resp. $C^{\nu}_{\lambda,\mu}$).

Now, we present some results on the numbers $c^{\nu}_{(\lambda,a),0}$. These are the cardinalities of the semistandard tableaux of shape ν, weight a such that $w(T) \in W(\lambda, a)$. We use the notation and some results from Section 3.6.1 and Section 3.7.1.

Lemma 6.1.26 *Let (λ, a) be a pair of partitions for m, with $\lambda = (\lambda_1, \lambda_2, \dots, \dots, \lambda_k)$, $\mu = 0$ (in particular, $n = m$) and ν a partition of n.*

(i) *If $\lambda = 0$, then $c^{\nu}_{(0,a),0} = |\mathrm{STab}(\nu, a)|$ is the number of semistandard tableaux of shape ν and weight a; in the notation of Corollary 3.7.11 (and of Corollary 6.2.21), $c^{\nu}_{(0,a),0} = K(\nu, a)$.*

(ii) *If* $\nu \not\trianglerighteq a$, *then* $c^{\nu}_{(\lambda,a),0} = 0$.
(iii) $c^{\nu}_{\lambda,0} = \delta_{\nu,\lambda}$. *Moreover, the unique* $T \in C^{\lambda}_{\lambda,0}$ *is the tableau with all the* j*'s in the row* $j = 1, 2, \ldots, k$.
(iv) *If* $\lambda \vdash m$ *and* $a = (\lambda, n-m)$ *we have*

$$c^{\nu}_{(\lambda,(\lambda,n-m)),0} = \begin{cases} 1 & \text{if } \nu/\lambda \text{ is totally disconnected} \\ 0 & \text{otherwise.} \end{cases}$$

Proof (i) It follows immediately from the definition of $c^{\nu}_{(0,a),0}$.

(ii) It is an easy consequence of Lemma 3.7.3.

(iii) Suppose that $T \in C^{\lambda}_{\lambda,0}$. Since T is semistandard, a number j can only appear on a row i with $i \leq j$ (along the columns, the numbers are strictly increasing). Suppose that some j appears in a row i with $i < j$, and suppose that j is minimal number satisfying this condition. Then in the rows i', with $i' < i$ there is no $j-1$, by minimality. Also in row i there is no $j-1$ on the right of j (since T is semistandard, so that, along the rows, the numbers are weakly increasing). Then in $w(T)$ the corresponding j-symbol is bad, and this is a contradiction. Therefore all the j's must appear in row j, for $j = 1, 2, \ldots, k$. In particular, $C^{\nu}_{\lambda,0} = \emptyset$ for $\nu \neq \lambda$ and $C^{\lambda}_{\lambda,0}$ only contains the tableau with all the j's in row j, so that $c^{\lambda}_{\lambda,0} = 1$.

(iv) If $T \in C^{\nu}_{(\lambda,(\lambda,n-m)),0}$, then the boxes containing the number $k+1$ form a totally disconnected skew shape (recall that, since T is semistandard, along the columns the numbers are stricly increasing). Such a skew shape is of the form ν/μ for some partition μ. Deleting the boxes containing the $(k+1)$'s, we remain with a semistandard tableau S of shape μ such that $w(S) \in W(\lambda, \lambda)$. From (iii) we deduce that $\lambda = \mu$ and that S is the unique tableau in $C^{\lambda}_{\lambda,0}$. □

Remark 6.1.27 Note that (iv) in the previous lemma may be also expressed in the following form: $c^{\nu}_{(\lambda,(\lambda,n-m)),0} = c^{\nu}_{(n-m),\lambda}$. Compare this with Pieri's rule (Corollary 3.5.14).

Let (λ, a) be a pair of partitions for m and suppose that $\lambda_j < a_j$ and $\lambda_{j-1} = a_{j-1}$ so that we may apply R_j and A_j. For $T \in C^{\nu}_{(\lambda,a),\mu} \setminus C^{\nu}_{A_j(\lambda,a),\mu}$ we denote by $\widetilde{\Phi}(T)$ the tableau obtained by replacing the number j in each bad j-box, by the number $j-1$. Similarly, for $S \in C^{\nu}_{R_j(\lambda,a),\mu}$ we denote by $\widetilde{\Psi}(S)$ the tableau obtained by replacing the numbers $j-1$ in the first $a_j - \lambda_j$ j-unpaired $(j-1)$-boxes, by the number j. In particular we have

$$w(\widetilde{\Phi}(T)) = \Phi(w(T)) \quad \text{and} \quad w(\widetilde{\Psi}(S)) = \Psi(w(S)).$$

The following is the Littlewood–Richardson tableaux version of James combinatorial theorem for words (Theorem 6.1.24).

Theorem 6.1.28 (James combinatorial theorem for Littlewood–Richardson tableaux) *The map*

$$\widetilde{\Phi} : C^{\nu}_{(\lambda,a),\mu} \setminus C^{\nu}_{A_j(\lambda,a),\mu} \to C^{\nu}_{R_j(\lambda,a),\mu}$$

is a bijection and its inverse is $\widetilde{\Psi}$.

Proof By virtue of Theorem 6.1.24, we only need to prove that

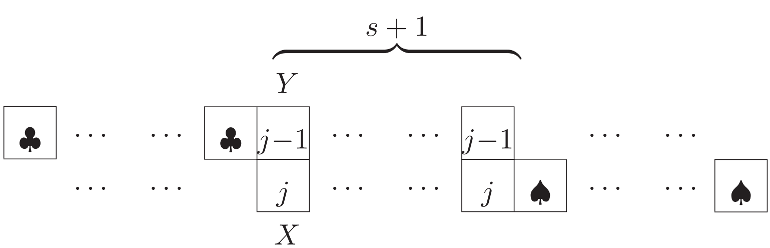
$$T \text{ is semistandard if and only if } \widetilde{\Phi}(T) \text{ is semistandard} \tag{6.9}$$

(note that, $\widetilde{\Phi}(T)$ may be defined even if T is not semistandard).

Suppose that T is semistandard. Then all j-symbols in a row of T appear consecutively in a single block. Moreover, this block is divided into two sub-blocks: the left-one (possibly empty) formed by the bad j-symbols and the right-one (possibly empty) formed by the good j-symbols. This shows that the rows of $\widetilde{\Phi}(T)$ are weakly increasing. Let us show that the columns are stricly increasing.

We prove it by contradiction. Note that the only situation that may produce a column in $\widetilde{\Phi}(T)$ which is not strictly increasing is the following: there exists a bad j-box (which therefore becomes a $(j-1)$-box in $\widetilde{\Phi}(T)$), denoted by X, with a $(j-1)$-box, denoted by Y, immediately above. Denote by $s \geq 0$ the number of j-boxes on the right of X. Since T is semistandard, there are at least other s $(j-1)$-boxes on the right of Y.

Therefore we should find this situation:



where the ♣'s indicate numbers smaller than j and the ♠'s numbers greater than j. Therefore the word $w(T)$ looks like

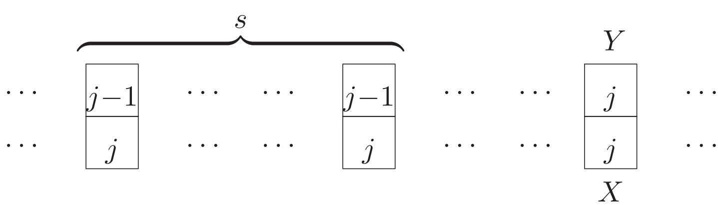
$$\cdots \overbrace{(j-1)\cdots(j-1)}^{s+1} \clubsuit \cdots \clubsuit \spadesuit \cdots \spadesuit \overbrace{j \cdots j}^{s+1} \cdots \tag{6.10}$$

Now, all the $(j-1)$-symbols in $w(T)$ are good. But, on the right of a sequence of at least $s+1$ consecutive good $(j-1)$-symbols, the next $s+1$ j-symbols are all good as well. Therefore, the j-box X should be good, contradicting our assumptions. We have shown the "only if" part of (6.9).

Suppose now that T is a tableau such that $w(T) \in W(\lambda, a) \setminus W(A_j(\lambda, a))$ and that $\widetilde{\Phi}(T)$ is semistandard. If a row of T is not weakly increasing, then it necessarily contains a bad j-box immediately followed by a $(j-1)$-box. But this is impossible because in $w(T)$ all $(j-1)$-symbols are good and we would be in the same situation of (6.10) (with $s = 0$ and without the subword $\clubsuit \cdots \clubsuit\spadesuit \cdots \spadesuit$). Thus, the j-symbol is necessarily good. This shows that the rows are weakly increasing.

Suppose now that a column in T is not strictly increasing. Again, this column necessarily contains a good j-box, call it X, with immediately above a bad j-box, call it Y. Let s denote the number of $(j-1)$-boxes on the left of Y. As X is good we necessarily have $s \geq 1$. We claim that each $(j-1)$-box on the left of Y has a good j-box immediately below. Indeed, X is a j-box also in $\widetilde{\Phi}(T)$ (recall that X is good in T) and $\widetilde{\Phi}(T)$ is semistandard.

Therefore we find the following situation:



and the word $w(T)$ looks like:

$$\cdots \underset{b}{j} * * * \cdots * \overbrace{\underset{g}{(j-1)} \cdots \underset{g}{(j-1)}}^{s} * * * \cdots * \underset{g}{j} * * * \cdots * \overbrace{\underset{g}{j} \cdots \underset{g}{j}}^{s} \cdots$$

where each $*$ is a symbol different from $j-1$. This clearly contradicts the definition of a good/bad symbol (we have a bad j-symbol followed by exactly s good $(j-1)$-symbols and at least $s+1$ other good j-symbols). Therefore, the columns of T are strictly increasing and this ends the proof. $\square$

Corollary 6.1.29 *We have*

$$c^{\nu}_{(\lambda,a),\mu} = c^{\nu}_{A_j(\lambda,a),\mu} + c^{\nu}_{R_j(\lambda,a),\mu}.$$

Theorem 6.1.30 (James combinatorial theorem for pair of partitions) *Let (λ, a) be a pair of partitions for n. Then in any tree with root (λ, a) as in Example 6.1.16, the number of leaves of the form (ν, ν) is equal to $c^{\nu}_{(\lambda,a),0}$.*

Proof The proof is by induction on the number of the level of the tree. If the tree has just two levels, then we must have a situation as in Figure 6.10,

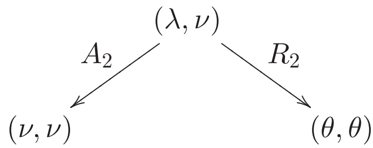


Figure 6.10

where ν and θ are partitions of n with $\nu = (\nu_1, \nu_2, \nu_3, \ldots)$, $\lambda = (\nu_1, \nu_2 - 1, \nu_3, \ldots)$ and $\theta = (\nu_1 + 1, \nu_2 - 1, \nu_3, \ldots)$. From Corollary 6.1.29 and Lemma 6.1.26.(iii),

$$c^{\nu}_{(\lambda,\nu),0} = c^{\nu}_{\nu,0} + c^{\nu}_{\theta,0} = 1 + 0 = 1$$

while similarly, again from Lemma 6.1.26.(iii),

$$c^{\theta}_{(\lambda,\nu),0} = c^{\theta}_{\nu,0} + c^{\theta}_{\theta,0} = 0 + 1 = 1.$$

Therefore the statement holds if the tree has exactly two levels. The general case is easily obtained by induction using again Corollary 6.1.29. □

Note that we may rewrite Theorem 6.1.24 in the following form. There is a natural bijection between $W(\lambda, a)$ and $W(A_j(\lambda, a)) \coprod W(R_j(\lambda, a))$, given by Φ and the inclusion $W(A_j(\lambda, a)) \subseteq W(\lambda, a)$. This means that in the tree in Figure 6.11 (compare with Example 6.1.16), there is a natural bijection between the first and the second level. We may then iterate this procedure, for instance like in Figure 6.6, and we associate a tree as in Figure 6.11.

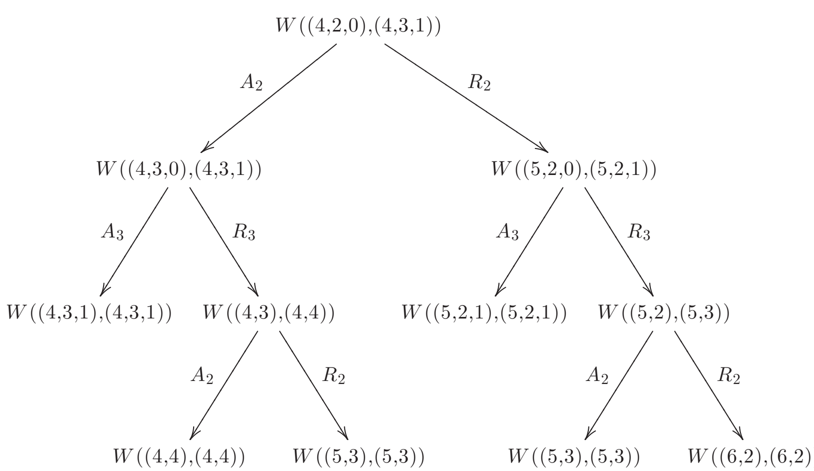


Figure 6.11

The following corollary collects the resuts from Proposition 6.1.14, Theorem 6.1.24, Theorem 6.1.30 and the construction in Figure 6.11.

Corollary 6.1.31 (James combinatorial theorem for words II) *Let (λ, a) be a pair of partitions for n. Construct a binary tree as in Figure 6.11. Then, the tree establishes a natural bijection*

$$W(\lambda, a) \cong \coprod_{\nu \vdash n} c^{\nu}_{(\lambda,a),0} W(\nu, \nu),$$

where $c^{\nu}_{(\lambda,a),0} W(\nu, \nu)$ stands for $c^{\nu}_{(\lambda,a),0}$ disjoint copies of $W(\nu, \nu)$.

Exercise 6.1.32 (1) State and discuss the tree-like version of Theorem 6.1.28.

(2) Prove that if (λ, a) is a pair of partitions for m, $\mu \vdash n - m$ and $\nu \vdash n$, then

$$c^{\nu}_{(\lambda,a),\mu} = \sum_{\eta \vdash m} c^{\eta}_{(\lambda,a),0} c^{\nu}_{\eta,\mu}.$$

6.1.5 The Littlewood–Richardson rule

Let $\mathcal{C}(\mathfrak{S}_n)$ be the vector space of central functions defined on $\mathfrak{S}_n$. We set

$$\mathcal{A} = \bigoplus_{n=0}^{\infty} \mathcal{C}(\mathfrak{S}_n)$$

where $\mathcal{C}(\mathfrak{S}_0) = \mathbb{C}$.

For two partitions $\lambda \vdash r$ and $\mu \vdash m$, we denote by χ^λ and χ^μ the characters of S^λ and S^μ, respectively. We define the product $\chi^\lambda \circ \chi^\mu$ as the character of the $\mathfrak{S}_{r+m}$-representation

$$\mathrm{Ind}^{\mathfrak{S}_{r+m}}_{\mathfrak{S}_r \times \mathfrak{S}_m} (S^\lambda \boxtimes \mathfrak{S}^\mu).$$

Then, we can extend the operation $\circ$ to the whole $\mathcal{A}$: for $\phi = \sum_{\lambda \vdash r} a_\lambda \chi^\lambda$ and $\psi = \sum_{\mu \vdash m} b_\mu \chi^\mu$, $a_\lambda, b_\mu \in \mathbb{C}$, we set

$$\phi \circ \psi = \sum_{\lambda \vdash r} \sum_{\mu \vdash m} a_\lambda b_\mu \chi^\lambda \circ \chi^\mu \in \mathcal{C}(\mathfrak{S}_{r+m}).$$

This is well defined because the characters form a basis for $\mathcal{A}$. This way, $\mathcal{A}$ becomes a *commutative graded algebra*. Indeed, $\mathcal{A}$ is an algebra and

$$\phi \in \mathcal{C}(\mathfrak{S}_r) \text{ and } \psi \in \mathcal{C}(\mathfrak{S}_m) \Rightarrow \phi \circ \psi \in \mathcal{C}(\mathfrak{S}_{r+m}).$$

For instance, for a composition $a = (a_1, a_2, \ldots, a_h)$, denoting by ψ^a the character of M^a, then we have

$$\psi^a = \psi^{(a_1)} \circ \psi^{(a_2)} \circ \cdots \circ \psi^{(a_h)} \equiv \chi^{(a_1)} \circ \chi^{(a_2)} \circ \cdots \circ \chi^{(a_h)}.$$

Let (λ, a) be a pair of partitions for a positive integer m. We associate with (λ, a) the linear map

$$\Theta^{(\lambda,a)}\colon \mathcal{A} \to \mathcal{A}$$

defined by setting, for $\mu \vdash n - m$ (with $n \geq m$),

$$\Theta^{(\lambda,a)}(\chi^\mu) = \sum_{\nu \vdash n} c^{\nu}_{(\lambda,a),\mu} \chi^\nu.$$

We simply write Θ^λ instead of $\Theta^{(\lambda,\lambda)}$. We now present some basic properties of the operators $\Theta^{(\lambda,a)}$.

Lemma 6.1.33

(i) *For any composition $a = (a_1, a_2, \ldots, a_h)$ we have*

$$\Theta^{(0,a)} = \Theta^{(0,(a_1))}\Theta^{(0,(a_2))} \cdots \Theta^{(0,(a_h))};$$

(ii) $\Theta^{(0,a)}(\chi^0) = \psi^a \equiv \psi^{(a_1)} \circ \psi^{(a_2)} \circ \cdots \circ \psi^{(a_h)}$;

(iii) *for any partition λ we have*

$$\Theta^\lambda(\chi^0) = \chi^\lambda;$$

(iv) *let (λ, a) be a pair of partitions with $\lambda_j < a_j$ and $\lambda_{j-1} = a_{j-1}$, then*

$$\Theta^{(\lambda,a)} = \Theta^{A_j(\lambda,a)} + \Theta^{R_j(\lambda,a)};$$

(v) *for any partition $\lambda \vdash n$ we have:*

$$\Theta^\lambda = \sum_{\substack{\eta \vdash n:\\ \eta \trianglerighteq \lambda}} H(\eta, \lambda)\Theta^{(0,\eta)},$$

where $(H(\eta, \lambda))$ is the inverse matrix of $(K(\eta, \lambda))$ (see Corollary 3.7.13).

Proof First of all, note that a tableau $T \in C^{\nu}_{(0,a),\lambda}$ may be constructed with the following procedure (compare with Section 3.7.2). We start by adding to the diagram of λ a_1 boxes containing 1 in such a way that no two 1's are on the same column and the resulting diagram has the shape of a partition $\nu^{(1)}$

with $\lambda \preceq \nu^{(1)} \preceq \nu$. We then continue by adding a_2 boxes containing 2, a_3 boxes containing 3, and so on. We then get a sequence of paritions

$$\lambda \preceq \nu^{(1)} \preceq \nu^{(2)} \preceq \cdots \preceq \nu^{(h-1)} \preceq \nu.$$

This means that

$$c^{\nu}_{(0,a),\lambda} = \sum_{\lambda \preceq \nu^{(1)} \preceq \nu^{(2)} \preceq \cdots \preceq \nu^{(h-1)} \preceq \nu} c^{\nu^{(1)}}_{(0,a_1),\lambda} c^{\nu^{(2)}}_{(0,a_2),\nu^{(1)}} \cdots c^{\nu}_{(0,a_h),\nu^{(h-1)}}.$$

It follows that $\Theta^{(0,a)}(\chi^{\lambda}) = \Theta^{(0,(a_h))}\Theta^{(0,(a_{h-1}))} \cdots \Theta^{(0,(a_1))}(\chi^{\lambda})$. Since Lemma 3.7.1 also holds for skew shape tableaux, we get

$$\Theta^{(0,a)} = \Theta^{(0,(a_h,a_{h-1},\dots,a_1))} = \Theta^{(0,a_1)}\Theta^{(0,a_2)} \cdots \Theta^{(0,a_h)}.$$

(ii) This is just a reformulation of the Young rule (Corollary 3.7.11; see also Corollary 6.2.21): from Lemma 6.1.26.(i) we have

$$\Theta^{(0,a)}(\chi^{0}) = \sum_{\nu \vdash n} c^{\nu}_{(0,a),0}\chi^{\nu} = \sum_{\substack{\nu \vdash n:\\ \nu \trianglerighteq a}} K(\nu, a)\chi^{\nu} = \psi^{a}.$$

(iii) This is an immediate consequence of Lemma 6.1.26.(iii).

(iv) This follows easily from Corollary 6.1.29.

Finally, from (iv), Theorem 6.1.30 and Lemma 6.1.26.(i), we deduce that, for $\eta \vdash n$,

$$\Theta^{(0,\eta)} = \sum_{\substack{\lambda \vdash n:\\ \lambda \trianglerighteq \eta}} K(\lambda, \eta)\Theta^{\lambda}$$

and therefore (v) follows from Corollary 3.7.13. □

We are now in position to state and prove the main result of the whole chapter.

Theorem 6.1.34 (James' proof of the Littlewood–Richardson rule) *Let λ be a partition of n and μ a partition of $n - m$. Then*

$$\Theta^{\lambda}(\chi^{\mu}) = \chi^{\mu} \circ \chi^{\lambda}.$$

Proof We have

$$
\begin{aligned}
\Theta^\lambda(\chi^\mu) &= \\
\text{(by Lemma 6.1.33.(iii))} &= \Theta^\lambda\Theta^\mu(\chi^0)\\
\text{(by Lemma 6.1.33.(v))} &= \sum_{\eta,\nu} H(\eta,\lambda)H(\nu,\mu)\Theta^{(0,\eta)}\Theta^{(0,\nu)}(\chi^0)\\
\text{(by Lemma 6.1.33.(i))} &= \sum_{\eta,\nu} H(\eta,\lambda)H(\nu,\mu)\Theta^{(0,(\eta_1))}\Theta^{(0,(\eta_2))}\cdots\Theta^{(0,(\eta_h))}\cdot\\
&\qquad \cdot\Theta^{(0,(\nu_1))}\Theta^{(0,(\nu_2))}\cdots\Theta^{(0,(\nu_k))}(\chi^0)\\
\text{(by Lemma 6.1.33.(i),(ii))} &= \sum_{\eta,\nu} H(\eta,\lambda)H(\nu,\mu)\psi^{(\eta_1)}\circ\psi^{(\eta_2)}\circ\cdots\circ\psi^{(\eta_h)}\circ\\
&\qquad \circ\psi^{(\nu_1)}\circ\psi^{(\nu_2)}\circ\cdots\circ\psi^{(\nu_k)}\\
&= \left(\sum_\eta H(\eta,\lambda)\psi^\eta\right)\circ\left(\sum_\nu H(\nu,\mu)\psi^\nu\right)\\
\text{(by Corollary 3.7.14)} &= \chi^\lambda\circ\chi^\mu.
\end{aligned}
$$

□

Corollary 6.1.35 (The Littlewood–Richardon rule) *Let $\lambda \vdash m$, $\mu \vdash n-m$ and $\nu \vdash n$ be three partitions.*

(i) *The multiplicity of S^ν in $\mathrm{Ind}_{\mathfrak{S}_m\times\mathfrak{S}_{n-m}}^{\mathfrak{S}_n}[S^\lambda \boxtimes S^\mu]$ is equal to $c^\nu_{\lambda,\mu}$, namely the number of skew tableaux of shape ν/μ, weight λ such that the associated word is a lattice permutation.*

(ii) *The multiplicity of $S^\lambda \boxtimes S^\mu$ in $\mathrm{Res}_{\mathfrak{S}_m\times\mathfrak{S}_{n-m}}^{\mathfrak{S}_n} S^\nu$ is equal to $c^\nu_{\lambda,\mu}$.*

6.2 Radon transforms, Specht modules and orthogonal decompositions of Young modules

6.2.1 Generalized Specht modules

We begin with an alternative description of the Young modules (see Section 3.6.2). Let $a = (a_1, a_2, \dots, a_h)$ be a composition of n. A (*bijective*) *tableau* of shape a is a bijective filling of the Young frame of shape a with the numbers $1, 2, \dots, n$. A tableau of shape $(3, 5, 2)$ is shown in Figure 6.12.

3	1	5		
7	8	4	6	2
10	9			

Figure 6.12

If T is a tableau of shape a and $\pi \in \mathfrak{S}_n$, then πT is the tableau obtained from T by replacing j with $\pi(j)$, $j = 1, 2, \ldots, n$. The *row stabilizer* $\mathcal{R}_T$ of T is the subgroup consisting of all permutations of $\pi \in \mathfrak{S}_n$ that globally fix the rows of T. In other words, $\pi \in \mathcal{R}_T$ if and only if T and πT have the same numbers on row i, $i = 1, 2, \ldots, h$. Similarly, the *column stabilizer* $\mathcal{C}_T$ of T is the subgroup consisting of all permutations of $\pi \in \mathfrak{S}_n$ that globally fix the columns of T. Clearly, $\mathcal{R}_T \cong \mathfrak{S}_{a_1} \times \mathfrak{S}_{a_2} \times \cdots \times \mathfrak{S}_{a_h}$ and $\mathcal{C}_T \cong \mathfrak{S}_{b_1} \times \mathfrak{S}_{b_2} \times \cdots \times \mathfrak{S}_{b_k}$, where b_i is the number of boxes in the ith column of T, $i = 1, 2, \ldots, k$, where $k = \max\{a_1, a_2 \ldots, a_h\}$. For instance, for the tableau in Figure 6.12 we have

$$\mathcal{R}_T = \mathfrak{S}_{\{1,3,5\}} \times \mathfrak{S}_{\{2,4,6,7,8\}} \times \mathfrak{S}_{\{10,9\}} \cong \mathfrak{S}_3 \times \mathfrak{S}_5 \times \mathfrak{S}_2$$

and

$$\mathcal{C}_T = \mathfrak{S}_{\{3,7,10\}} \times \mathfrak{S}_{\{1,8,9\}} \times \mathfrak{S}_{\{4,5\}} \times \mathfrak{S}_{\{2\}} \times \mathfrak{S}_{\{6\}} \cong \mathfrak{S}_3 \times \mathfrak{S}_3 \times \mathfrak{S}_2,$$

where $\mathfrak{S}_A$ denotes the symmetric group on the set A.

We now define an equivalence relation on the set of all tableaux of shape a. Given two such tableaux T_1 and T_2, we say that they are *row-equivalent* if there exists $\pi \in \mathcal{R}_{T_1}$ such that $T_2 = \pi T_1$. Thus, T_1 and T_2 are row-equivalent if and only if they have the same numbers in the ith row, for $i = 1, 2, \ldots, h$. Note that T_1 and T_2 are row-equivalent if and only if $\mathcal{R}_{T_1} = \mathcal{R}_{T_2}$.

The row-equivalence class of the tableau T is called a *tabloid* of shape a and is denoted by $\{T\}$. In other words, a tabloid is just a tableau with unordered row entries and we shall represent it as a tableau without vertical lines. For instance, the tabloid in Figure 6.13 corresponds to the tableau of Figure 6.12.

$$\begin{array}{lllll} \hline 3 & 1 & 5 & & \\ \hline 7 & 8 & 4 & 6 & 2 \\ \hline 10 & 9 & & & \\ \hline \end{array} \quad \equiv \quad \begin{array}{lllll} \hline 1 & 3 & 5 & & \\ \hline 2 & 4 & 6 & 7 & 8 \\ \hline 9 & 10 & & & \\ \hline \end{array}$$

Figure 6.13

Clearly, the set of all tabloids of shape a is the same thing as the space Ω_a of all compositions of $\{1, 2, \ldots, n\}$ of type a, introduced in Section 3.6.2: the tabloid $\{T\}$ corresponds to the composition $(A_1, A_2, \ldots, A_h)$ where A_i are the numbers on the ith row of $\{T\}$. For instance, the tabloid in Figure 6.13 corresponds to the composition $(\{1, 3, 5\}, \{2, 4, 6, 7, 8\}, \{9, 10\})$. In view of this identification, we shall denote by Ω_a the set of all tabloids of shape a. Clearly, $\Omega_a \cong \mathfrak{S}_n/(\mathfrak{S}_{a_1} \times \mathfrak{S}_{a_2} \times \cdots \times \mathfrak{S}_{a_h})$. This way, the Young module M^a

may be identified with the set of all formal linear combinations of tabloids:

$$M^a = \left\{ \sum_{\{T\}\in\Omega_a} f(\{T\})\{T\} : f \in L(\Omega_a) \right\} \tag{6.11}$$

with the action of $\mathfrak{S}_n$ given by:

$$\pi\left(\sum_{\{T\}\in\Omega_a} f(\{T\})\{T\}\right) = \sum_{\{T\}\in\Omega_a} f(\{T\})\{\pi T\} \equiv \sum_{\{T\}\in\Omega_a} (\pi f)(\{T\})\{T\}$$

(where, clearly, $(\pi f)(\{T\}) = f(\pi^{-1}\{T\}))$ for all $\pi \in \mathfrak{S}_n$.

Now suppose that (λ, a) is a pair of partitions for n, with $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_h)$ and $a = (a_1, a_2, \ldots, a_h)$. Let T be a tableau of shape a. We shall circle the numbers contained in those boxes of the diagram of λ. Figures 6.14 and 6.15 shown an example with T as given

$$T = \begin{array}{|c|c|c|c|c|} \hline 1 & 5 & 3 \\ \hline 4 & 7 & 6 & 2 & 8 \\ \hline 10 & 9 \\ \hline \end{array}$$

Figure 6.14

and $(\lambda, a) = ((3, 2, 0), (3, 5, 2))$.

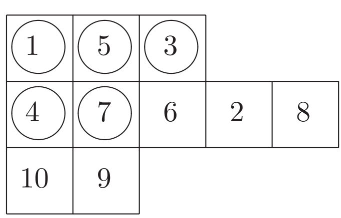


Figure 6.15

Definition 6.2.1 (1) Let (λ, a) be a pair of partitions for n and let T be a tableau of shape a. The *generalized polytaboid* of type (λ, a) associated with T is the element in M^a (cf. (6.11))

$$E_T^{\lambda,a} = \sum_{\pi} \varepsilon(\pi)\{\pi T\},$$

where the sum is over all $\pi \in \mathcal{C}_T$ such that π fixes all numbers outside the diagram of λ (that is, π only moves the circled numbers).

(2) Denote by $S^{\lambda,a}$ the subspace of M^a spanned by all generalized polytabloids $E_T^{\lambda,a}$, with T varying in the set of all tableaux of shape a. The space $S^{\lambda,a}$ is called the *generalized Specht module of type* (λ, a). If $a = \lambda$, $S^{\lambda,\lambda}$ is called a *Specht module*.

Note that, in particular, $S^{0,a} \equiv M^a$ and $S^{0,0} = \{0\}$.

Example 6.2.2 With T and (λ, a) as in Figure 5.1 and 5.2, respectively, we have that $E_T^{(3,2,0),(3,5,2)}$ is equal to the polytabloid in Figure 6.16.

$$\begin{array}{ccccc} \hline 1 & 3 & 5 & & \\ \hline 2 & 4 & 6 & 7 & 8 \\ \hline 9 & 10 & & & \\ \hline \end{array} \quad - \quad \begin{array}{ccccc} \hline 3 & 4 & 5 & & \\ \hline 1 & 2 & 6 & 7 & 8 \\ \hline 9 & 10 & & & \\ \hline \end{array}$$

$$- \quad \begin{array}{ccccc} \hline 1 & 3 & 7 & & \\ \hline 2 & 4 & 5 & 6 & 8 \\ \hline 9 & 10 & & & \\ \hline \end{array} \quad + \quad \begin{array}{ccccc} \hline 3 & 4 & 7 & & \\ \hline 1 & 2 & 5 & 6 & 8 \\ \hline 9 & 10 & & & \\ \hline \end{array}$$

Figure 6.16

Example 6.2.3 The space $S^{(n-1,1),(n-1,1)}$ is spanned by the generalized polytabloids of the form shown in Figure 6.17.

$$\begin{array}{ccccc} \hline h & j_1 & j_2 & \cdots & j_{n-2} \\ \hline k & & & & \\ \hline \end{array} \quad - \quad \begin{array}{ccccc} \hline k & j_1 & j_2 & \cdots & j_{n-2} \\ \hline h & & & & \\ \hline \end{array}$$

Figure 6.17

where $\{h, k, j_1, j_2, \ldots, j_{n-2}\} \equiv \{1, 2, \ldots, n\}$. $S^{(n-1,1),(n-1,1)}$ is the subspace of $M^{n-1,1} \equiv L(\mathfrak{S}_n/\mathfrak{S}_{n-1})$ (note that $\mathfrak{S}_n/\mathfrak{S}_{n-1} \equiv \{1, 2, \ldots, n\}$) spanned by all the differences $\delta_k - \delta_h$, with $h, k \in \{1, 2, \ldots, n\}$. Using the result in Example 1.4.5 (see also Example 1.4.10), it is easy to see that $S^{(n-1,1),(n-1,1)} \equiv S^{n-1,1}$ as $\mathfrak{S}_n$-representations. This is not incidental: we will show that $S^{\lambda,\lambda} \cong S^\lambda$ for every partition λ.

Exercise 6.2.4 Show that $S^{(n-2,1),(n-2,2)}$ is the subspace of $M^{n-2,2}$ spanned by all differences of the form

$$\delta_{\{k,j\}} - \delta_{\{h,j\}}$$

where h, k, j are distinct numbers in $\{1, 2, \ldots, n\}$.

Also show that $S^{(n-2,1),(n-2,2)} \cong S^{n-1,1} \oplus S^{n-2,2}$ as $\mathfrak{S}_n$-representation.

Hint. Use Example 1.4.10.

In the definition of $S^{\lambda,a}$, it is not restrictive to suppose that $\lambda_1 = a_1$. Indeed, the numbers in the last $\lambda_1 - \lambda_2$ boxes of the first row of an a-tableau T remain fixed in the construction of $E_T^{\lambda,a}$.

We now introduce an algorithm that associates, with each word $w = x_1x_2\cdots x_n$ of length n and type a, a bijective tableau T_w of shape a.

Definition 6.2.5 Consider a word $w = x_1x_2\cdots x_n$ of length n and type a. Starting with x_1, and then proceeding in order with $x_2, x_3, \ldots, x_n$, we define a tableau T_w as follows:

- if $x_i = j$ and x_i is good, we put i in the first unoccupied box of the jth row;
- if $x_i = j$ and x_i is bad, we put i in the last unoccupied box of the jth row.

For instance, with the sequence

$$\begin{matrix} 1 & 2 & 2 & 1 & 1 & 3 & 3 & 2 & 2 & 3 \\ g & g & b & g & g & g & b & g & g & g \end{matrix}$$

the associated tableau T_w is the following

$$\begin{array}{|c|c|c|c}\hline 1 & 4 & 5 & \multicolumn{1}{c}{} \\ \hline 2 & 8 & 9 & \multicolumn{1}{c|}{3} \\ \hline 6 & 10 & 7 & \multicolumn{1}{c}{} \\ \cline{1-3}\end{array}\ .$$

Lemma 6.2.6 *The correspondence*

$$\begin{array}{ccc} W(0,a) & \to & \Omega_a \\ w & \mapsto & \{T_w\}, \end{array}$$

that associates with each word $w \in W(0,a)$ the tabloid $\{T_w\}$ containing T_w, is a bijection.

Proof The word $w = x_1x_2\cdots x_n$ is associated with the tabloid that has i in row x_i for all $i = 1, 2, \ldots, n$. □

Suppose $w \in W(\lambda, a)$. Then the part of T_w consisting of the boxes of λ is standard.

We now introduce an order on the set Ω_a of all a-tabloids. If $\{T_1\}, \{T_2\} \in \Omega_a$ and $\{T_1\} \neq \{T_2\}$ we write $\{T_1\} < \{T_2\}$ if $i_0 := \max\{i \in \{1, 2, \ldots, n\} : i$ occupies different rows in T_1 and $T_2\}$ is in a row of $\{T_1\}$ which is higher than the corresponding row of $\{T_2\}$. This is a total order.

For instance, the total order in $\Omega_{(2,2)}$ is shown in Figure 6.18.

$$\begin{array}{c}\underline{3\,4}\\\underline{1\,2}\end{array} < \begin{array}{c}\underline{2\,4}\\\underline{1\,3}\end{array} < \begin{array}{c}\underline{1\,4}\\\underline{2\,3}\end{array} < \begin{array}{c}\underline{2\,3}\\\underline{1\,4}\end{array} < \begin{array}{c}\underline{1\,3}\\\underline{2\,4}\end{array} < \begin{array}{c}\underline{1\,2}\\\underline{3\,4}\end{array}$$

Figure 6.18

Lemma 6.2.7 *The vectors $E^{\lambda,a}_{T_w}$, $w \in W(\lambda, a)$, are linearly independent in $S^{\lambda,a}$.*

Proof We first introduce a numbering $w_1, w_2, \ldots, w_h$ ($h = |W(\lambda, a)|$) of the words in $W(\lambda, a)$ such that

$$\{T_{w_1}\} < \{T_{w_2}\} < \cdots < \{T_{w_h}\}$$

with respect to the total order on Ω_a.

We observe that if $w \in W(\lambda, a)$ then $\{T_w\}$ is the gratest tabloid appearing in the sum that defines $E^{\lambda,a}_{T_w}$. Indeed, T_w is standard inside λ and therefore if $\pi \in \mathcal{C}_{T_w}$ does not move the numbers outside the diagram of λ, then the following holds: if i is the largest number moved by π then it is in a row of $\pi(T_w)$ which is higher than the corresponding row of T_w. This means that, for $k = h, h-1, \ldots, 2$, the vector $E^{\lambda,a}_{T_{w_k}}$ is not a linear combination of the vectors $E^{\lambda,a}_{T_{w_1}}, E^{\lambda,a}_{T_{w_2}}, \ldots, E^{\lambda,a}_{T_{w_{k-1}}}$. □

Recalling Definition 6.1.12 we have:

Lemma 6.2.8 *The inclusion $S^{A_j(\lambda,a)} \leq S^{\lambda,a}$ always holds.*

Proof This is obvious if $A_j(\lambda, a) = (0, 0)$. If $A_j(\lambda, a)$ is not trivial and T is a tableau of shape a, we have

$$E^{A_j(\lambda,a)}_T = \sum_{\theta\in\Theta} \varepsilon(\theta)\theta E^{\lambda,a}_T,$$

where Θ is a set of representatives for the cosets of the subgroup of $\mathcal{C}_T$ which fixes the numbers outside the diagram of λ in the subgroup of $\mathcal{C}_T$ which fixes the numbers outside the diagram of μ, if $A_j(\lambda, a) = (\mu, a)$. □

6.2.2 A family of Radon transforms

Definition 6.2.9 Let $a = (a_1, a_2, \ldots, a_m)$ be a composition of n. Given $j \in \{1, 2, \ldots, m\}$ suppose that $a_j > 0$ and consider the composition $b = (a_1, a_2, \ldots, a_{j-2}, a_{j-1} + a_j - v, v, a_{j+1}, \ldots, a_m)$, where $0 \leq v \leq a_j$. We define an intertwining operator

$$\mathcal{R}_{j,v} : M^a \to M^b$$

by setting, for all $\{T\} \in \Omega_a$,

$$\mathcal{R}_{j,v}\{T\} = \sum \{S\}$$

where the sum is over all $\{S\} \in \Omega_b$ such that

- the jth row of $\{S\}$ is a v-subset of the jth row of $\{T\}$;
- for $k \neq j, j-1$, the kth row of $\{S\}$ coincides with kth row of $\{T\}$.

In other words, $\mathcal{R}_{j,v}\{T\}$ is the sum of all tabloids obtained by raising $a_j - v$ numbers from the jth to the $(j-1)$st row of $\{T\}$.

Exercise 6.2.10 Show that $\mathcal{R}_{j,v}$ is a Radon transform in the sense of Section 3.6.4 and write down the corresponding matrix in $\mathcal{M}_{a,b}$.

Example 6.2.11 Let $a = (4, 4)$, $j = 2$, $v = 2$, so that $b = (6, 2)$. If $\{T\} \in M^{(4,4)}$ is as given in Figure 6.19,

$$\{T\} = \begin{array}{l} \hline 1\ 3\ 5\ 7 \\ \hline 2\ 4\ 6\ 8 \\ \hline \end{array}$$

Figure 6.19

then we have $\mathcal{R}_{2,2}\{T\}$ as shown in Figure 6.20.

$$\mathcal{R}_{2,2}\{T\} = \begin{array}{l} \hline 1\ 3\ 5\ 7\ 2\ 4 \\ \hline 6\ 8 \\ \hline \end{array} + \begin{array}{l} \hline 1\ 3\ 5\ 7\ 2\ 6 \\ \hline 4\ 8 \\ \hline \end{array} + \begin{array}{l} \hline 1\ 3\ 5\ 7\ 2\ 8 \\ \hline 4\ 6 \\ \hline \end{array}$$

$$+ \begin{array}{l} \hline 1\ 3\ 5\ 7\ 4\ 6 \\ \hline 2\ 8 \\ \hline \end{array} + \begin{array}{l} \hline 1\ 3\ 5\ 7\ 4\ 8 \\ \hline 2\ 6 \\ \hline \end{array} + \begin{array}{l} \hline 1\ 3\ 5\ 7\ 6\ 8 \\ \hline 2\ 4 \\ \hline \end{array}$$

Figure 6.20

The key lemma is the following.

Lemma 6.2.12 *Let (λ, a) be a pair of partitions for n. Suppose that $\lambda_{j-1} = a_{j-1}$ and $\lambda_j < a_j$. Then*

$$\mathcal{R}_{j,\lambda_j} S^{\lambda,a} = S^{R_j(\lambda,a)}$$

and

$$\mathcal{R}_{j,\lambda_j} S^{A_j(\lambda,a)} = 0.$$

Proof Let T be an a-tableau and let $E_T^{\lambda,a}$ be the associated generalized polytabloid. Since $\mathcal{R}_{j,\lambda_j}$ commutes with the action of $\mathfrak{S}_n$, we have:

$$\begin{aligned} \mathcal{R}_{j,\lambda_j} E_T^{\lambda,a} &= \mathcal{R}_{j,\lambda_j} \sum \varepsilon(\pi)\pi\{T\} \\ &= \sum \varepsilon(\pi)\pi \mathcal{R}_{j,\lambda_j}\{T\}, \end{aligned} \tag{6.12}$$

where, as usual, the sum is over all $\pi \in \mathcal{C}_T$ that fix all the numbers outside the diagram of λ. But

$$\mathcal{R}_{j,\lambda_j}\{T\} = \{T_1\} + \sum \{S\} \tag{6.13}$$

where T_1 is obtained by moving from the jth to the $(j-1)$st row all the numbers outside the diagram of λ (that is, the numbers in the boxes $\lambda_j + 1, \lambda_j + 2, \ldots, a_j$ in the jth row of T) and the sum runs over all other tabloids appearing in the definition of $\mathcal{R}_{j,\lambda_j}\{T\}$. This means that, in the notation of (6.12),

$$\sum \varepsilon(\pi)\pi\{T_1\} = E_{T_1}^{R_j(\lambda,a)}. \tag{6.14}$$

On the other hand, for every $\{S\}$ in (6.13), there exists a number inside the diagram of λ, say the number x, that has been moved to the $(j-1)$st row. If y is the number immediately above x in T, then the transposition $\pi_1 := (x \to y \to x)$ appears in the sum $\sum \varepsilon(\pi)\pi$ of (6.12) and, clearly, $\pi_1\{S\} = \{S\}$.

Therefore,

$$\begin{aligned} \sum \varepsilon(\pi)\pi\{S\} &= \frac{1}{2}\Big[\sum \varepsilon(\pi)\pi\{S\} + \sum \varepsilon(\pi\pi_1)\pi\pi_1\{S\}\Big] \\ &= \frac{1}{2}\Big[\sum \varepsilon(\pi)\pi\{S\} - \sum \varepsilon(\pi)\pi\{S\}\Big] \\ &= 0. \end{aligned} \tag{6.15}$$

In conclusion, from (6.12), (6.13), (6.14) and (6.15) we deduce that

$$\mathcal{R}_{j,\lambda_j} E_T^{\lambda,T} = E_{T_1}^{R_j(\lambda,a)}$$

and therefore

$$\mathcal{R}_{j,\lambda_j} S^{\lambda,a} = S^{R_j(\lambda,a)}.$$

Arguing as in (6.15), it is easy to prove that also $\mathcal{R}_{j,\lambda_j} S^{A_j(\lambda,a)} = 0$. Indeed, if $A_j(\lambda, a) = (\mu, a)$, there is always an x inside the diagram of μ that is moved by $\mathcal{R}_{j,\lambda_j}$ to the $(j-1)$st row. □

We are now in position to prove the fundamental result of this section. Its proof is based on the deep combinatorial results in Section 6.1.3.

Theorem 6.2.13

(i) *We have*

$$\dim S^{\lambda,a} = |W(\lambda, a)| \tag{6.16}$$

and the basis for $S^{\lambda,a}$ *is given by the set* $\{E^{\lambda,a}_{T_w} : w \in W(\lambda, a)\}$ *in Lemma 6.2.7.*

(ii) *In the notation of Lemma 6.2.12 we have:*

$$S^{\lambda,a} \cap \mathrm{Ker}\mathcal{R}_{j,\lambda_j} = S^{A_j(\lambda,a)}.$$

Proof First of all, note that (6.16) holds when $\lambda = 0$: this is an obvious consequence of Lemma 6.2.6 (now $E^{0,a}_{T_w} \equiv \{T_w\}$, so that the same lemma ensures that the set $\{E^{0,a}_{T_w} : w \in W(0, a)\}$ is a basis of $S^{0,a} \equiv M^a$). We can now prove (i) by induction, starting from the pair of partitions $(0, b)$ as the base and using Proposition 6.1.17. The inductive step consists in showing that if (i) holds true for $S^{\lambda,a}$, then it also holds for $S^{R_j(\lambda,a)}$ and $S^{A_j(\lambda,a)}$.

Now, if $\dim S^{\lambda,a} = |W(\lambda, a)|$, then

$$\begin{aligned}
|W(\lambda, a)| &= \dim S^{\lambda,a}\\
\text{(by Lemma 6.2.12)}\quad &\geq \dim S^{R_j(\lambda,a)} + \dim S^{A_j(\lambda,a)}\\
\text{(by Lemma 6.2.7)}\quad &\geq |W(R_j(\lambda, a))| + |W(A_j(\lambda, a))|\\
\text{(by Theorem 6.1.24)}\quad &= |W(\lambda, a)|
\end{aligned}$$

and therefore each inequality above is necessarily an equality. In particular, the inductive step follows:

$$\dim S^{R_j(\lambda,a)} = |W(R_j(\lambda, a))|$$

and

$$\dim S^{A_j(\lambda,a)} = |W(A_j(\lambda, a))|.$$

Finally, note that the equality $\dim S^{\lambda,a} = \dim S^{R_j(\lambda,a)} + \dim S^{A_j(\lambda,a)}$, coupled with Lemma 6.2.12, gives (ii). □

We now recall an elementary fact of linear algebra. Let V and W be two complex vector spaces with Hermitian scalar products $\langle \cdot, \cdot \rangle_V$ and $\langle \cdot, \cdot \rangle_W$, respectively. Let $T : V \to W$ be a linear operator and denote by $T^* : W \to V$ its adjoint. Then

$$V = \mathrm{Ker}T \bigoplus \mathrm{Ran}T^*,$$

where $\mathrm{Ran}T^* = \{T^*w : w \in W\}$ and the sum is orthogonal.

Indeed,

$$\begin{aligned} v \in \mathrm{Ker}T &\Leftrightarrow \langle Tv, w\rangle_W = 0 \text{ for all } w \in W \\ &\Leftrightarrow \langle v, T^* w\rangle_V = 0 \text{ for all } w \in W \\ &\Leftrightarrow v \in (\mathrm{Ran}T^*)^\perp. \end{aligned}$$

Moreover, from the analogous decomposition $W = \mathrm{Ker}T^* \oplus \mathrm{Ran}T$ it follows that T is surjective if and only if T^* is injective.

Note also that if $V_1 \leq V$ and $TV_1 = W_1 \leq W$, in general we have $(T|_{V_1})^* \neq T^*|_{W_1}$. Indeed, take $V = W = \mathbb{C}^2$, $T(z_1, z_2) = (z_2, z_1 + z_2)$, $V_1 = \{(z, 0) : z \in \mathbb{C}\}$ and $W_1 = \{(0, z) : z \in \mathbb{C}\}$. Then $TV_1 = W_1$, $T = T^*$ but $T^* W_1 \neq V_1$.

From these considerations applied to the restriction of $\mathcal{R}_{j,\lambda_j}$ to $S^{\lambda,a}$, and from Theorem 6.2.13, we immediately deduce the following:

Corollary 6.2.14 *We have the isomorphism (of $\mathfrak{S}_n$-representations)*

$$S^{\lambda,a} \cong S^{R_j(\lambda,a)} \bigoplus S^{A_j(\lambda,a)} \tag{6.17}$$

with orthogonal direct sum.

Moreover,

$$S^{A_j(\lambda,a)} \cong \mathrm{Ker}\left(\mathcal{R}_{j,\lambda_j}|_{S^{\lambda,a}}\right)$$

and

$$S^{R_j(\lambda,a)} \cong \mathrm{Ran}\left[\left(\mathcal{R}_{j,\lambda_j}|_{S^{\lambda,a}}\right)^*\right].$$

Remark 6.2.15 In the notation of Definition 6.2.9, the adjoint $\mathcal{R}^*_{j,v}$ of $\mathcal{R}_{j,v}$ has the following simple expression: for each $\{S\} \in \Omega_b$,

$$\mathcal{R}^*_{j,v}\{S\} = \sum \{T\}$$

where the sum runs over all $\{T\} \in \Omega_a$ such that:

- the jth row of $\{S\}$ is a v-subset of the jth row of $\{T\}$;
- for $k \neq j - 1, j$, the kth row of $\{S\}$ coincides with the kth row of $\{T\}$.

Indeed, with this definition we clearly have

$$\langle \mathcal{R}^*_{j,v}\{S\}, \{T\}\rangle_{M^a} = \langle \{S\}, \mathcal{R}_{j,v}\{T\}\rangle_{M^b}.$$

However, in general, $\left(\mathcal{R}_{j,\lambda_j}|_{S^{\lambda,a}}\right)^*$ is not the restriction of $\mathcal{R}^*_{j,\lambda_j}$ to $S^{R_j(\lambda,a)}$.

6.2.3 Decomposition theorems

In this section, we collect several fundamental decomposition theorems that are consequences of Theorem 6.2.13 and Corollary 6.2.14. We begin with the decomposition of an arbitrary generalized Specht module.

Theorem 6.2.16 *Let* (λ, a) *be a pair of partitions for* n. *We have*

$$S^{\lambda,a} \cong \bigoplus_{\nu \vdash n} c^{\nu}_{(\lambda,a),0} S^{\nu,\nu}$$

where $c^{\nu}_{(\lambda,a),0}$ *is the number of all Littlewood–Richardson tableaux of shape* ν *and type* (λ, a) *(see Definition 6.1.25).*

Proof This follows from repeated applications of Corollary 6.2.14, taking into account Theorem 6.1.30. □

Lemma 6.2.17 *Let* $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k) \vdash m$, $n > m$, *and denote by* $S^{(n-m)}$ *the trivial representation of* $\mathfrak{S}_{n-m}$. *Then*

$$\mathrm{Ind}^{\mathfrak{S}_n}_{\mathfrak{S}_m \times \mathfrak{S}_{n-m}} \left[S^{\lambda,\lambda} \boxtimes S^{(n-m)} \right] \cong S^{\lambda,(\lambda,n-m)}, \tag{6.18}$$

where $(\lambda, n-m) = (\lambda_1, \lambda_2, \dots, \lambda_k, n-m)$.

Proof For each $A \in \Omega_{m,n-m}$ (so that A is an $(n-m)$-subset of $\{1, 2, \dots, n\}$), let $[S^{\lambda,(\lambda,n-m)}]_A$ denote the subspace spanned by all generalized polytabloids $E_T^{\lambda,(\lambda,n-m)}$ such that the last row of T is occupied by the numbers in A. Then, clearly,

$$[S^{\lambda,(\lambda,n-m)}]_A \cong S^{\lambda,\lambda} \boxtimes S^{(n,n-m)}$$

with respect to the action of the stabilizer $\mathfrak{S}_m \times \mathfrak{S}_{n-m}$ of the $(n-m)$-subset A. Moreover,

$$S^{\lambda,(\lambda,n-m)} = \bigoplus_{A \in \Omega_{m,n-m}} [S^{\lambda,(\lambda,n-m)}]_A$$

and this proves (6.18) (cf. Lemma 1.6.2). □

Corollary 6.2.18 (Pieri's rule) *Let* $\lambda \vdash m$, $n > m$ *and denote by* $S^{(n-m)}$ *the trivial representation of* $\mathfrak{S}_{n-m}$. *Then*

$$\mathrm{Ind}^{\mathfrak{S}_n}_{\mathfrak{S}_m \times \mathfrak{S}_{n-m}} \left[S^{\lambda,\lambda} \boxtimes S^{(n-m)} \right] = \bigoplus_{\nu \vdash n} c^{\nu}_{(n-m),\lambda} S^{\nu,\nu}$$

where $c^{\nu}_{(n-m),\lambda}$ *is as in Definition 6.1.25 (and in Lemma 6.1.26.(iv)).*

Proof Since

$$\begin{aligned} S^{\lambda,(\lambda,n-m)} &\cong \bigoplus_{\nu\vdash n} c^{\nu}_{(\lambda,(\lambda,n-m)),0} S^{\nu,\nu} \quad \text{(Theorem 6.2.16)} \\ &= \bigoplus_{\nu\vdash n} c^{\nu}_{(n-m),\lambda} S^{\nu,\nu} \quad \text{(Remark 6.1.27)} \end{aligned}$$

from Lemma 6.2.17 the corollary follows. □

As a particular case, we get:

Corollary 6.2.19 (Branching rule) *If* $\lambda \vdash n-1$ *then*

$$\mathrm{Ind}^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}\times\mathfrak{S}_1}[S^{\lambda,\lambda} \boxtimes S^{(1)}] = \bigoplus_{\substack{\nu\vdash n\\ \nu\to\lambda}} S^{\nu,\nu}.$$

Corollary 6.2.20 (Specht modules) *For* $n \geq 1$ *and every* $\lambda \vdash n$*,*

$$S^{\lambda,\lambda} \cong S^{\lambda}$$

as $\mathfrak{S}_n$*-representations.*

Proof This follows from Corollary 3.3.12 and Corollary 6.2.19. □

In particular, we may now rewrite the decomposition in Theorem 6.2.16 as follows:

$$S^{\lambda,a} = \bigoplus_{\nu\vdash n} c^{\nu}_{(\lambda,a),0} S^{\nu}.$$

Also, clearly, Corollary 6.2.18 coincides with Pieri's rule (cf. Corollary 3.5.14).

Corollary 6.2.21 (Young's rule) *For any composition* a *of* n *we have*

$$M^a = \bigoplus_{\substack{\lambda\vdash n:\\ \lambda\trianglerighteq a}} K(\lambda,a) S^{\lambda}$$

where $K(\lambda,a)$ *is the number of semistandard tableaux of shape* λ *and weight* a*.*

Proof We have $S^{0,a} \equiv M^a$ and $c^{\lambda}_{(0,a),0} = K(\lambda,a)$ (see Lemma 6.1.26). □

We now give an application of Theorem 6.2.16. We simply indicate by λ a pair of partitions of the form (λ,λ).

Example 6.2.22 From the tree in Figure 6.21

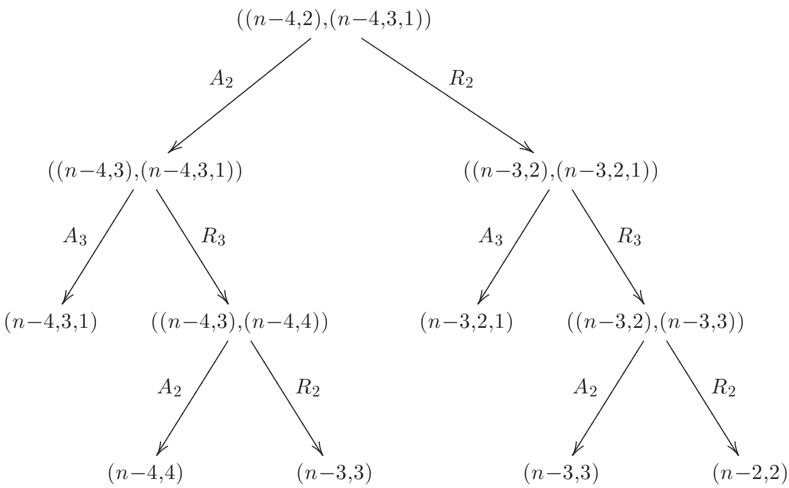


Figure 6.21

we get the following decomposition:

$$S^{(n-4,2),(n-4,3,1)} \cong S^{n-4,3,1} \oplus S^{n-4,4} \oplus 2S^{n-3,3} \oplus S^{n-3,2,1} \oplus S^{n-2,2}. \quad (6.19)$$

Note also that Corollary 6.2.14 ensures that (6.19) is an orthogonal decomposition: the isotypic component $2S^{n-3,3}$ is given as the orthogonal direct of two copies of $S^{n-3,3}$ corresponding with the two distinct paths of the tree in Figure 6.21 ending at a vertex labelled by $(n-3, 3)$.

Let $a = (a_1, a_2, \ldots, a_m)$, $j \in \{1, 2, \ldots, m\}$ and suppose that $a_j > 0$. We set

$$\mathcal{R}_j := \mathcal{R}_{j,a_j-1} \colon M^a \to M^b \quad (6.20)$$

where

$$b = (a_1, a_2, \ldots, a_{j-2}, a_{j-1} + 1, a_j - 1, a_{j+1}, \ldots, a_m)$$

as in Definition 6.2.9. Actually, we think of $\mathcal{R}_j$ as an operator defined on the direct sum of all M^a's with $a_j > 0$, so that we can write $\mathcal{R}_j^k$ instead of the more cumbersome expression $\mathcal{R}_{j,a_j-k}\mathcal{R}_{j,a_j-k+1} \cdots \mathcal{R}_{j,a_j-1}$.

Lemma 6.2.23 *Let $a = (a_1, a_2, \ldots, a_m)$, $j \in \{1, 2, \ldots, m\}$ and suppose that $a_j > 0$. For $v = 1, 2, \ldots, a_j$ consider the operators*

$$\mathcal{R}_{j,v} \colon M^a \to M^{b^{(v)}}$$

where $b^{(v)} = (a_1, a_2, \ldots, a_{j-2}, a_{j-1} + a_j - v, v, a_{j+1}, \ldots, a_m)$ *as in Definition* 6.2.9*. Then, for* $v = 1, 2, \ldots, a_j$*, we have*

$$\mathcal{R}_{j,v} = \frac{1}{(a_j - v)!}(\mathcal{R}_j)^{a_j - v}.$$

Proof Let $\{T\} \in \Omega_a$. Then $\mathcal{R}_{j,v}\{T\}$ is the sum of all tableaux that may be obtained by raising $a_j - v$ numbers from the jth to the $(j-1)$st row, while $(\mathcal{R}_j)^{a_j - v}\{T\}$ is the sum of all tableaux that may be obtained by repeating $a_j - v$ times the following procedure: take a number from the jth row and raise it to the $(j-1)$st. Clearly, a tableau $\{S\} \in \Omega_{b^{(v)}}$ appears in the first sum if and only it appears in the second. However, in the first sum it appears exactly once, while in the second one it appears exactly $(a_j - v)!$ times (we have $a_j - v$ different choices for the first raised number, $a_j - v - 1$ choices for the second one, and so on). □

Theorem 6.2.24 (James' intersection kernel theorem) *Let* $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ *be a partition of* n *and let* $\mathcal{R}_j$ *be as in* (6.20)*. Then*

$$S^\lambda = M^\lambda \cap \left(\bigcap_{j=2}^{k} \mathrm{Ker}\mathcal{R}_j \right).$$

Proof Let us show that

$$S^\lambda = M^\lambda \cap \left(\bigcap_{j=2}^{k} \left(\bigcap_{v=0}^{\lambda_j - 1} \mathrm{Ker}\mathcal{R}_{j,v} \right) \right).$$

Indeed, if we construct a tree for $M^\lambda \equiv S^{0,\lambda}$ (as in Example 6.2.22), then S^λ appears at the end of exactly one path, which is labelled only by A_j's, that is of the form

$$\xrightarrow{A_2}\xrightarrow{A_2}\cdots\xrightarrow{A_2}\xrightarrow{A_3}\xrightarrow{A_3}\cdots\xrightarrow{A_3}\cdots\cdots\xrightarrow{A_k}\xrightarrow{A_k}\cdots\xrightarrow{A_k}.$$

Moreover, from Lemma 6.2.23 it follows that

$$\bigcap_{v=0}^{\lambda_j - 1} \mathrm{Ker}\mathcal{R}_{j,v} = \mathrm{Ker}\mathcal{R}_j$$

and this ends the proof. □

6.2.4 The Gelfand–Tsetlin bases for M^a revisited

In this section, we analyze the decomposition of M^a in Theorem 3.7.10 along the lines of Theorem 6.2.16. We continue to denote by $\mathcal{R}_j$ the Radon transform $\mathcal{R}_{j,a_j-1}$, as in (6.20). We first treat two particular cases.

Proposition 6.2.25
(i) *For* $0 \le t \le k \le n/2$ *we have*

$$S^{(n-k,t),(n-k,k)} = M^{n-k,k} \cap \operatorname{Ker}\left(\mathcal{R}_2^{k-t+1}\right) \cong \bigoplus_{j=t}^{k} S^{n-j,j}$$

(for $t = k$*, this is a particular case of Theorem 6.2.24).*
(ii) *The decomposition*

$$M^{n-k,k} = \bigoplus_{t=0}^{k} (\mathcal{R}_2^*)^{k-t}\left[M^{n-t,t} \cap \operatorname{Ker}\mathcal{R}_2\right]$$

coincides with (3.72). In other words, $(\mathcal{R}_2^*)^{k-t}\left[M^{n-t,t} \cap \mathcal{R}_2\right]$ *coincides with the subspace of* $M^{n-k,k}$ *which is isomorphic to* $S^{n-t,t}$*.*

Proof Consider the tree in Figure 6.22.

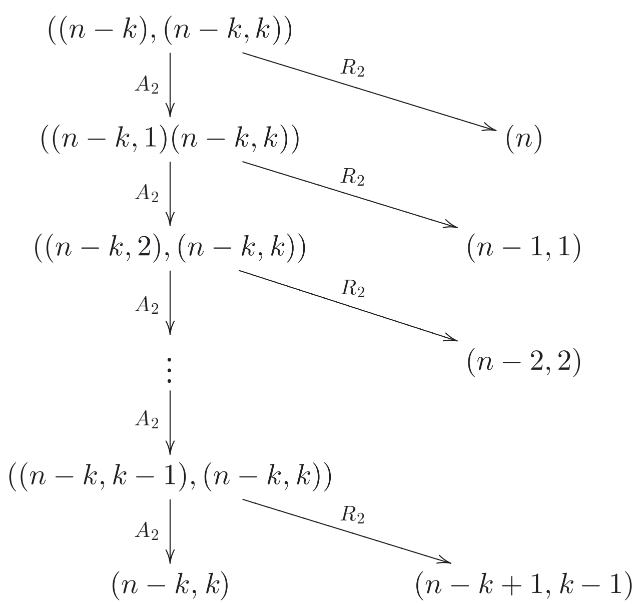


Figure 6.22

Then from Theorem 6.2.13 and Theorem 6.2.16, it follows that

$$S^{(n-k,t),(n-k,k)} = M^{n-k,k} \cap \left(\bigcap_{v=0}^{t-1} \mathrm{Ker}\mathcal{R}_{2,v} \right) \cong \bigoplus_{j=t}^{k} S^{n-j,j}.$$

Since $\mathcal{R}_{2,v} = \frac{1}{(k-v)!}\mathcal{R}_2^{k-v}$, we have

$$\bigcap_{v=0}^{t-1} \mathrm{Ker}\mathcal{R}_{2,v} \equiv \mathrm{Ker}\mathcal{R}_{2,t-1} \equiv \mathrm{Ker}\mathcal{R}_2^{k-t+1}$$

and (i) is proved.

Arguing as in Corollary 6.2.14, we get

$$M^{n-k,k} = \left[M^{n-k,k} \cap \mathrm{Ker}\mathcal{R}_{2,t}\right] \bigoplus \left[\mathcal{R}_{2,t}^*(M^{n-t,t})\right].$$

Since $S^{n-t,t}$ is not contained in $M^{n-k,k} \cap \mathrm{Ker}\mathcal{R}_{2,t} \cong S^{((n-k,t+1),(n-k,k))} \cong \bigoplus_{j=t+1}^{k} S^{n-j,j}$, the subspace in $M^{n-k,k}$ isomorphic to $S^{n-t,t}$ comes from $\mathcal{R}_{2,t}^*(M^{n-t,t})$. Therefore, since $M^{n-t,t} \cap \mathrm{Ker}\mathcal{R}_2 \cong S^{n-t,t}$ and (taking the adjoints in Lemma 6.2.23) $\mathcal{R}_{2,t}^* = \frac{1}{(k-t)!}(\mathcal{R}_2^*)^{k-t}$, we necessarily have $S^{n-t,t} = (\mathcal{R}_2^*)^{k-t}\left[M^{n-t,t} \cap \mathrm{Ker}\mathcal{R}_2\right] \leq M^{n-k,k}$. □

An elementary proof of Proposition 6.2.25 can be found in Chapter 6 of our monograph [20] and a generalization is in Chapter 8 therein. These facts are based on the papers of Delsarte [25], Dunkl [31, 32, 33] and Stanton [114].

Example 6.2.26 Consider the Young module $M^{n-2,1,1}$ and the associated tree in Figure 6.23.

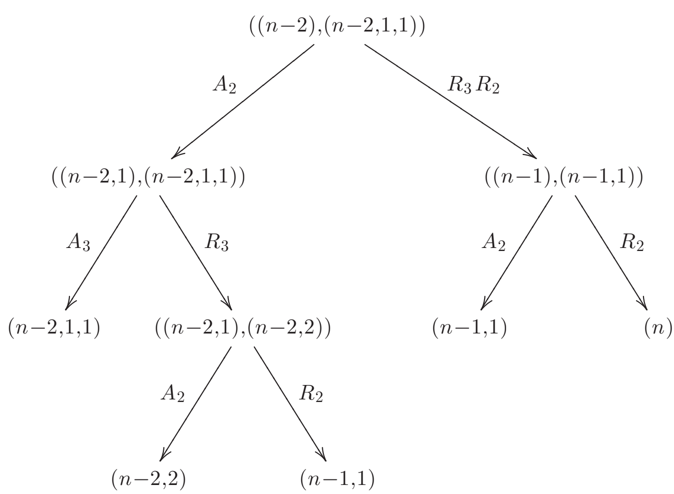


Figure 6.23

Note that in $((n-2),(n-2,1,1)) \xrightarrow{R_3 R_2} ((n-1),(n-1,1))$ we have omitted the intermediate (trivial) step in $((n-2),(n-2,1,1)) \xrightarrow{R_2} ((n-1),(n-1,0,1)) \xrightarrow{R_3} ((n-1),(n-1,1))$. In any case, we deduce the following decomposition

$$M^{n-2,1,1} \cong S^{n-2,1,1} \bigoplus S^{n-2,2} \bigoplus 2S^{n-1,1} \bigoplus S^{(n)}.$$

Note that we also have an orthogonal decomposition of the isotypic component $2S^{n-1,1}$. Moreover, by Lemma 6.2.17 and from the first level of the tree in Figure 5.6, we deduce the following decomposition:

$$\begin{aligned} M^{n-2,1,1} &\cong S^{(n-2,1),(n-2,1,1)} \oplus S^{(n-1),(n-1,1)} \\ &\cong S^{(n-2,1),(n-2,1,1)} \oplus M^{n-1,1} \\ &\cong \mathrm{Ind}_{\mathfrak{S}_{n-1}\times\mathfrak{S}_1}^{\mathfrak{S}_n}\left[S^{n-2,1} \boxtimes S^{(1)}\right] \oplus \mathrm{Ind}_{\mathfrak{S}_{n-1}\times\mathfrak{S}_1}^{\mathfrak{S}_n}\left[S^{(n-1)} \boxtimes S^{(1)}\right]. \end{aligned}$$

This is precisely the $M^{n-2,1,1}$ case of the decomposition used in Theorem 3.7.10 to get the Gelfand–Tsetlin decomposition of a general Young module.

The following lemma is an easy generalization of Lemma 6.2.17.

Lemma 6.2.27 *Let λ be a partition of m and let a be a composition of $n-m$. Then*

$$\mathrm{Ind}_{\mathfrak{S}_m\times\mathfrak{S}_{n-m}}^{\mathfrak{S}_n}[S^\lambda \boxtimes M^a] \cong S^{\lambda,(\lambda,a)}.$$

Our next purpose is to analyze the decomposition of M^a for an arbitrary composition $a=(a_1,a_2,\dots,a_h)$ of n. First of all, we have to draw a tree for the pair of partitions $((a_1),a)$. Recalling the condition "$\lambda_{j-1}=a_{j-1}$" in Definition 6.1.12, the tree starts with an application of A_2 and R_2. Clearly, there are several trees that may be constructed in order to decompose M^a. We choose a particular tree according to the following definition.

Definition 6.2.28 Given a pair of partitions $((a_1),a)$, the associated *Gelfand–Tsetlin tree* (*GZ-tree*) is the tree constructed using the following rule: at each step (that is, at each node of the tree) we use the operators A_j and R_j, with the smallest index j (among the operators that are applicable to that node).

For instance, given the pair of partitions represented by the tableau in Figure 6.24, we apply the operators A_2 and R_2 (note that A_4 and R_4 are also applicable).

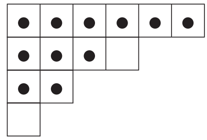


Figure 6.24

In the GZ-tree of $((a_1), a)$, a *special node* is a node of type $(\mu, (\mu, b))$, where $\mu = (\mu_1, \mu_2, \ldots, \mu_k)$ is a partition with $\mu_k > 0$ and $b = (b_1, b_2, \ldots, b_m)$ a composition. The nonnegative integer m is called the *height* of the special node.

In drawing the GZ-tree of $((a_1), a)$, we also adopt the following convention: in a special node $(\mu, (\mu, b))$ with $b_1 = 0$, we simply replace b with $(b_2, b_3, \ldots, b_m)$, as in Example 6.2.26. In particular, every node of the form (μ, μ) is special.

Lemma 6.2.29 *For a special node $(\mu, (\mu, b))$ of the GZ-tree of $((a_1), a)$ one always has*

$$b = (b_1, b_2, \ldots, b_m) = (a_{h-m+1}, a_{h-m+2}, \ldots, a_m).$$

In other words, in the diagram of $(\mu, (\mu, b))$, the undotted rows (corresponding to b) coincide with the last m rows of the diagram of a.

Proof The proof is quite obvious: applying the operators A_j and R_j with the smallest index j corresponds to modifying the diagram of the pair of partitions in the highest modificable row. We limit ourselves to give an example: we show how the diagram of $((4, 3), (4, 3, 2, 2))$ is modified when applying the rule in Definition 6.2.28, and, accordingly, we determine the subsequent special node.

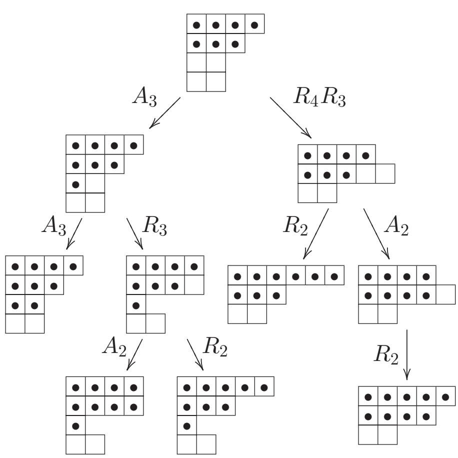


Figure 6.25

Every leaf in the tree in Figure 6.25 has ☐☐ in the last row. □

The following lemma is just a rephrasing of Pieri's rule.

Lemma 6.2.30 *Suppose that* $(\mu, (\mu, b))$ *and* $(\nu, (\nu, c))$ *are two pair of partitions that appear as special nodes in the GZ-tree of* $((a_1), a)$ *and that their heights are respectively* m *and* $m-1$ *(so that* $c = (b_2, b_3, \ldots, b_m)$*). Take a node labelled by* $(\mu, (\mu, b))$ *and let* $\mathcal{T}$ *be the tree below it. Then*

- $(\nu, (\nu, c))$ *appears exactly once in* $\mathcal{T}$ *if* ν/μ *is totally disconnected;*
- $(\nu, (\nu, c))$ *does not appear in* $\mathcal{T}$ *if* ν/μ *is not totally disconnected.*

Remark 6.2.31 Lemma 6.2.30 has the following consequence:

$$S^{\mu,(\mu,b)} \cong \bigoplus S^{\nu,(\nu,c)}$$

where the sum runs over all $\nu \vdash k + b_1$ such that ν/μ is totally disconnected. Note also that (if $b_1 + b_2 + \cdots + b_m = n - k$) by (6.18), Lemma 6.2.27 and transitivity of the induction (cf. Proposition 1.6.6) we have

$$\begin{aligned} S^{\mu,(\mu,b)} &\cong \mathrm{Ind}_{\mathfrak{S}_k \times \mathfrak{S}_{n-k}}^{\mathfrak{S}_n} \left[S^\mu \boxtimes M^b\right] \\ &\cong \mathrm{Ind}_{\mathfrak{S}_k \times \mathfrak{S}_{n-k}}^{\mathfrak{S}_n} \mathrm{Ind}_{\mathfrak{S}_k \times S_{b_1} \times S_{n-k-b_1}}^{\mathfrak{S}_k \times \mathfrak{S}_{n-k}} \left[S^\mu \boxtimes S^{(b_1)} \boxtimes M^c\right] \\ &\cong \mathrm{Ind}_{\mathfrak{S}_k \times S_{b_1} \times S_{n-k-b_1}}^{\mathfrak{S}_n} \left[S^\mu \boxtimes S^{(b_1)} \boxtimes M^c\right] \\ &\cong \mathrm{Ind}_{\mathfrak{S}_{k+b_1} \times \mathfrak{S}_{n-k-b_1}}^{\mathfrak{S}_n} \left\{\left[\mathrm{Ind}_{\mathfrak{S}_k \times \mathfrak{S}_{b_1}}^{\mathfrak{S}_{k+b_1}} \left(S^\mu \boxtimes S^{(b_1)}\right)\right] \boxtimes M^c\right\} \end{aligned} \tag{6.21}$$

and that the decomposition of $\mathrm{Ind}_{\mathfrak{S}_k \times \mathfrak{S}_{b_1}}^{\mathfrak{S}_{k+b_1}} \left(S^\mu \boxtimes S^{b_1}\right)$ is given by Pieri's rule (Corollary 6.2.18).

Just note that in the construction of the tree associated with $((a_1), a)$, two boxes raised from the same row and which become dotted cannot belong to the same column (see the example in Figure 6.25).

The *reduced* GZ-tree of $((a_1), a)$ is obtained from the Gelfand–Tsetlin tree by deleting all the nonspecial nodes: two special nodes labelled by $(\mu, (\mu, b))$ and $(\nu, (\nu, c))$ are connected by an oriented edge $(\mu, (\mu, b)) \to (\nu, (\nu, c))$ if and only if the conditions in Lemma 6.2.30 are satisfied and ν/μ is totally disconnected. This means that in the GZ-tree of $((a_1), a)$ there is an oriented path from $(\mu, (\mu, b))$ to $(\nu, (\nu, c))$ and all the intermediate vertices are nonspecial. If the edges of a given path are labelled, for instance, by

$$\xrightarrow{R_{m-1}} \xrightarrow{A_{m-2}} \xrightarrow{R_{m-2}} \xrightarrow{A_{m-3}}$$

then, the corresponding edge in the reduced tree is labelled by $A_{m-3}R_{m-2}\cdot$ $\cdot A_{m-2}R_{m-1}$.

Example 6.2.32 The tree in Figure 6.26 is the reduced GZ-tree for $((n-3), (n-3, 2, 1))$ (we suppose $n \geq 6$).

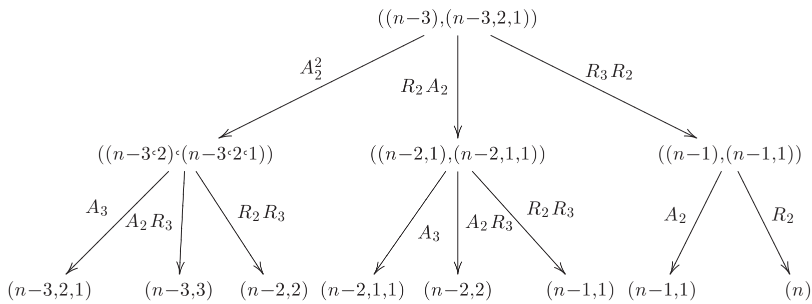


Figure 6.26

In particular, we obtain

$$M^{n-3,2,1} \cong S^{(n)} \oplus 2S^{n-1,1} \oplus 2S^{n-2,2} \oplus S^{n-3,3} \oplus S^{n-2,1,1} \oplus S^{n-3,,2,1}$$

together with an orthogonal decomposition of the isotypic components $2S^{n-1,1}$ and $2S^{n-2,2}$.

Theorem 6.2.33 *The decomposition of M^a obtained via the GZ-tree of $((a_1), a)$ coincided with the orthogonal decomposition obtained in Theorem 3.7.10 by means of the GZ-basis for the $\mathfrak{S}_a$-invariant vectors in M^a. Moreover, if $\mu \vdash n$, $\mu \trianglerighteq a$, $T \in \mathrm{STab}(\mu, a)$ and $\mu = \nu^{(h)} \Rightarrow \nu^{(h-1)} \Rightarrow \cdots \Rightarrow \nu^{(1)} = (a_1)$ is the path in the reduced Young poset $\mathbb{Y}_a$ associated with T, then the subspace S_T^μ in Theorem 3.7.10 coincides with the S^μ-subspace of M^a corresponding to the following path in the reduced GZ-tree of $((a_1), a)$:*

$$(\nu^{(1)}, a) \equiv ((a_1), a) \to (\nu^{(2)}, (\nu^{(2)}, b^{(2)})) \to \cdots$$
$$\cdots \to (\nu^{(h-1)}, (\nu^{(h-1)}, b^{(h-1)})) \to (\nu^{(h)}, \nu^{(h)}) \equiv \mu.$$

Proof Compare (6.21) with (3.78). □

Corollary 6.2.34 *With the same notation as Theorem 6.2.33, there exists a chain of subspaces*

$$V^{(1)} \equiv M^a \geq V^{(2)} \geq V^{(3)} \geq \cdots \geq V^{(h)} \equiv S_T^\mu$$

such that

$$
\begin{aligned}
V^{(j)} &\cong S^{\nu^{(j)},(\nu^{(j)},b^{(j)})} \\
&\cong \mathrm{Ind}_{\mathfrak{S}_{k_j}\times\mathfrak{S}_{n-k_j}}^{\mathfrak{S}_n}\left[S^{\nu^{(j)}} \boxtimes M^{b^{(j)}}\right],
\end{aligned}
$$

where $k_j = a_1 + a_2 + \cdots + a_j$, *for* $j = 1, 2, \ldots, h$.

Remark 6.2.35 Observe that one may construct trees for $((a_1), a)$ which are not of GZ-type. Therefore, the corresponding orthogonal decompositions of M^a into irreducible representations (that is, orthogonal decomposition of the isotypic components) given by Corollary 6.2.14 do not come from GZ-bases.

Exercise 6.2.36 Construct a non-GZ-decomposition of $M^{n-3,2,1}$.

7
Finite dimensional $*$-algebras

In the present chapter we give an exposition on finite dimensional semisimple algebras over $\mathbb{C}$ and their representation theory. We need the representation theory of finite dimensional algebras mainly to apply it to the commutant of a representation of a finite group. We adopt an anusual approach (inspired by Letac's course [82]): we work with $*$-closed subalgebras of $\mathrm{End}(V)$, where V is a finite dimensional Hermitian vector space, and we call them (finite dimensional) $*$-algebras. Our approach is concrete and concise; in particular, we do not need any particular knowledge of the theory of associative algebras, of ring theory nor of Wedderburn theory. In addition to the above mentioned notes by G. Letac, our treatment is inspired by the monographs by Shilov [110], by Goodman and Wallach [49], the lecture notes by A. Ram [106], the course by Clerc [21] and the book by Goodman, de la Harpe and Jones [48]. More algebraic expositions may be found in the books by Lang [76], Alperin and Bell [3] and Procesi [103]. An elementary book entirely devoted to finite dimensional algebras is Farenick's [36].

7.1 Finite dimensional algebras of operators

7.1.1 Finite dimensional $*$-algebras

Let V be a finite dimensional vector space over $\mathbb{C}$ endowed with a scalar product $\langle\cdot,\cdot\rangle$. We denote by $\mathrm{End}(V)$ the algebra of all linear operators $T : V \to V$.

A *subalgebra* of $\mathrm{End}(V)$ is a linear subspace $\mathcal{A} \leq \mathrm{End}(V)$ which is closed under multiplication of operators: $T, S \in \mathcal{A} \Rightarrow TS \in \mathcal{A}$. If $I_V \in \mathcal{A}$, where I_V is the identity of $\mathrm{End}(V)$, we say that $\mathcal{A}$ is unital. The algebra $\mathcal{A}$ is said to be commutative (or Abelian) if $AB = BA$ for all $A, B \in \mathcal{A}$. Finally, $\mathcal{A}$ is *self-adjoint* if $T \in \mathcal{A} \Rightarrow T^* \in \mathcal{A}$, where T^* is the adjoint of T.

Definition 7.1.1 A *(finite dimensional) $*$-algebra* of operators over V is a unital, self-adjoint subalgebra of $\mathrm{End}(V)$.

Remark 7.1.2 From an abstract point of view, a C*-algebra is an algebra $\mathcal{A}$ over the complex field $\mathbb{C}$ endowed with an involution (see Section 1.2.3) and a map $\|\cdot\| : \mathcal{A} \to [0, +\infty)$, called a *$*$-norm*, satisfying the following axioms:

- $\|A + B\| \leq \|A\| + \|B\|$;
- $\|\alpha A\| = |\alpha| \cdot \|A\|$;
- $\|A\| = 0$ if and only if $A = 0$;
- $\mathcal{A}$ is complete with respect to $\|\cdot\|$;
- $\|I_A\| = 1$ if A is unital;
- $\|AB\| \leq \|A\| \cdot \|B\|$;
- $\|A^*A\| = \|A\|^2$,

for all $A, B \in \mathcal{A}$ and $\alpha \in \mathbb{C}$. Note that from the last two axioms, one immediately deduces that $\|\cdot\|$ is a *$*$-norm*, that is, $\|A^*\| = \|A\|$ for all $A \in \mathcal{A}$. By a celebrated theorem of Gelfand, Naimark and Segal, any abstract C*-algebra $\mathcal{A}$ is $*$-isomorphic to a (closed) $*$-subalgebra of the algebra $\mathcal{B}(H)$ of all bounded linear operators on a Hilbert space H. Note that if H is finite dimensional, every linear operator is automatically bounded (i.e. continuous) and $\mathcal{B}(H) = \mathrm{End}(H)$. For more on this we refer to the monographs by Arveson [5], Connes [22], Conway [23] and Dixmier [29].

Let $\mathcal{A} \subseteq \mathrm{End}(V)$ be a subalgebra and let $W \leq V$ be a subspace.

A vector $w \in W$ is *cyclic* (for W) if $\mathcal{A}w := \{Tw : T \in \mathcal{A}\} = W$.

The subspace W is *$\mathcal{A}$-invariant* when $Tw \in W$ for all $T \in \mathcal{A}$ and $w \in W$. An $\mathcal{A}$-invariant subspace W is *trivial* when $W = V$ or $W = \{0\}$. Suppose that W is invariant. We say that $W \neq \{0\}$ is *$\mathcal{A}$-irreducible* if its invariant subspaces are $\{0\}$ and W itself only.

Lemma 7.1.3 (Complete reduction) *Let V be a finite dimensional unitary space and $\mathcal{A} \leq \mathrm{End}(V)$ a $*$-algebra of operators on V. Then there exists an orthogonal decomposition*

$$V = \bigoplus_{j \in J} W_j \tag{7.1}$$

where J is a (finite) set of indices such that every W_j is $\mathcal{A}$-invariant and irreducible.

Proof We proceed by induction on $n = \dim(V)$. If $\dim(V) = 1$ then V is irreducible and there is nothing to prove. Suppose that the statement holds for all unitary spaces of dimension less than or equal to $n - 1$. Suppose that

$\dim(V) = n$. If V is irreducible there is nothing to prove. Otherwise, there is a nontrivial $\mathcal{A}$-invariant subspace $W \leq V$. Let us prove that its orthogonal $W^\perp := \{v \in V : \langle v, w\rangle = 0, \forall w \in W\}$ is also $\mathcal{A}$-invariant. Indeed, if $T \in \mathcal{A}$ and $v \in W^\perp$ then, for any $w \in W$,

$$\langle Tv, w\rangle = \langle v, T^*w\rangle = 0$$

as $T^* \in \mathcal{A}$ and $T^*w \in \mathcal{A}W \subseteq W$. We thus have the orthogonal decomposition

$$V = W \oplus W^\perp$$

into $\mathcal{A}$-invariant subspaces. As $\dim(W), \dim(W^\perp) \leq n - 1$, by the inductive hypothesis we have

$$W = \bigoplus_{i \in I} W_i \quad \text{and} \quad W^\perp = \bigoplus_{i' \in I'} W_{i'}$$

with I and I' finite sets of indices and every W_i and $W_{i'}$ $\mathcal{A}$-invariant and irreducible. Setting $J = I \coprod I'$, we deduce (7.1). □

7.1.2 Burnside's theorem

In this section, we collect some general results on arbitrary nontrivial subalgebras of $\mathrm{End}(V)$.

In the following, once the algebra $\mathcal{A}$ is specified, we shall omit the prefix $\mathcal{A}$- in front of the terms invariant and irreducible.

Lemma 7.1.4 *Let $\mathcal{A}$ be a subalgebra of* $\mathrm{End}(V)$ *and $W \leq V$ be an invariant subspace. Suppose that $\mathcal{A}W \neq \{0\}$. Then W is irreducible if and only if every nontrivial vector in W is cyclic.*

Proof Suppose that W is irreducible. First of all, note that the set $\{v \in W : \mathcal{A}v = 0\}$ is $\mathcal{A}$-invariant. Since it cannot be all of W, we necessarily have

$$\{v \in W : \mathcal{A}v = 0\} = \{0\}. \tag{7.2}$$

Let $w \in W$ be a nontrivial vector. The subspace $\mathcal{A}w$ is clearly invariant and contained in W by the invariance of W. As $0 \neq w$, we have $\mathcal{A}w \neq \{0\}$ by (7.2) and therefore, by irreducibility, $\mathcal{A}w = W$. This shows that w is cyclic.

Conversely, suppose that every nontrivial vector in W is cyclic. Consider a nontrivial invariant subspace U in W. Pick a nontrivial vector $u \in U$. Then $W = \mathcal{A}u \subseteq U$ (the equality follows from cyclicity of u, and the inclusion from invariance of U) and therefore $U = W$. This shows that W is irreducible. □

Note that if $\mathcal{A}W = \{0\}$, then W is irreducible if and only if $\dim W = 1$. Moreover, if this is the case, W does not contain cyclic vectors.

Let $v \in V$ and $f \in V'$ (the dual space of V). We define the operator $T_{f,v} \in \mathrm{End}(V)$ by setting $T_{f,v}(w) = f(w)v$ (compare with Section 1.2.5). Such an operator is called *atomic*. Denoting by $\mathrm{rk}(T) = \dim T(V)$ the *rank* of an operator $T \in \mathrm{End}(V)$, we clearly have $\mathrm{rk}(T_{f,v}) = 1$, provided that $v \neq 0$ and $f \neq 0$.

Exercise 7.1.5 Let $T \in \mathrm{End}(V)$.

(1) Show that $\mathrm{rk}(T) = 1$ if and only if T is atomic and non-trivial.
(2) Show that the atomic operators span $\mathrm{End}(V)$.

Let V be a vector space and denote, as usual, by V' its dual space. We say that a subset $F \subset V'$ *separates* the points in V if for all $v \in V \setminus \{0\}$ there exists $f \in F$ such that $f(v) \neq 0$.

Lemma 7.1.6 *Suppose that W' is a subspace of V' that separates the points in V. Then $W' = V'$.*

Proof Suppose, by contradiction, that $W' \subsetneq V'$, and let $f' \in V' \setminus W'$. Let $\phi \in V''$ (V'' is the bidual of V) be such that $\phi(f') = 1$ and $\phi(f) = 0$ for all $f \in W'$. As V'' is canonically isomorphic to V, this amounts to say that there exists $v \in V$ such that $f'(v) = 1$ (so that $v \neq 0$) and $f(v) = 0$ for all $f \in W'$, contradicting the hypothesis on W'. □

There is a fundamental theorem of Burnside that characterizes irreducible representations of nontrivial subalgebras of $\mathrm{End}(V)$. We present the short proof of I. Halperin and P. Rosenthal in [55].

Theorem 7.1.7 (Burnside) *Let $\mathcal{A}$ be a subalgebra of* $\mathrm{End}(V)$*, $\mathcal{A} \neq \{0\}$. Then the space V is irreducible for $\mathcal{A}$ if and only if $\mathcal{A} =$* $\mathrm{End}(V)$*.*

Proof First of all, we prove that V is irreducible for $\mathrm{End}(V)$. Fix $u, v \in V$, with $v \neq 0$. Let $f \in V'$ be such that $f(v) = 1$. Then the atomic operator $T = T_{f,u}$ satisfies $Tv = u$. This shows that $\mathrm{End}(V)v = V$, that is, v is cyclic. From Lemma 7.1.4 we deduce that V is irreducible.

For the converse, we reproduce the proof in [55]. Let $\mathcal{A}$ be a subalgebra of $\mathrm{End}(V)$ and suppose that V is $\mathcal{A}$-irreducible.

First step We show that $\mathcal{A}$ contains an operator T_0 of rank $\mathrm{rk}(T_0) = 1$. Set $d = \min\{\mathrm{rk}(T) : 0 \neq T \in \mathcal{A}\}$. Suppose, by contradiction, that $d > 1$. Take $T_0 \in \mathcal{A}$ such that $\mathrm{rk}(T_0) = d$ and $v_1, v_2 \in V$ such that $T_0 v_1$ and $T_0 v_2$ are linearly independent. Since V is irreducible for $\mathcal{A}$ and $T_0 v_1$ is nontrivial, by Lemma 7.1.4 there exists $A \in \mathcal{A}$ such that $AT_0 v_1 = v_2$. It follows that the vectors $T_0 A T_0 v_1$ and $T_0 v_1$ are linearly independent, that is, the operator $T_0 A T_0 - \lambda T_0 \in \mathcal{A}$ is nontrivial for every $\lambda \in \mathbb{C}$. Set $W = T_0(V)$. Then there exists $\lambda_0 \in \mathbb{C}$ such that the operator $T_0 A - \lambda_0 I_W : W \to W$ has a nontrivial kernel (because $\mathbb{C}$ is algebraically closed) and therefore $W' := (T_0 A T_0 - \lambda_0 T_0)(V) = (T_0 A - \lambda_0 I_W)(W) \subsetneq W$. But then $\text{rk}(T_0 A T_0 - \lambda_0 T_0) = \dim(W') < \dim(W) = \text{rk}(T_0) = d$, contradicting the minimality of d. Therefore d must be equal to 1.

Second step We show that $\mathcal{A}$ contains all atomic operators in $\text{End}(V)$. From the first step we know that there exist $v_0 \in V$ and $f_0 \in V'$, both nontrivial, such that $T_{f_0, v_0} \in \mathcal{A}$. For any $A \in \mathcal{A}$, define $f_A \in V'$ by setting $f_A(v) = f_0(Av)$, for all $v \in V$. Set $W' = \{f \in V' : T_{f, v_0} \in \mathcal{A}\}$. Since $T_{f_0, v_0} A = T_{f_A, v_0}$, we have that $W' \supset \{f_A : A \in \mathcal{A}\}$. Let us show that $W' = V'$. By Lemma 7.1.6, it suffices to show that for all $v \in V \setminus \{0\}$ there exists $f \in W'$ such that $f(v) \neq 0$. Let $u \in V$ be such that $f_0(u) \neq 0$. By Lemma 7.1.4, there exists $A \in \mathcal{A}$ such that $Av = u$. But then for $f = f_A \in W'$ we have

$$f(v) = f_A(v) = f_0(Av) = f_0(u) \neq 0.$$

Now, $W' = V'$ means that $T_{f, v_0} \in \mathcal{A}$ for all $f \in V'$. On the other hand, $AT_{f, v_0} = T_{f, Av_0}$ and since $\{Av_0 : A \in \mathcal{A}\} = V$, this implies that $T_{f,v} \in \mathcal{A}$ for all $v \in V$ and $f \in V'$. Since the atomic operators span $\text{End}(V)$ (cf. Exercise 7.1.5) this shows that $\mathcal{A} = \text{End}(V)$. □

7.2 Schur's lemma and the commutant

7.2.1 Schur's lemma for a linear algebra

Let V and U be two complex vector spaces. Let $\mathcal{A}$ be a subalgebra of $\text{End}(V)$. A *representation* of $\mathcal{A}$ on U is a linear map

$$\rho : \mathcal{A} \to \text{End}(U)$$

such that $\rho(TS) = \rho(T)\rho(S)$, for all $S, T \in \mathcal{A}$. In other words, ρ is an algebra homomorphism. If $\mathcal{A}$ is unital, we require that $\rho(I_V) = I_U$.

To emphasize the role of the space U, a representation $\rho : \mathcal{A} \to \text{End}(U)$ will be also denoted by the pair (ρ, U) or simply by U.

The representation ρ is *trivial* if $\rho(T) = 0$ for all $T \in \mathcal{A}$. Note that a representation of a unital algebra cannot be trivial as $\rho(I_V) = I_U$. The *kernel* of ρ is the subspace $\text{Ker}\rho = \{T \in \mathcal{A} : \rho(T) = 0\}$. Clearly, $\text{Ker}\rho$ is a subalgebra (indeed a two-sided ideal) of $\mathcal{A}$. We say that ρ is *faithful* if $\text{Ker}\rho = \{0\}$. A subspace $W \leq U$ is $\mathcal{A}$-*invariant* if $\rho(T)w \in W$ for all $T \in \mathcal{A}$ and $w \in W$. A nontrivial representation (ρ, U) is *irreducible* if $\{0\}$ and U are the only $\mathcal{A}$-invariant subspaces of U.

Note also that a subspace $W \leq U$ is $\mathcal{A}$-invariant if and only if it is $\rho(\mathcal{A})$-invariant, where $\rho(\mathcal{A}) := \{\rho(T) : T \in \mathcal{A}\}$. Therefore, the following proposition is an immediate consequence of Burnside's theorem (Theorem 7.1.7).

Proposition 7.2.1 *A nontrivial representation (ρ, U) of $\mathcal{A}$ is irreducible if and only if $\rho(\mathcal{A}) = \mathrm{End}(U)$.*

Corollary 7.2.2 *Let $\mathcal{A}$ be a subalgebra of* $\mathrm{End}(V)$. *If $\mathcal{A}$ is commutative, then every irreducible representation of $\mathcal{A}$ is one-dimensional.*

Proof If (ρ, U) is $\mathcal{A}$-irreducible, then $\rho(\mathcal{A}) = \mathrm{End}(U)$ is commutative, and this forces $\dim(U) = 1$. □

Let (ρ, U) and (σ, W) be two representations of the algebra $\mathcal{A}$. An *intertwining operator* is a linear map $T : U \to W$ such that

$$T\rho(A) = \sigma(A)T \tag{7.3}$$

for all $A \in \mathcal{A}$. We denote by $\mathrm{Hom}_{\mathcal{A}}(\rho, \sigma)$, or $\mathrm{Hom}_{\mathcal{A}}(U, W)$, the space of all operators that intertwine ρ and σ. When $U = W$ then $\mathrm{Hom}_{\mathcal{A}}(U, U)$, also denoted $\mathrm{End}_{\mathcal{A}}(U)$, is a subalgebra of $\mathrm{End}(U)$. An *isomorphism* between (ρ, U) and (σ, W) is an intertwining operator T that is also a bijection between U and W. When such isomorphism exists, we say that ρ and σ are *equivalent* and we write $\rho \sim \sigma$ (or $U \sim W$) to indicate that they are equivalent.

The following is the analogue of the classical Schur lemma (cf. Lemma 1.2.1) in the context of the representation theory of finite dimensional algebras. Its proof, as well as the proof of several other analogues, can be deduced, *mutatis mutandis* from the corresponding arguments in Chapter 1. We present the proof as a sort of paradigm, while for the other analogues, we leave it as an exercise.

Theorem 7.2.3 (Schur's lemma) *Let $\mathcal{A}$ be a subalgebra of* $\mathrm{End}(V)$ *and let (ρ, U) and (σ, W) be two irreducible representations of $\mathcal{A}$.*

(i) *If $T \in \mathrm{Hom}_{\mathcal{A}}(\rho, \sigma)$ then either $T = 0$ or T is an isomorphism.*
(ii) $\mathrm{Hom}_{\mathcal{A}}(\rho, \rho) = \mathbb{C}I_U$.

Proof (i) If $T \in \mathrm{Hom}_{\mathcal{A}}(\rho, \sigma)$, then $\mathrm{Ker}T \leq U$ and $\mathrm{Im}T \leq W$ are $\mathcal{A}$-invariant. Therefore, by irreducibility, either $\mathrm{Ker}T = U$, so that $T \equiv 0$, or $\mathrm{Ker}T = \{0\}$, and necessarily $\mathrm{Im}T = W$, so that T is an isomorphism.

(ii) Let $T \in \mathrm{Hom}_{\mathcal{A}}(\rho, \rho)$. Since $\mathbb{C}$ is algebraically closed, T has at least one eigenvalue: there exists $\lambda \in \mathbb{C}$ such that $\mathrm{Ker}(T - \lambda I_U)$ is nontrivial. But $T - \lambda I_U \in \mathrm{Hom}_{\mathcal{A}}(\rho, \rho)$ so that by part (i), necessarily $T - \lambda I_U \equiv 0$, in other words $T = \lambda I_U$. □

We now suppose that $\mathcal{A} \le \mathrm{End}(V)$ is a $*$-algebra. A $*$-*representation* of $\mathcal{A}$ on a unitary space W is a representation (σ, W) that satisfies the following additional condition:

$$\sigma(A^*) = \sigma(A)^*,$$

for all $A \in \mathcal{A}$.

Proposition 7.2.4 *Let (ρ, U) and (σ, W) be two irreducible $*$-representations of $\mathcal{A}$ and let $T \in \mathrm{Hom}_{\mathcal{A}}(\rho, \sigma)$ be an isomorphism. Then, there exists $\alpha \in \mathbb{C}$ such that αT is an isometry.*

Proof For all $A \in \mathcal{A}$ we have, in virtue of (7.3),

$$\rho(A)T^* = \rho(A^*)^*T^* = (T\rho(A^*))^* = (\sigma(A^*)T)^* = T^*\sigma(A^*)^* = T^*\sigma(A),$$

that is, $T^* \in \mathrm{Hom}_{\mathcal{A}}(\sigma, \rho)$, and it is an isomorphism as well. It then follows that $T^*T \in \mathrm{Hom}_{\mathcal{A}}(\rho, \rho)$ and, being nontrivial, there exists $\lambda \neq 0$, necessarily $\lambda > 0$, as T^*T is a positive operator, such that $T^*T = \lambda I_U$. Setting $\alpha = 1/\sqrt{\lambda}$, we have that αT is an isometry. □

7.2.2 The commutant of a $*$-algebra

Let $\mathcal{A} \le \mathrm{End}(V)$ be a finite dimensional $*$-algebra. The *commutant* of $\mathcal{A}$ is the $*$-algebra $\mathcal{A}' := \mathrm{End}_{\mathcal{A}}(V)$, that is,

$$\mathcal{A}' = \{S \in \mathrm{End}(V) : ST = TS \text{ for all } T \in \mathcal{A}\}.$$

We now use Schur's lemma (Theorem 7.2.3) in order to determine the structure of $\mathcal{A}'$. The decomposition (7.1) can be written, grouping together the equivalent representations, in the form

$$V = \bigoplus_{\lambda \in \Lambda} m_\lambda W_\lambda,$$

where Λ is a set of pairwise inequivalent representations of $\mathcal{A}$ and the positive integer m_λ is the multiplicity of W_λ in V. More precisely, there exist injective operators $I_{\lambda,1}, I_{\lambda,2}, \dots, I_{\lambda,m_\lambda} \in \mathrm{Hom}_{\mathcal{A}}(W_\lambda, V)$, necessarily linearly independent, such that

$$m_\lambda W_\lambda = \bigoplus_{j=1}^{m_\lambda} I_{\lambda,j} W_\lambda \tag{7.4}$$

and the decomposition is orthogonal. By Proposition 7.2.4, we may also suppose that $I_{\lambda,j} : W_\lambda \to I_{\lambda,j} W_\lambda$ is an isometry.

Denote by $E_{\lambda,j}$ the orthogonal projection $E_{\lambda,j} : V \to I_{\lambda,j} W_\lambda$. Clearly, $\sum_{\lambda\in\Lambda} \sum_{j=1}^{m_\lambda} E_{\lambda,j} = I_V$. Moreover, $E_{\lambda,j} \in \mathcal{A}'$ for all $\lambda \in \Lambda$ and $j = 1, 2, \ldots, m_\lambda$. Indeed, if $v \in V$ and $A \in \mathcal{A}$, then

$$Av = A \sum_{\lambda\in\Lambda} \sum_{j=1}^{m_\lambda} E_{\lambda,j} v = \sum_{\lambda\in\Lambda} \sum_{j=1}^{m_\lambda} A E_{\lambda,j} v$$

and, since each $I_{\lambda,j} W_\lambda$ is $\mathcal{A}$-invariant, we have that $AE_{\lambda,j} v \in I_{\lambda,j} W$. This implies that $AE_{\lambda,j} v = E_{\lambda,j} Av$ for all $v \in V$, and therefore $AE_{\lambda,j} = E_{\lambda,j} A$, showing that $E_{\lambda,j} \in \mathcal{A}'$.

Proposition 7.2.5 *The set* $\{I_{\lambda,1}, I_{\lambda,2}, \ldots, I_{\lambda,m_\lambda}\}$ *is a basis for the space* $\mathrm{Hom}_{\mathcal{A}}(W_\lambda, V)$*. In particular,* $m_\lambda = \mathrm{dimHom}_{\mathcal{A}}(W_\lambda, V)$ *and it does not depend on the chosen decomposition.*

Proof Exercise (see the proof of Lemma 1.2.5). □

Corollary 7.2.6 *If* $V = \bigoplus_{\lambda\in\Lambda'} m'_\lambda U_\lambda$ *is another decomposition of* U *into* $\mathcal{A}$*-irreducible inequivalent representations, then necessarily* $\Lambda' = \Lambda$ *and we have* $m'_\lambda = m_\lambda$ *and* $m'_\lambda U_\lambda = m_\lambda W_\lambda$ *for all* $\lambda \in \Lambda$.

Proof Exercise (see the proof of Corollary 1.2.6). □

Definition 7.2.7 The positive integer m_λ is called the *multiplicity* of λ for $\mathcal{A}$ (or of W_λ in V). Moreover, the invariant subspace $m_\lambda W_\lambda$ is the λ-*isotypic component* of λ for $\mathcal{A}$. Finally, $V = \bigoplus_{\lambda\in\Lambda} m_\lambda W_\lambda$ is called the $\mathcal{A}$-*isotypic decomposition* of V.

Let $V = \bigoplus_{\lambda\in J} m_\lambda W_\lambda$ be the $\mathcal{A}$-isotypic decomposition of V.

For all $\lambda \in \Lambda$ and $1 \le i, j \le m_\lambda$ there exist non-trivial operators $T^\lambda_{i,j} \in \mathrm{Hom}_{\mathcal{A}}(V, V)$ such that

$$\mathrm{Im} T^\lambda_{i,j} = I_{\lambda,i} W_\lambda$$
$$\mathrm{Ker} T^\lambda_{i,j} = V \ominus I_{\lambda,j} W_\lambda$$

and

$$T^\lambda_{i,j} T^\theta_{s,t} = \delta_{\lambda,\theta} \delta_{j,s} T^\lambda_{i,t}. \tag{7.5}$$

We may construct the operators $T^\lambda_{k,j}$ in the following way. Denote by $I^{-1}_{\lambda,j}$ the inverse of $I_{\lambda,j} : W_\lambda \to I_{\lambda,j} W_\lambda \le V$. Then we may set

$$T^\lambda_{k,j} v = \begin{cases} I_{\lambda,k} I^{-1}_{\lambda,j} v & \text{if } v \in I_{\lambda,j} W_\lambda \\ 0 & \text{if } v \in V \ominus I_{\lambda,j} W_\lambda. \end{cases}$$

In particular,

$$T^{\lambda}_{j,j} \equiv E_{\lambda,j}.$$

Moreover, by Schur's lemma (Theorem 7.2.3) we have

$$\mathrm{Hom}_{\mathcal{A}}(I_{\lambda,j}W_{\lambda}, I_{\lambda,i}W_{\lambda}) = \mathbb{C}T^{\lambda}_{i,j}.$$

We shall use these operators to study the structure of the commutant $\mathcal{A}'$ of the finite dimensional $*$-algebra $\mathcal{A} \leq \mathrm{End}(V)$.

Theorem 7.2.8 *The set*

$$\{T^{\lambda}_{i,j} : \lambda \in \Lambda, 1 \leq i, j \leq m_{\lambda}\} \tag{7.6}$$

is a vector space basis for $\mathcal{A}'$. Moreover, the map

$$\mathcal{A}' \to \bigoplus_{\lambda\in\Lambda} M_{m_\lambda,m_\lambda}(\mathbb{C})$$

$$T \mapsto \bigoplus_{\lambda\in\Lambda} \left(\alpha^{\lambda}_{i,j}\right)_{i,j=1}^{m_\lambda}$$

where the $\alpha^{\lambda}_{i,j}$'s are the coefficients of T with respect to the basis (7.6):

$$T = \sum_{\lambda\in\Lambda}\sum_{i,j=1}^{m_\lambda} \alpha^{\lambda}_{i,j}T^{\lambda}_{i,j},$$

is a $*$-isomorphism (see Section 1.2.3).

Proof Exercise (see the proof of Theorem 1.2.14). □

Consider an algebra of the form $\mathcal{A} = \oplus_{\lambda\in\Lambda}\mathrm{End}(V_\lambda)$, where V_λ is a finite dimensional complex vector space for $\lambda \in \Lambda$ and Λ is a finite set of indices. Each $A \in \mathcal{A}$ may be written in the form $A = \oplus_{\lambda\in\Lambda}A_\lambda$, with $A_\lambda \in \mathrm{End}(V_\lambda)$. For $\rho \in \Lambda$ define the $\mathcal{A}$-representation (σ_ρ, V_ρ) by setting

$$\sigma_\rho(A) = A_\rho,$$

for all $A = \oplus_{\lambda\in\Lambda}A_\lambda \in \mathcal{A}$. Also, for $\rho \in \Lambda$ define $I_\rho \in \mathcal{A}$ by setting

$$I_\rho v = \begin{cases} v & \text{if } v \in V_\rho \\ 0 & \text{if } v \in V_\lambda \text{ with } \lambda \neq \rho. \end{cases}$$

Lemma 7.2.9 *The representations σ_ρ, $\rho \in \Lambda$, are irreducible and pairwise inequivalent.*

Proof Irreducibility follows from Proposition 7.2.1. Let $T \in \mathrm{Hom}_{\mathcal{A}}(V_\lambda, V_\rho)$ with $\lambda \neq \rho$. Then, for all $v \in V_\lambda$ we have

$$Tv = T\sigma_\lambda(I_\lambda)v = \sigma_\rho(I_\lambda)Tv = 0$$

since $Tv \in V_\rho$. It follows that $T \equiv 0$ and $V_\lambda \not\sim V_\rho$. □

In other words, the theorem says that the representations of $\mathcal{A}$ on V_λ and V_ρ cannot be equivalent because the two blocks of the algebra $\mathcal{A}$ that act on these spaces are different. In Theorem 7.2.8 we showed that the commutant of $\mathcal{A}$ is isomorphic to the direct sum of full matrix algebras. Thus Lemma 7.2.9 gives $|\Lambda|$ inequivalent representations of $\mathcal{A}'$. In the following sections we shall prove, as a consequence of the double commutant theorem (Theorem 7.3.2), that indeed every finite dimensional $*$-algebra $\mathcal{A}$ is isomorphic to the direct sum of full matrix algebras and in Section 7.4 we shall prove that Lemma 7.2.9 provides all the irreducible representations of $\mathcal{A}$.

7.3 The double commutant theorem and the structure of a finite dimensional $*$-algebra

In this section we shall prove the fundamental theorem on the structure of a finite dimensional $*$-algebra $\mathcal{A} \leq \mathrm{End}(V)$, namely, that $\mathcal{A}$ is isomorphic to a direct sum of full matrix algebras. Also, we shall study the actions of both $\mathcal{A}$ and $\mathcal{A}'$ on V. We first need two basic tools: the tensor product of algebras and of their representations, and the double commutant theorem.

7.3.1 Tensor product of algebras

Let V and W be two finite dimensional unitary spaces. There is a natural isomorphism

$$\Phi : \mathrm{End}(V) \otimes \mathrm{End}(W) \to \mathrm{End}(V \otimes W)$$

where

$$\Phi(T \otimes S)(v \otimes w) = Tv \otimes Sw,$$

for all $T \in \mathrm{End}(V)$, $S \in \mathrm{End}(W)$, $v \in V$ and $w \in W$, and then Φ is extended by linearity.

It is easy to check that Φ is injective and since $\dim[\mathrm{End}(V)\otimes\mathrm{End}(W)] = (\dim V)^2(\dim W)^2 = \dim[\mathrm{End}(V\otimes W)]$, is also surjective. Alternatively, one can check that a basis of $\mathrm{End}(V)\otimes\mathrm{End}(W)$ is mapped by Φ onto a basis of $\mathrm{End}(V\otimes W)$. In what follows, Φ will be usually omitted and $\mathrm{End}(V)\otimes\mathrm{End}(W)$ identified with $\mathrm{End}(V\otimes W)$.

Now, if $\mathcal{A}\leq\mathrm{End}(V)$ and $\mathcal{B}\leq\mathrm{End}(W)$ are two subalgebras, their tensor product is defined as the subalgebra $\Phi(\mathcal{A}\otimes\mathcal{B})\leq\mathrm{End}(V\otimes W)$ and we simply denote it by $\mathcal{A}\otimes\mathcal{B}$. Note that if $A_1, A_2\in\mathcal{A}$ and $B_1, B_2\in\mathcal{B}$, then we have

$$(A_1\otimes B_1)\cdot(A_2\otimes B_2) = A_1A_2\otimes B_1B_2.$$

If (σ, U) is a representation of $\mathcal{A}$ and (ρ, Y) is a representation of $\mathcal{B}$, then their *tensor product* is just the representation $\sigma\boxtimes\rho$ of $\mathcal{A}\otimes\mathcal{B}$ on $U\otimes Y$ given by setting

$$(\sigma\boxtimes\rho)(A\otimes B) = \sigma(A)\otimes\rho(B),$$

for all $A\in\mathcal{A}$, $B\in\mathcal{B}$.

The proof of the following proposition is adapted from [106].

Proposition 7.3.1 *Let $\mathcal{A}$ and $\mathcal{B}$ be two algebras. If (σ, U) is an irreducible representation of $\mathcal{A}$ and (ρ, Y) is an irreducible representation of $\mathcal{B}$ then their tensor product $(\sigma\boxtimes\rho, U\otimes Y)$ is an irreducible representation of $\mathcal{A}\otimes\mathcal{B}$. Moreover, every irreducible representation of $\mathcal{A}\otimes\mathcal{B}$ is of this form.*

Proof Observe that

$$\begin{aligned}
[(\sigma\boxtimes\rho)(\mathcal{A}\otimes\mathcal{B})] &= \langle\sigma(A)\otimes\rho(B) : A\in\mathcal{A}, B\in\mathcal{B}\rangle\\
&= \sigma(\mathcal{A})\otimes\rho(\mathcal{B})\\
\text{(by Proposition 7.2.1)}\quad &= \mathrm{End}(U)\otimes\mathrm{End}(Y)\\
&= \mathrm{End}(U\otimes Y)
\end{aligned}$$

and therefore, again by Proposition 7.2.1, $(\sigma\otimes\rho, U\otimes Y)$ is irreducible.

Conversely, suppose that $\mathcal{A}\leq\mathrm{End}(V)$ and $\mathcal{B}\leq\mathrm{End}(W)$ and let (η, Z) be an irreducible representation of $\mathcal{A}\otimes\mathcal{B}$. Consider the $\mathcal{A}$-representation (σ, Z) defined by $\sigma(A) = \eta(A\otimes I_W)$ for all $A\in\mathcal{A}$. Let $X\leq Z$ be a $\mathcal{A}$-invariant and $\mathcal{A}$-irreducible subspace. Consider the $\mathcal{B}$-representation $(\rho, \mathrm{Hom}_{\mathcal{A}}(X, Z))$ defined as follows

$$\rho(B)\cdot T = \eta(I_V\otimes B)T$$

for all $B \in \mathcal{B}$ and $T \in \mathrm{Hom}_{\mathcal{A}}(X, Z)$. It is well defined: if $A \in \mathcal{A}$ we have

$$\begin{aligned}\sigma(A)[\rho(B)T] &\equiv \eta(A \otimes I_W)\eta(I_V \otimes B)T \\ &= \eta(A \otimes B)T \\ &= \eta(I_V \otimes B)\eta(A \otimes I_W)T \\ &= \eta(I_V \otimes B)T\eta(A \otimes I_W) \\ &\equiv [\rho(B)T]\sigma(A),\end{aligned}$$

and therefore $\rho(B)T \in \mathrm{Hom}_{\mathcal{A}}(X, Z)$. Consider a nontrivial $\mathcal{B}$-invariant and $\mathcal{B}$-irreducible subspace J in $\mathrm{Hom}_{\mathcal{A}}(X, Z)$. Then the map

$$\begin{aligned}\Psi : X \otimes J &\to Z \\ v \otimes T &\mapsto Tv\end{aligned}$$

is a linear isomorphism of $\mathcal{A} \otimes \mathcal{B}$-representations. Indeed, $\Psi \in \mathrm{Hom}_{\mathcal{A}\otimes\mathcal{B}}(X \otimes J, Z)$:

$$\begin{aligned}[\eta(A \otimes B) \circ \Psi](v \otimes T) &= \eta(A \otimes B)(Tv) \\ &= \eta(I_V \otimes B)\eta(A \otimes I_W)Tv \\ &= \eta(I_V \otimes B)T\eta(A \otimes I_W)v \\ &= \rho(B)T\sigma(A)v \\ &= \Psi(\sigma(A)v \otimes \rho(B)T) \\ &= [\Psi \circ (\sigma(A) \otimes \rho(B))](v \otimes T),\end{aligned}$$

for all $A \in \mathcal{A}$, $B \in \mathcal{B}$, $v \in X$ and $T \in \mathrm{Hom}_{\mathcal{A}}(X, Z)$. Moreover, Ψ is nontrivial because if $T \in J$, $T \neq 0$, then by Schur's lemma and $\mathcal{A}$-irreducibility of X, we have that $T(X) \cong X$ is nontrivial. Finally, by another application of Schur's lemma and $\mathcal{A} \otimes \mathcal{B}$-irreducibility of both $X \otimes J$ (by the first part of the proof) and Z, Ψ is an isomorphism. □

7.3.2 The double commutant theorem

In this section, we prove the finite dimensional version of the celebrated von Neumann double commutant theorem.

Theorem 7.3.2 (Double commutant theorem) *Let $\mathcal{A} \leq \mathrm{End}(V)$ be a $*$-algebra of operators on a finite dimensional unitary space V. Then*

$$\mathcal{A}'' := (\mathcal{A}')' = \mathcal{A}$$

that is, the commutant of $\mathcal{A}'$ coincides with $\mathcal{A}$.

Proof The inclusion $\mathcal{A} \subseteq \mathcal{A}''$ is trivial: if $T \in \mathcal{A}$ and $S \in \mathcal{A}'$, then T commutes with S and therefore $T \in \mathcal{A}''$.

Let us prove the reverse inclusion, namely $\mathcal{A}'' \subseteq \mathcal{A}$.

Claim 1 If $T \in \mathcal{A}''$ and $v \in V$, then there exists $S \in \mathcal{A}$ such that $Sv = Tv$.

Proof of the claim Indeed, $\mathcal{A}v$ is an $\mathcal{A}$-invariant subspace and therefore the orthogonal projection $E : V \to \mathcal{A}v$ belongs to $\mathcal{A}'$ (also the orthogonal $(\mathcal{A}v)^\perp$ is $\mathcal{A}$-invariant; see the proof of Lemma 7.1.3). We conclude by observing that

$$Tv = TI_V v = TEI_V v = TEv = ETv \in \mathrm{Im}E = \mathcal{A}v. \qquad \square$$

We now use a clever trick of von Neumann to deduce the theorem from the claim. Set $n = \dim V$. Let $V^{\oplus^n}$ denote the direct sum of n copies of V. We identify $V^{\oplus^n}$ with the set of all n-tuples $v = (v_1, v_2, \ldots, v_n)$ with $v_i \in V$ and $\mathrm{End}(V^{\oplus^n})$ with the set of all $n \times n$ matrices $T = (T_{i,j})_{i,j=1}^n$ with $T_{i,j} \in \mathrm{End}(V)$. Clearly if $T \in \mathrm{End}(V^{\oplus^n})$ and $(v_1, v_2, \ldots, v_n) \in V^{\oplus^n}$, we have

$$T(v_1, v_2, \ldots, v_n) = \Big(\sum_{j_1=1}^n T_{1,j_1} v_{j_1}, \sum_{j_2=1}^n T_{2,j_2} v_{j_2}, \ldots, \sum_{j_n=1}^n T_{n,j_n} v_{j_n}\Big).$$

Let $\mathcal{B}$ be the $*$-algebra on $V^{\oplus^n}$ consisting of all operators of the form

$$S = (\delta_{i,j} A)_{i,j=1}^n$$

with $A \in \mathcal{A}$, so that $S(v_1, v_2, \ldots, v_n) = (Av_1, Av_2, \ldots, Av_n)$ for all $(v_1, v_2, \ldots, v_n) \in V^{\oplus^n}$. Then, the commutant $\mathcal{B}'$ of $\mathcal{B}$ is formed by all operators $T = (T_{i,j})_{i,j=1}^n$ such that $T_{i,j} \in \mathcal{A}'$.

Claim 2 The double commutant $\mathcal{B}''$ consists of all operators of the form $R = (\delta_{i,j} B)_{i,j=1}^n$ with $B \in \mathcal{A}''$.

Proof of the claim The operators of the form $E_{(h,k)} = (I_V \delta_{i,h} \delta_{j,k})_{i,j=1}^n$ belong to $\mathcal{B}'$. An operator $R = (R_{i,j})_{i,j=1}^n$ commutes with all the $E_{(h,k)}$, $h, k = 1, 2, \ldots, n$, if and only if it is of the form $R = (B\delta_{i,j})_{i,j=1}^n$ with $B \in \mathrm{End}(V)$. Moreover, R commutes with the operators $(A\delta_{i,j})_{i,j=1}^n$, $A \in \mathcal{A}'$, (which belong to $\mathcal{B}'$) if and only if $B \in \mathcal{A}''$. $\square$

We now apply the first claim to the algebras $\mathcal{B}$ and $\mathcal{B}''$. Let $v_1, v_2, \ldots, v_n$ constitute a basis for V and consider the vector $v = (v_1, v_2, \ldots, v_n) \in V^{\oplus^n}$. If $A_1 \in \mathcal{A}''$ then $T = (A_1 \delta_{i,j})_{i,j=1}^n \in \mathcal{B}''$ and therefore there exists $S = (A_2 \delta_{i,j})_{i,j=1}^n \in \mathcal{B}$ (so that $A_2 \in \mathcal{A}$) such that $Sv = Tv$. But this means that $A_2 v_j = A_1 v_j$ for all $j = 1, 2, \ldots, n$, that is, $A_1 = A_2$ (as $\{v_1, v_2, \ldots, v_n\}$ is a basis). We have shown that if $A \in \mathcal{A}''$ then $A \in \mathcal{A}$, that is, $\mathcal{A}'' = \mathcal{A}$. □

7.3.3 Structure of finite dimensional ∗-algebras

Suppose now that $\mathcal{A}$ is a $*$-algebra of operators on a finite dimensional V and that

$$V = \bigoplus_{\lambda \in \Lambda} m_\lambda W_\lambda$$

is the $\mathcal{A}$-isotypic decomposition of V. Since $\mathcal{A} = \mathcal{A}''$, by Theorem 7.2.8, $\mathcal{A}$ is $*$-isomorphic to a direct sum of full matrix algebras. We now give a precise formulation of this fact. Consider the representatation (η, V) of $\mathcal{A}' \otimes \mathcal{A}$ defined by

$$\eta(B \otimes A)v = BAv (= ABv)$$

for all $B \in \mathcal{A}'$, $A \in \mathcal{A}$ and $v \in V$. Let $\lambda \in \Lambda$. Set $d_\lambda = \dim W_\lambda$ and $Z_\lambda = \mathrm{Hom}_{\mathcal{A}}(W_\lambda, V)$ (so that $m_\lambda = \dim Z_\lambda$, by Proposition 7.2.5). The algebra $\mathcal{A}'$ acts on Z_λ in the obvious way: if $T \in Z_\lambda$, $B \in \mathcal{A}'$ then $BT \in Z_\lambda$; indeed for $w \in W_\lambda$ and $A \in \mathcal{A}$ we have

$$(BT)\lambda(A)w = BATw = A(BT)w. \tag{7.7}$$

We now show that the isotypic component $m_\lambda W_\lambda$ is $(\mathcal{A}' \otimes \mathcal{A})$-invariant. For, recalling (7.4) so that the generic element in $m_\lambda W_\lambda$ is of the form Tw with $T \in \mathrm{Hom}_{\mathcal{A}}(W_\lambda, V)$ and $w \in W_\lambda$, we have, for all $B \in \mathcal{A}'$ and $A \in \mathcal{A}$,

$$\eta(B \otimes A)(Tw) = BATw = (BT)\lambda(A)w \in m_\lambda W_\lambda$$

where the inclusion follows from the fact that $BT \in Z_\lambda$. We then denote by $(\eta_\lambda, m_\lambda W_\lambda)$ the representation of $\mathcal{A}' \otimes \mathcal{A}$ defined by setting $\eta_\lambda(B \otimes A) = \eta(B \otimes A)|_{m_\lambda W_\lambda}$ for all $B \in \mathcal{A}'$ and $A \in \mathcal{A}$.

Consider the linear map

$$\begin{aligned} \mathcal{T}_\lambda : Z_\lambda \otimes W_\lambda &\to m_\lambda W_\lambda \\ T \otimes w &\mapsto Tw \end{aligned}$$

and the representation $(\sigma_\lambda, Z_\lambda \otimes W_\lambda)$ of $\mathcal{A}' \otimes \mathcal{A}$ defined by

$$\sigma_\lambda(B \otimes A)(T \otimes w) = BT \otimes \lambda(A)w,$$

for all $A \in \mathcal{A}$, $B \in \mathcal{A}'$, $T \in Z_\lambda$ and $w \in W_\lambda$. That is, σ_λ is the tensor product of the representation of $\mathcal{A}'$ on Z_λ with λ.

Lemma 7.3.3 $\mathcal{T}_\lambda \in \mathrm{Hom}_{\mathcal{A}' \otimes \mathcal{A}}(\sigma_\lambda, \eta_\lambda)$ *and it is an isomorphism.*

Proof It is easy to see that $\mathcal{T}_\lambda$ intertwines the two representations, for:

$$\begin{aligned}\eta_\lambda(B\otimes A)[\mathcal{T}_\lambda(T\otimes w)] &= \eta_\lambda(B\otimes A)[Tw]\\ &= BA(Tw)\\ &= BT\lambda(A)w\\ &= \mathcal{T}_\lambda[BT\otimes\lambda(A)w]\\ &= \mathcal{T}_\lambda[\sigma_\lambda(B\otimes A)(T\otimes w)]\end{aligned}$$

for all $A\in\mathcal{A}$, $B\in\mathcal{A}'$, $T\in Z_\lambda$ and $w\in W_\lambda$.

Let now $I_{\lambda,1}, I_{\lambda,2},\dots,I_{\lambda,m_\lambda}$ be the injective operators which constitute a basis for the space Z_λ (cf. Proposition 7.2.5). Then, every operator $T\in Z_\lambda$ may be written in the form $T=\sum_{j=1}^{m_\lambda}\alpha_j I_{\lambda,j}$ and, if $w\neq 0$ and $\mathcal{T}_\lambda(T\otimes w)=0$, then $\sum_{j=1}^{m_\lambda}\alpha_j I_{\lambda,j}w=0$, which implies $\alpha_1=\alpha_2=\cdots=\alpha_{m_\lambda}=0$ (recall that the operators $I_{\lambda,j}$ realize an orthogonal decomposition of $m_\lambda W_\lambda$). This shows that $\mathcal{T}_\lambda$ is injective. Since $\dim(Z_\lambda\otimes W_\lambda)=m_\lambda d_\lambda=\dim(m_\lambda W_\lambda)$, $\mathcal{T}_\lambda$ is also surjective. □

Fix an orthonormal basis $w_{\lambda,1}, w_{\lambda,2},\dots,w_{\lambda,d_\lambda}$ of W_λ and for $j=1,2,\dots,d_\lambda$, consider the linear operator

$$\begin{aligned}\mathcal{S}_{\lambda,j}: Z_\lambda &\to\ Z_\lambda\otimes W_\lambda\cong m_\lambda W_\lambda\ \subset V\\ T&\mapsto T\otimes w_{\lambda,j}\overset{\mathcal{T}_\lambda}{\mapsto} Tw_{\lambda,j}\end{aligned}\tag{7.8}$$

Since the operators $I_{\lambda,i}$, are isometric and determine an orthogonal decomposition, the vectors

$$I_{\lambda,i}w_{\lambda,j}\quad i=1,2,\dots,m_\lambda,\ \ j=1,2,\dots,d_\lambda$$

constitute an orthonormal basis of $m_\lambda W_\lambda$. Therefore,

$$m_\lambda W_\lambda=\bigoplus_{j=1}^{d_\lambda}\mathcal{S}_{\lambda,j}Z_\lambda\tag{7.9}$$

is another orthogonal decomposition of $m_\lambda W_\lambda$. It is also obvious that $\mathcal{S}_{\lambda,j}\in\mathrm{Hom}_{\mathcal{A}'}(Z_\lambda,V)$.

Theorem 7.3.4 (Double commutant theory) *Set* $Z_\lambda=\mathrm{Hom}_{\mathcal{A}}(W_\lambda,V)$ *for* $\lambda\in\Lambda$.

(i) *The linear isomorphism*

$$V\cong\bigoplus_{\lambda\in\Lambda}(Z_\lambda\otimes W_\lambda),\tag{7.10}$$

given explicitly by the linear map

$$\bigoplus_{\lambda\in\Lambda}(Z_\lambda \otimes W_\lambda) \ni \sum_{\lambda\in\Lambda} T_\lambda \otimes w_\lambda \mapsto \sum_{\lambda\in\Lambda} \mathcal{T}_\lambda(T_\lambda \otimes w_\lambda) = \sum_{\lambda\in\Lambda} T_\lambda w_\lambda,$$

determines a decomposition of V *into irreducible* $(\mathcal{A}' \otimes \mathcal{A})$-*representations (and this decomposition is multiplicity-free).*

(ii) *With respect to the isomorphism* (7.10) *we have*

$$\mathcal{A} \cong \bigoplus_{\lambda\in\Lambda} \left[I_{Z_\lambda} \otimes \mathrm{End}(W_\lambda)\right] \tag{7.11}$$

and

$$\mathcal{A}' \cong \bigoplus_{\lambda\in\Lambda} \left[\mathrm{End}(Z_\lambda) \otimes I_{W_\lambda}\right],$$

where I_{Z_λ} *and* I_{W_λ} *are the identity operators on* Z_λ *and* W_λ*, respectively.*

(iii) *The* $\mathcal{A}'$*-isotypic decomposition of* V *is given by*

$$V = \bigoplus_{\lambda\in\Lambda} \bigoplus_{j=1}^{d_\lambda} \mathcal{S}_{\lambda,j} Z_\lambda \cong \bigoplus_{\lambda\in\Lambda} d_\lambda Z_\lambda.$$

Proof Given $\rho \in \Lambda$, every operator $T \in \mathrm{Hom}_{\mathcal{A}}(W_\rho, V) \equiv Z_\rho$ may be written in the form

$$T = \sum_{k=1}^{m_\rho} \beta_{\rho,k} I_{\rho,k},$$

with $\beta_{\rho,k} \in \mathbb{C}$. Also, every $B \in \mathcal{A}'$ may be written in the form

$$B = \sum_{\lambda\in\Lambda} \sum_{i,j=1}^{m_\lambda} \alpha^\lambda_{i,j} T^\lambda_{i,j}$$

where $\alpha^\lambda_{i,j} \in \mathbb{C}$ and the operators $T^\lambda_{i,j} \in \mathrm{Hom}_{\mathcal{A}}(I_{\lambda,j} W_\lambda, I_{\lambda,i} W_\lambda)$ (cf. Theorem 7.2.8). Since $T^\lambda_{i,j} I_{\rho,k} = \delta_{\lambda,\rho}\delta_{j,k} I_{\rho,i}$, we have

$$BT = \sum_{i=1}^{m_\rho} \left(\sum_{j=1}^{m_\rho} \alpha^\rho_{i,j} \beta_{\rho,j} \right) I_{\rho,i}. \tag{7.12}$$

This shows that $\mathcal{A}'$ acts on Z_ρ as $M_{m_\rho,m_\rho}(\mathbb{C}) \subset \bigoplus_{\lambda\in\Lambda} M_{m_\lambda,m_\lambda}(\mathbb{C}) \cong \mathcal{A}'$ (cf. Theorem 7.2.8) on $\mathbb{C}^{m_\rho}$. From Lemma 7.2.9 we deduce that the $\mathcal{A}'$-representations Z_λ, $\lambda \in \Lambda$, are irreducible and pairwise inequivalent. This proves (iii). Moreover, from Proposition 7.3.1 and Lemma 7.3.3 we immediately get (i). The second isomorphism in (ii) is just a reformulation of Theorem 7.2.8, while the first one follows from (iii), the double commutant theorem (Theorem 7.3.2) and, again, Theorem 7.2.8: indeed, $\mathcal{A}$ is the commutant of $\mathcal{A}'$ and (iii) is the isotypic decomposition of V under $\mathcal{A}'$. □

Corollary 7.3.5 (Structure of finite dimensional $*$-algebras) *Every finite dimensional $*$-algebra is a direct sum of full matrix algebras: indeed, with the above notation we have a $*$-isomorphism*

$$\mathcal{A} \cong \bigoplus_{\lambda\in\Lambda} \mathrm{End}(W_\lambda) \cong \bigoplus_{\lambda\in\Lambda} M_{d_\lambda,d_\lambda}(\mathbb{C}). \tag{7.13}$$

Remark 7.3.6 If $|\Lambda| \geq 2$, then the representation η of $\mathcal{A}' \otimes \mathcal{A}$ on V is not faithful. Indeed, take $\lambda_1, \lambda_2 \in \Lambda$, $\lambda_1 \neq \lambda_2$, and $T \in \mathcal{A}$, $S \in \mathcal{A}'$ such that $T = I_{W_{\lambda_1}}$ on W_{λ_1}, $T \equiv 0$ on W_λ for $\lambda \neq \lambda_1$, $S = I_{Z_{\lambda_2}}$ on Z_{λ_2}, $S \equiv 0$ on Z_λ for $\lambda \neq \lambda_2$. Then, $\eta(S \otimes T) = 0$.

Example 7.3.7 Consider the subalgebra $\mathcal{A}$ of $M_{10,10}(\mathbb{C})$ consisting of all matrices of the form

$$\begin{pmatrix} A & & & \\ & A & & \\ & & B & \\ & & & B \end{pmatrix}$$

with $A \in M_{2,2}(\mathbb{C})$ and $B \in M_{3,3}(\mathbb{C})$. Then $\mathcal{A} \cong M_{2,2}(\mathbb{C}) \oplus M_{3,3}(\mathbb{C})$ and $\mathcal{A}' \cong M_{2,2}(\mathbb{C}) \oplus M_{2,2}(\mathbb{C})$. Moreover, the decomposition (7.10) is given by

$$\mathbb{C}^{10} \cong (\mathbb{C}^2 \otimes \mathbb{C}^2) \oplus (\mathbb{C}^2 \otimes \mathbb{C}^3).$$

Exercise 7.3.8 (1) Let V and W be two vector spaces with bases $\{v_1, v_2, \ldots, v_n\}$ and $\{w_1, w_2, \ldots, w_m\}$, respectively. Let $A \in \mathrm{End}(V)$, $B \in \mathrm{End}(W)$ and denote by $(a_{i,j})_{i,j=1}^n$ the matrix representing A. Show that in the basis $\{v_k \otimes w_\ell : 1 \leq k \leq n, 1 \leq \ell \leq m\}$ the operator $A \otimes B$ is represented by the block matrix

$$\begin{pmatrix} a_{1,1}B & a_{1,2}B & \cdots & a_{1,n}B \\ a_{2,1}B & a_{2,2}B & \cdots & a_{2,n}B \\ \cdots & \cdots & \cdots & \cdots \\ a_{n,1}B & a_{n,2}B & \cdots & a_{n,n}B \end{pmatrix}.$$

(2) Use (1) to prove that for a finite dimensional $*$-algebra of the form

$$\mathcal{A} \cong \bigoplus_{i=1}^{k} \left(I_{m_i} \otimes M_{d_i,d_i}(\mathbb{C})\right)$$

we have

$$\mathcal{A}' \cong \bigoplus_{i=1}^{k} \left(M_{m_i,m_i}(\mathbb{C}) \otimes I_{d_i}\right).$$

7.3.4 Matrix units and central elements

Let $\mathcal{A}$ be a $*$-algebra of operators on a finite dimensional space V and denote by $V = \oplus_{\lambda\in\Lambda} m_\lambda W_\lambda$ the $\mathcal{A}$-isotypic decomposition of V. Let $\{w_{\lambda,1}, w_{\lambda,2}, \dots$ $\dots, w_{\lambda,d_\lambda}\} \subset W_\lambda$ be an orthonormal basis. Using the first isomorphism in (7.13) we can define a set of operators $\{E^\lambda_{i,j} : 1 \leq i, j \leq d_\lambda, \lambda \in \Lambda\}$ in $\mathcal{A}$ by setting

$$E^\lambda_{i,j} w_{\theta,k} = \delta_{\lambda,\theta}\delta_{j,k} w_{\theta,i}. \tag{7.14}$$

Clearly, under the second isomorphism in (7.13), the corresponding matrix representation of $E^\lambda_{i,j}$ has 1 in the (i, j)-position in the direct summand corresponding to λ and 0 in all other positions. We call the operators $E^\lambda_{i,j}$ the *matrix units* associated with the (orthonormal) bases $\{w_{\lambda,j} : 1 \leq j \leq d_\lambda\}$, $\lambda \in \Lambda$. Moreover, $\{E^\lambda_{i,j} : 1 \leq i, j \leq d_\lambda, \lambda \in \Lambda\}$ is a vector space basis for $\mathcal{A}$. The diagonal operators $E^\lambda_{i,i}$, $1 \leq i \leq d_\lambda$, $\lambda \in \Lambda$ are called the *primitive idempotents* associated with the (orthonormal) bases $\{w_{\lambda,j} : 1 \leq j \leq d_\lambda\}, \lambda \in \Lambda$.

The following proposition is an immediate consequence of Theorem 7.3.4.

Proposition 7.3.9 *Let $\mathcal{S}_{\lambda,j}$ be as in* (7.9)*. Then we have*

$$\mathcal{S}_{\lambda,j} Z_\lambda = E^\lambda_{j,j} V.$$

In particular,

$$V = \bigoplus_{\lambda\in\Lambda} \bigoplus_{j=1}^{d_\lambda} E^\lambda_{j,j} V$$

is an orthogonal decomposition of V into $\mathcal{A}'$-irreducible subspaces, with

$$E^\lambda_{j,j} V \equiv Z_\lambda \otimes w_{\lambda,j} \cong Z_\lambda$$

for all $j = 1, 2, \dots, d_\lambda$, $\lambda \in \Lambda$.

Proof From (7.8) and (7.9) it follows that, under the isomorphism (7.10), $\mathcal{S}_{\lambda,j} Z_\lambda$ corresponds to $Z_\lambda \otimes w_{\lambda,j}$. Keeping into account (7.14), an application of (7.11) ends the proof. □

Clearly, we can use a set of primitive idempotents of $\mathcal{A}'$ to get a decomposition of V into $\mathcal{A}$-irreducible representations.

The *center* of $\mathcal{A}$ is the set $\mathcal{C}(\mathcal{A}) = \{A \in \mathcal{A} : AB = BA, \forall B \in \mathcal{A}\}$. We consider $\mathbb{C}^m$ as a commutative algebra with pointwise product: $(\alpha_1, \alpha_2, \dots, \alpha_m) \cdot (\beta_1, \beta_2, \dots, \beta_m) = (\alpha_1\beta_1, \alpha_2\beta_2, \dots, \alpha_m\beta_m)$ for $\alpha_1, \alpha_2, \dots, \alpha_m, \beta_1, \beta_2, \dots, \beta_m \in \mathbb{C}$. Moreover, we set $C^\lambda = \sum_{j=1}^{d_\lambda} E^\lambda_{j,j}$. The following proposition is obvious, but it is important and we state it explicitely.

Proposition 7.3.10 *We have* $\mathcal{C}(\mathcal{A}) \equiv \mathcal{C}(\mathcal{A}') \equiv \mathbb{C}^{|\Lambda|}$. *Moreover,* $\{C^\lambda : \lambda \in \Lambda\}$ *is a basis for* $\mathcal{C}(\mathcal{A})$.

7.4 Ideals and representation theory of a finite dimensional $*$-algebra

In this section, we conclude the description of the representation theory of finite dimensional $*$-algebras. We end with a revisitation of the representation theory of a finite group.

7.4.1 Representation theory of $\mathrm{End}(V)$

Let V be a finite dimensional vector space. Let $\mathcal{A}$ be a subalgebra of $\mathrm{End}(V)$. A *left* (resp. *right*) *ideal* $\mathcal{I}$ in $\mathcal{A}$ is a linear subspace such that if $A \in \mathcal{A}$, $B \in \mathcal{I}$ then $AB \in \mathcal{I}$ (resp. $BA \in \mathcal{I}$). A *twosided ideal* is a linear subspace which is both a left and a right ideal. The *trivial* ideals are $\{0\}$ and $\mathcal{A}$.

Proposition 7.4.1 *Let V be a finite dimensional vector space.*

(i) $\mathrm{End}(V)$ *does not contain nontrivial twosided ideals.*
(ii) *The natural representations of* $\mathrm{End}(V)$ *on V is the unique irreducible representation of* $\mathrm{End}(V)$*, up to equivalence.*

Proof (i) Let $n = \dim V$ and denote by $E_{i,j} \in M_{n,n}(\mathbb{C}) \cong \mathrm{End}(V)$ the matrix with 1 in position (i, j) and all the other entries equal to zero. Let $\mathcal{I} \neq \{0\}$ be a twosided ideal in $M_{n,n}(\mathbb{C})$. Consider in $\mathcal{I}$ a nonzero element $A = (a_{i,j})_{i,j=1}^n$. Fix a nonzero entry of A, say $a_{h,k}$. Then

$$E_{h,k} = \frac{1}{a_{h,k}} E_{h,h} A E_{k,k} \in \mathcal{I}$$

and for all $i, j = 1, 2, \dots, n$

$$E_{i,j} = E_{i,h} E_{h,k} E_{k,j} \in \mathcal{I}.$$

Therefore $\mathcal{I} = M_{n,n}(\mathbb{C})$.

(ii) Let (σ, W) be an irreducible representation of $\mathrm{End}(V)$. Fix $f \in V'$, $f \neq 0$, and for every $v \in V$ define $T_v \in \mathrm{End}(V)$ by

$$T_v u = f(u)v$$

for all $u \in V$ (that is, in the notation of Section 7.1.2, $T_v = T_{f,v}$). We have

$$AT_v = T_{Av} \tag{7.15}$$

and therefore the subspace

$$\widetilde{V} = \{T_v : v \in V\} \leq \mathrm{End}(V)$$

is $\mathrm{End}(V)$-invariant and isomorphic to the natural representation on V. Fix also $v_0 \in V, v_0 \neq 0$. The set $\mathrm{Ker}(\sigma) = \{A \in \mathrm{End}(V) : \sigma(A) = 0\}$ is a twosided ideal in $\mathrm{End}(V)$; from (i) and the fact that $I_V \notin \mathrm{Ker}(\sigma)$, we deduce that $\mathrm{Ker}(\sigma) = \{0\}$. Therefore there exists $w_0 \in W$ such that

$$\sigma(T_{v_0})w_0 \neq 0. \tag{7.16}$$

Define a linear map $\phi : V \to W$ by setting

$$\phi(v) = \sigma(T_v)w_0$$

for all $v \in V$. The map ϕ intertwines the natural representation of $\mathrm{End}(V)$ on V with the representation (σ, W). Indeed, for all $A \in \mathrm{End}(V)$ and $v \in V$, we have

$$\sigma(A)\phi(v) = \sigma(AT_v)w_0 = \sigma(T_{Av})w_0 = \phi(Av),$$

where the third equality follows from (7.15). Moreover, by (7.16) ϕ is nontrivial; by Schur's lemma (i), we conclude that ϕ is an isomorphism. □

Exercise 7.4.2 Translate the proof of (ii) in the setting of a full matrix algebra $M_{n,n}(\mathbb{C})$.

Hint. Take the space $\widetilde{V}$ consistig of all matrices whose columns $2, 3, \ldots, n$ are identically zero.

An algebra $\mathcal{A}$ isomorphic to $\mathrm{End}(V)$ for some vector space V is said to be *simple*. See also Remark 7.4.6.

7.4.2 Representation theory of finite dimensional ∗-algebras

Let $\mathcal{A}$ be a finite dimensional ∗-algebra. Let $\{E^\lambda_{j,j} : \lambda \in \Lambda, j = 1, 2, \dots, d_\lambda\}$ be a set of primitive idempotents for $\mathcal{A}$ as in Section 7.3.4 and set

$$E^\lambda = \sum_{i=1}^{d_\lambda} E^\lambda_{i,i}.$$

Then the map $A \mapsto AE^\lambda \equiv E^\lambda A$ is the projection from $\mathcal{A}$ onto the subalgebra isomorphic to $\mathrm{End}(W_\lambda)$, and $\sum_{\lambda\in\Lambda} E^\lambda = I_V$.

Proposition 7.4.3 *Take a subset $\Lambda_0 \subseteq \Lambda$ and set*

$$\mathcal{B}_\lambda = \begin{cases} \{0\} & \text{if } \lambda \notin \Lambda_0 \\ \mathrm{End}(W_\lambda) & \text{if } \lambda \in \Lambda_0. \end{cases} \tag{7.17}$$

Then $\bigoplus_{\lambda\in\Lambda} \mathcal{B}_\lambda$ is a twosided ideal of $\mathcal{A}$ and every twosided ideal of $\mathcal{A}$ is of this form, for a suitable Λ_0.

Proof Clearly, every subspace of the form $\bigoplus_{\lambda\in\Lambda} \mathcal{B}_\lambda$ is a twosided ideal. Conversely, suppose that $\mathcal{I}$ is a twosided ideal in $\mathcal{A}$. Set $\Lambda_0 = \{\lambda \in \Lambda : \mathrm{End}(W_\lambda) \cap \mathcal{I} \neq \{0\}\}$. Since $\mathrm{End}(W_\lambda) \cap \mathcal{I}$ is a twosided ideal in $\mathrm{End}(W_\lambda)$, by Proposition 7.4.1 if $\lambda \in \Lambda_0$ then $\mathrm{End}(W_\lambda) \subset \mathcal{I}$. It follows that if we define $\mathcal{B}_\lambda$ as in (7.17), then $\bigoplus_{\lambda\in\Lambda} \mathcal{B}_\lambda \subset \mathcal{I}$. On the other hand, if $A \in \mathcal{I}$ then $AE^\lambda \in \mathrm{End}(W_\lambda) \cap \mathcal{I}$ and therefore

$$A = A \sum_{\lambda\in\Lambda} E^\lambda = \sum_{\lambda\in\Lambda} AE^\lambda = \sum_{\lambda\in\Lambda_0} AE^\lambda \in \bigoplus_{\lambda\in\Lambda} \mathcal{B}_\lambda$$

so that $\mathcal{I} = \bigoplus_{\lambda\in\Lambda} \mathcal{B}_\lambda$. □

The following corollary is also an obvious consequence of Theorem 7.3.4, but it seems more convenient to deduce it from the more simple considerations that have led to Proposition 7.4.3.

Corollary 7.4.4 *If an algebra $\mathcal{A}$ is the direct sum of full matrix algebras as in* (7.18)*, then this decomposition is unique (up to a permutation of the summands).*

If $\mathcal{A}$ and $\mathcal{B}$ are finite dimensional associative algebras, an *isomorphism* between $\mathcal{A}$ and $\mathcal{B}$ is a linear bijection $\varphi : \mathcal{A} \to \mathcal{B}$ such that $\varphi(TS) = \varphi(T)\varphi(S)$ for all $T, S \in \mathcal{A}$.

Corollary 7.4.5 *Let $\mathcal{A} \cong \oplus_{\lambda\in\Lambda} M_{d_\lambda,d_\lambda}(\mathbb{C})$ and $\mathcal{B} \cong \oplus_{\rho\in R} M_{d_\rho,d_\rho}(\mathbb{C})$. Then $\mathcal{A}$ and $\mathcal{B}$ are isomorphic if and only if there exists a bijection $\theta : \Lambda \to R$ such that $d_\lambda = d_{\theta(\lambda)}$ for all $\lambda \in \Lambda$. In particular, two finite dimensional ∗-algebras are isomorphic if and only if they are ∗-isomorphic.*

The (unique) decomposition of $\mathcal{A}$ in (7.18) is the decomposition of $\mathcal{A}$ as a *direct sum of simple algebras*.

Remark 7.4.6 An associative algebra is *simple* when it does not contain nontrivial twosided ideals. The *Jacobian radical* of an algebra $\mathcal{A}$ is $J(A) = \cap_\sigma \mathrm{Ker}(\sigma)$, where the intersection is over all irreducible representations σ. An algebra $\mathcal{A}$ is *semisimple* when $J(\mathcal{A}) = \{0\}$. Weddeburn proved the following fundamental theorem for algebras over $\mathbb{C}$.

Theorem 7.4.7 *Let $\mathcal{A}$ be an associative algebra over $\mathbb{C}$.*

(i) *$\mathcal{A}$ is simple if and only if it is isomorphic to* $\mathrm{End}(V)$ *for a finite dimensional vector space V.*
(ii) *$\mathcal{A}$ is semisimple if and only if it is isomorphic to an algebra of the form* $\bigoplus_{\lambda\in\Lambda} \mathrm{End}(W_\lambda)$.

For the proof we refer to Shilov's monograph [110]; see also the book by Alperin and Bell [3].

Let σ_λ be the natural representation of $\mathcal{A}$ on W_λ (cf. (7.18)). We have already seen that σ_λ, $\lambda \in \Lambda$, is a set of irreducible pairwise inequivalent representations of $\mathcal{A}$ (Lemma 7.2.9). We now show that this set is also complete.

Proposition 7.4.8 *Every irreducible representation (σ, W) of $\mathcal{A}$ is isomorphic to one of the σ_λ's.*

Proof Set $\mathrm{Ker}\sigma = \{A \in \mathcal{A} : \sigma(A) = 0\}$. Then $\mathrm{Ker}\sigma$ is a twosided ideal in $\mathcal{A}$ and therefore, by Proposition 7.4.3, there exists $\Lambda_0 \subseteq \Lambda$ such that $\mathrm{Ker}\sigma = \bigoplus_{\lambda\in\Lambda} \mathcal{B}_\lambda$. Let θ be the restriction of σ to $\bigoplus_{\lambda\in\Lambda\setminus\Lambda_0} \mathrm{End}(W_\lambda)$. Then $\mathrm{Ker}\theta = \{0\}$ and therefore the map

$$\theta : \bigoplus_{\lambda\in\Lambda\setminus\Lambda_0} \mathrm{End}(W_\lambda) \to \mathrm{End}(W)$$

is injective; by Burnside's lemma it is also surjective and therefore θ is an isomorphism.

Proposition 7.4.1.(i) and Proposition 7.4.3 (applied to $\bigoplus_{\lambda\in\Lambda\setminus\Lambda_0} \mathrm{End}(W_\lambda)$) force $|\Lambda \setminus \Lambda_0| = 1$. Let $\Lambda \setminus \Lambda_0 = \{\rho\}$. Then Proposition 7.4.1.(ii) forces θ to be equivalent to the restriction of σ_ρ to $\mathrm{End}(W_\rho)$ and in turn σ to be equivalent to σ_ρ. □

From now on, we denote by $\widehat{\mathcal{A}}$ a complete set of pairwise nonequivalent irreducible representations of $\mathcal{A}$. Therefore we can write (7.13) as follows

$$\mathcal{A} \cong \bigoplus_{\lambda\in\widehat{\mathcal{A}}} \mathrm{End}(W_\lambda) \cong \bigoplus_{\lambda\in\widehat{\mathcal{A}}} M_{d_\lambda,d_\lambda}. \tag{7.18}$$

We shall also use $\widehat{\mathcal{A}}$ to denote a complete set of irreducible $\mathcal{A}'$-representations via the *natural pairing* $W_\lambda \leftrightarrow Z_\lambda$ given by Theorem 7.3.4.

7.4.3 The Fourier transform

Let (σ, W) be an irreducible representation of $\mathcal{A}$. The *Fourier transform* of $A \in \mathcal{A}$ evaluated at σ is just the operator $\sigma(A) \in \mathrm{End}(W)$.

The *Fourier transform* on $\mathcal{A}$ is the map

$$\begin{aligned} \mathcal{A} &\to \mathcal{B} := \bigoplus_{\lambda \in \widehat{\mathcal{A}}} \mathrm{End}(W_\lambda) \leq \mathrm{End}\left(\bigoplus_{\lambda \in \widehat{\mathcal{A}}} W_\lambda\right) \\ A &\mapsto \oplus_{\lambda \in \widehat{\mathcal{A}}} \sigma_\lambda(A). \end{aligned} \tag{7.19}$$

The following proposition is an immediate consequence of Corollary 7.3.5, Corollary 7.4.4 and Proposition 7.4.8.

Proposition 7.4.9

(i) *The Fourier transform in (7.19) is an isomorphism between $\mathcal{A}$ and the subalgebra $\mathcal{B}$ of* $\mathrm{End}(\bigoplus_{\lambda \in \widehat{\mathcal{A}}} W_\lambda)$. *Moreover, $\bigoplus_{\lambda \in \widehat{\mathcal{A}}} W_\lambda$ is the (multiplicity-free) decomposition into $\mathcal{B}$-irreducible representations.*

(ii) *If $\rho\colon \mathcal{A} \to \mathcal{B} \leq \mathrm{End}(U)$ is an isomorphism, then there exists a decomposition of* $\mathrm{End}(U)$ *into irreducible $\mathcal{B} \cong \mathcal{A}$-representations*

$$U \cong \bigoplus_{\lambda \in \widehat{\mathcal{A}}} m_\lambda W_\lambda,$$

with $m_\lambda \geq 1$ for all $\lambda \in \widehat{\mathcal{A}}$.

7.4.4 Complete reducibility of finite dimensional $*$-algebras

To complete the picture, we want now to show that every finite dimensional representation of a finite dimensional $*$-algebra may be written as a direct sum of irreducible representations. Our exposition is based on Shilov's monograph [110].

Lemma 7.4.10 *Let V be a finite dimensional vector space, $\mathcal{A}$ a subalgebra of* $\mathrm{End}(V)$ *and (σ, W) a finite dimensional representation of $\mathcal{A}$. Suppose that in W there is a finite set $W_1, W_2, \ldots, W_m$ of invariant irreducible subspaces such that $W = \langle W_1, W_2, \ldots, W_m \rangle$. Then, there exists a subset $H \subseteq \{1, 2, \ldots, m\}$ such that $W = \bigoplus_{i \in H} W_i$ with direct sum.*

Proof Suppose that $K \subseteq \{1, 2, \ldots, m\}$ and that $W = \bigoplus_{i \in K} W_i$ is a direct sum. If $j \notin K$, then $\left(\bigoplus_{k \in K} W_k\right) \cap W_j$ is an invariant subspace contained in W_j and therefore it is either equal to $\{0\}$ or to W_j. We can construct a sequence

$H_1, H_2, \ldots, H_m$ of subsets of $\{1, 2, \ldots, m\}$ starting with $H_1 = \{1\}$ and taking

$$H_{j+1} = \begin{cases} H_j & \text{if } \left(\bigoplus_{i \in H_j} W_i\right) \cap W_{j+1} = W_{j+1} \\ H_j \cup \{j+1\} & \text{if } \left(\bigoplus_{i \in H_j} W_i\right) \cap W_{j+1} = \{0\}. \end{cases}$$

Finally, we have

$$\bigoplus_{i \in H_m} W_i = W$$

with direct sum. □

In the following lemma, $\{E^{\lambda}_{i,j} : i, j = 1, 2, \ldots, d_\lambda, \lambda \in \widehat{\mathcal{A}}\}$ is a fixed set of matrix units for $\mathcal{A}$ (cf. Section 7.3.4).

Lemma 7.4.11 *Let $\mathcal{A}$ be as in (7.18) (with Λ replaced by $\widehat{\mathcal{A}}$). Suppose that (σ, W) is a representation of $\mathcal{A}$ and that there exist $w \in W$, $\rho \in \widehat{\mathcal{A}}$ and $1 \leq t \leq d_\rho$ such that $w_0 := \sigma(E^{\rho}_{t,t})w \neq 0$. Then the subspace $U = \sigma(\mathcal{A})w_0 \equiv \{\sigma(A)w_0 : A \in \mathcal{A}\}$ is $\mathcal{A}$-irreducible.*

Proof Let $A, B, C \in \mathcal{A}$, then they may expressed in the form

$$\begin{aligned} A &= \textstyle\sum_{\lambda \in \widehat{\mathcal{A}}} \sum_{i,j=1}^{d_\lambda} \alpha^{\lambda}_{i,j} E^{\lambda}_{i,j} \\ B &= \textstyle\sum_{\lambda \in \widehat{\mathcal{A}}} \sum_{i,j=1}^{d_\lambda} \beta^{\lambda}_{i,j} E^{\lambda}_{i,j} \\ C &= \textstyle\sum_{\lambda \in \widehat{\mathcal{A}}} \sum_{k,j=1}^{d_\lambda} \gamma^{\lambda}_{k,j} E^{\lambda}_{k,j}. \end{aligned}$$

Since $w_0 := \sigma(E^{\rho}_{t,t})w$, if we set $u = \sigma(A)w_0$ and $z = \sigma(B)w_0$, then we have

$$u = \sum_{i=1}^{d_\rho} \alpha^{\rho}_{i,t} \sigma(E^{\rho}_{i,t})w, \qquad z = \sum_{i=1}^{d_\rho} \beta^{\rho}_{i,t} \sigma(E^{\rho}_{i,t})w$$

and

$$\sigma(C)u = \sum_{k=1}^{d_\rho} \left(\sum_{j=1}^{d_\rho} \gamma^{\rho}_{k,j} \alpha^{\rho}_{j,t} \right) \sigma(E^{\rho}_{k,t})w.$$

This amounts to saying that if $u, z \in U$, with $u \neq 0$, then it is always possible to find $C \in \mathcal{A}$ such that $\sigma(C)u = z$ (this follows from the irreducibility of $\mathbb{C}^{d_\rho}$ under $M_{d_\rho,d_\rho}(\mathbb{C})$). Therefore, any $u \in U$, with $u \neq 0$, is cyclic and U is irreducible. □

We are now in position to prove that every finite dimensional representation of a $*$-algebra is a direct sum of irreducible representations.

Theorem 7.4.12 *Let $\mathcal{A}$ be a finite dimensional $*$-algebra. Then every finite dimensional representation of $\mathcal{A}$ may be written as a direct sum of irreducible representations.*

Proof Suppose that $\mathcal{A}$ is decomposed as in (7.18) (with Λ replaced by $\widehat{\mathcal{A}}$) and let (σ, W) be a representation of $\mathcal{A}$. Let $\{w_1, w_2, \dots, w_m\}$ be a basis for W. Since $I = \sum_{\lambda\in\widehat{\mathcal{A}}}\sum_{i=1}^{d_\lambda} E^\lambda_{i,i}$ is the identity of $\mathcal{A}$, then $\sum_{\lambda\in\widehat{\mathcal{A}}}\sum_{i=1}^{d_\lambda}\sigma(E^\lambda_{i,i}) = \sigma(I) = I_W$ and therefore the set

$$\{\sigma(E^\lambda_{i,i})w_t : \lambda \in \widehat{\mathcal{A}},\ i = 1, 2, \dots, d_\lambda,\ t = 1, 2, \dots, m\}$$

spans W. By Lemma 7.4.11 we deduce that if $\sigma(E^\lambda_{i,i})w_t \neq 0$ then $\sigma(\mathcal{A}E^\lambda_{i,i})w_t$ is irreducible. Therefore, the nontrivial subspaces of the form $\sigma(\mathcal{A}E^\lambda_{i,i})w_t$ are irreducible and span W. From Lemma 7.4.10 we deduce that W may be written as a direct sum of irreducible representations. □

An associative algebra with the property that every representation may be written as a direct sum of irreducible representation is called *completely reducible*, see [49].

7.4.5 The regular representation of a $*$-algebra

Let $\mathcal{A} \cong \oplus_{\sigma\in\widehat{\mathcal{A}}}\mathrm{End}(W_\sigma)$ be a finite dimensional $*$-algebra.

The *left regular representation* of $\mathcal{A}$ is the representation $\lambda\colon \mathcal{A} \to \mathrm{End}(\mathcal{A})$ given by left mutiplication: $\lambda(A)B = AB$ for all $A, B \in \mathcal{A}$.

We endow $\mathrm{End}(V)$ with the Hilbert–Schmidt scalar product defined by

$$\langle A, B\rangle_{HS} = tr(B^*A)$$

for all $A, B \in \mathrm{End}(V)$. Then the left regular representation is a $*$-representation:

$$\begin{aligned}\langle\lambda(A)B, C\rangle_{HS} &= \langle AB, C\rangle_{HS}\\ &= tr(C^*AB)\\ &= tr[(A^*C)^*B]\\ &= \langle B, A^*C\rangle_{HS}\\ &= \langle B, \lambda(A^*)C\rangle_{HS},\end{aligned}$$

which shows that $\lambda(A)^* = \lambda(A^*)$ for all $A \in \mathcal{A}$.

The *right regular representation* of $\mathcal{A}$ is the representation $\rho\colon \mathcal{A} \to \mathrm{End}(\mathcal{A})$ given by right mutiplication by: $\rho(A)B = BA^T$ for all $A, B \in \mathcal{A}$. Note that $\rho(A)$ belongs to $\mathrm{Hom}_{\mathcal{A}}(\lambda, \lambda)$, the commutant of the left regular representation of $\mathcal{A}$, for all $A \in \mathcal{A}$.

Proposition 7.4.13

(i) *Let λ denote the left regular representation of $\mathcal{A}$. Then, the map*

$$\begin{aligned}\mathcal{A} &\to \mathrm{Hom}_{\mathcal{A}}(\lambda, \lambda)\\ A &\mapsto \rho(A)\end{aligned}$$

is a $*$-isomorphism of $*$-algebras (see Section 1.2.3).

(ii) *$\mathcal{A} \cong \bigoplus_{\sigma\in\widehat{\mathcal{A}}} d_\sigma W_\sigma$ is the decomposition of $\mathcal{A}$ with respect to the left regular representation.*

(iii) *$\mathcal{A} \cong \bigoplus_{\sigma\in\widehat{\mathcal{A}}}(W_\sigma \otimes W_\sigma)$ is the decomposition of $\mathcal{A}$ with respect to the $\mathcal{A} \otimes \mathcal{A}$-representation $\eta\colon \mathcal{A}\otimes\mathcal{A} \to \mathrm{End}(\mathcal{A})$ given by $\eta(A\otimes B)C = ACB^T$, for all $A, B, C \in \mathcal{A}$.*

Proof (i) First of all note that $\rho\colon \mathcal{A} \to \mathrm{Hom}_{\mathcal{A}}(\lambda, \lambda)$ is an injective algebra homomorphism. Let us show that it is also surjective.

Let $T \in \mathrm{Hom}_{\mathcal{A}}(\lambda, \lambda)$ and set $A = T(I_V) \in \mathcal{A}$. Then, for every $B \in \mathcal{A}$,

$$T(B) = T(BI_V) = BT(I_V) = BA = \rho(A^T)B,$$

that is, $T = \rho(A^T)$.

Finally,

$$\begin{aligned}\langle\rho(A)B, C\rangle_{HS} &= \langle BA^T, C\rangle_{HS}\\ &= tr(C^*BA^T)\\ &= tr(A^TC^*B)\\ &= tr\{[C(A^T)^*]^*B\}\\ &= tr\{[C(A^*)^T]^*B\}\\ &= \langle B, C(A^*)^T\rangle_{HS}\\ &= \langle B, \rho(A^*)C\rangle_{HS},\end{aligned}$$

which shows that $\rho(A)^* = \rho(A^*)$ for all $A \in \mathcal{A}$. Thus ρ is a $*$-isomorphism.

(ii) and (iii) are obvious reformulations of (i) and (ii) in Theorem 7.3.4. The details are left to the reader: it suffices to examine the case $|\widehat{\mathcal{A}}| = 1$ (that is $\mathcal{A} \cong \mathrm{End}(W)$) and then note that each block $\mathrm{End}(W_\sigma)$ is invariant under λ, ρ and η. □

7.4.6 Representation theory of finite groups revisited

Let G be a finite group. The group algebra $L(G) = \{f\colon G \to \mathbb{C}\}$ is a $*$-algebra with involution given by $\check{f}(g) = \overline{f(g^{-1})}$ for all $f \in L(G)$ and $g \in G$ (see Section 1.5.1).

Any unitary representation (σ, V) of G may be uniquely extended to a representation $(\sigma_{L(G)}, V)$ of the group algebra $L(G)$ by setting

$$\sigma_{L(G)}(f) = \sum_{g \in G} f(g)\sigma(g)$$

for all $f \in L(G)$. Indeed, $\sigma(\check{f}) = \sigma(f)^*$ for all $f \in L(G)$. Conversely, given any $*$-algebra representation (σ, U) of the group algebra $L(G)$, then the map $\sigma_G : G \to \mathrm{End}(U)$ given by $\sigma_G(g) = \sigma(\delta_g)$ for all $g \in G$ yields a unitary representation (σ_G, U) of the group G.

Therefore there is a natural bijection between the set of all unitary representations of G and the set of all $*$-algebra representations of $L(G)$. Moreover, this bijection preserves irreducibility.

It turns out that many of the results in Chapters 1 and 2 might be deduced from the results in the present chapter.

As an example, in the discussion below we establish a connection between Corollary 2.1.6 and Proposition 7.4.13. After that, we shall discuss the representation theory of the commutant (cf. Section 1.2).

Let $\sigma \in \widehat{G}$ and denote by M^σ the linear span of the matrix coefficients of the irreducible representation σ. Denote by $\sigma' \in \widehat{G}$ the adjoint of σ. A slight different formulation of Corollary 2.1.6 is the following:

$$L(G) \cong \bigoplus_{\sigma \in \widehat{G}} M^{\sigma'} \cong \bigoplus_{\sigma \in \widehat{G}} \sigma \otimes \sigma'$$

is the decomposition of $L(G)$ into irreducible representations with respect to the action η of $G \times G$ on $L(G)$ given by

$$[\eta(g_1, g_2)f](g) = f(g_1^{-1} g g_2)$$

for all $g, g_1, g_2 \in G$ and $f \in L(G)$.

Let us show why here we have $\sigma \otimes \sigma'$ (and not $\sigma \otimes \sigma$, as in Proposition 7.4.13.(iii)). The right regular representation $\rho : G \to \mathrm{End}(L(G))$ is given by $\rho(g)f = f * \delta_{g^{-1}}$ for all $g \in G$ and $f \in L(G)$. It corresponds to the $L(G)$-representation given by $\rho(\varphi)f = f * \varphi^\flat$, where $\varphi^\flat(g) = \varphi(g^{-1})$, for all $f, \phi \in L(G)$ and $g \in G$. Note that the map $L(G) \ni \varphi \to \varphi^\flat \in L(G)$ is an anti-automorphism which does not preserve the decomposition $L(G) \cong \bigoplus_{\sigma \in \widehat{G}} M^\sigma$. Indeed, from the elementary property of the matrix coefficients $\varphi^\sigma_{i,j}(g^{-1}) = \overline{\varphi^\sigma_{j,i}(g)} = \varphi^{\sigma'}_{j,i}(g)$, it follows that the map $\varphi \to \varphi^\flat$ switches M^σ with $M^{\sigma'}$. On the other hand, the transposition used in Proposition 7.4.13 now coincides with the map $\varphi^\sigma_{i,j} \mapsto \varphi^\sigma_{j,i}$.

We end this section with a discussion of Proposition 7.3.9 in the setting of a group algebra.

Let G be a finite group and let (σ, V) be a unitary representation of G. Let us set $\mathcal{A} = \{\sigma(f) : f \in L(G)\}$. Then $\mathcal{A}' = \mathrm{End}_G(V)$ and we can apply Theorem 7.3.4 to $\mathcal{A}$ in order to decompose V under the action of $\mathrm{End}_G(V)$.

Theorem 7.4.14 *Let $V = \oplus_{\rho\in R} m_\rho W_\rho$ be the decomposition of V into G-irreducibles, with $R \subseteq \widehat{G}$ and, for all $\rho \in R$, $m_\rho > 0$. For $\rho \in R$ we set $Z_\rho = \mathrm{Hom}_G(W_\rho, V)$.*

(i) *Under the action of $\mathrm{End}_G(V) \otimes L(G)$ the space V decomposes in the following multiplicity-free way:*

$$V \cong \bigoplus_{\rho\in R} (Z_\rho \otimes W_\rho); \tag{7.20}$$

(ii) *If $e_j^\rho : \rho \in \widehat{G}$ and $j = 1, 2, \ldots, d_\rho$ are as in Proposition 1.5.17, then*

$$V = \bigoplus_{\rho\in R} \bigoplus_{j=1}^{d_\rho} \sigma(e_j^\rho) V$$

is an orthogonal decomposition of V into $\mathrm{End}_G(V)$-irreducible representations with $\sigma(e_j^\rho)V \cong Z_\rho$, for all $j = 1, 2, \ldots, d_\rho$ and $\rho \in R$.

Proof Just note that the elements $\sigma(e_j^\rho)$, $j = 1, 2, \ldots, d_\rho$, $\rho \in R$, are the primitive idempotents (cf. Proposition 7.3.9) of $\mathcal{A} = \{\sigma(f) : f \in L(G)\}$ associated with the orthonormal bases of the spaces W_ρ, $\rho \in R$, as in Proposition 1.5.17.(iii). □

7.5 Subalgebras and reciprocity laws

In this section, we collect some results that connect the operations of restriction and induction for a representation with the operations of taking the commutant and the centralizer of a subalgebra.

7.5.1 Subalgebras and Bratteli diagrams

Let $\mathcal{A}$ and $\mathcal{B}$ be two finite dimensional $*$-algebras and suppose that $\mathcal{B} \subseteq \mathcal{A}$. Note that this implies that $\mathcal{A}$ and $\mathcal{B}$ have the same identity since they are $*$-algebras of operators on the same vector space. We also say that $\mathcal{B}$ is a *subalgebra* of $\mathcal{A}$.

Let (ρ, V) be a representation of $\mathcal{A}$. Its *restriction* to $\mathcal{B}$ is given by

$$[\mathrm{Res}^{\mathcal{A}}_{\mathcal{B}} \rho](T) = \rho(T)$$

for all $T \in \mathcal{B}$.

For $(\lambda, W_\lambda) \in \widehat{\mathcal{A}}$ and $(\rho, V_\rho) \in \widehat{\mathcal{B}}$, we denote by $m_{\lambda,\rho}$ the multiplicity of ρ in $\mathrm{Res}^{\mathcal{A}}_{\mathcal{B}}\lambda$, so that

$$\mathrm{Res}^{\mathcal{A}}_{\mathcal{B}} W_\lambda \cong \bigoplus_{\rho \in \widehat{\mathcal{B}}} m_{\lambda,\rho} V_\rho.$$

The $\widehat{\mathcal{A}} \times \widehat{\mathcal{B}}$ matrix

$$\mathfrak{M}(\mathcal{A}, \mathcal{B}) = (m_{\lambda,\rho})_{\lambda \in \widehat{\mathcal{A}}, \rho \in \widehat{\mathcal{B}}}$$

is called the *inclusion matrix* for the pair $\mathcal{B} \subseteq \mathcal{A}$ (see [48]). Note that the integers $m_{\lambda,\rho}$ satisfy the following condition:

$$\sum_{\rho \in \widehat{\mathcal{B}}} m_{\lambda,\rho} d_\rho = d_\lambda$$

for all $\lambda \in \widehat{\mathcal{A}}$.

The *Bratteli diagram* (or *branching graph*) of the inclusion $\mathcal{B} \subseteq \mathcal{A}$ is the bipartite labelled multi-graph described as follows. The set of vertices is $\widehat{\mathcal{A}} \coprod \widehat{\mathcal{B}}$, each vertex θ is labelled by d_θ. Moreover, two vertices $\lambda \in \widehat{\mathcal{A}}$ and $\rho \in \widehat{\mathcal{B}}$ are connected by exactly $m_{\lambda,\rho}$ edges. Pictorially, we display the vertices on two horizontal lines, with the vertices in $\widehat{\mathcal{A}}$ on the top row and the vertices in $\widehat{\mathcal{B}}$ on the bottom row (note that in [48] the positions of $\widehat{\mathcal{A}}$ and $\widehat{\mathcal{B}}$ are inverted).

Example 7.5.1

(i) From the representation theory of $\mathfrak{S}_n$ developed in Chapter 2 (in particular the Young poset in Figure 3.12 and the branching rule, Corollary 3.3.11) we get the Bratteli diagram in Figure 5.1 for the inclusion $L(\mathfrak{S}_4) \subseteq L(\mathfrak{S}_5)$.

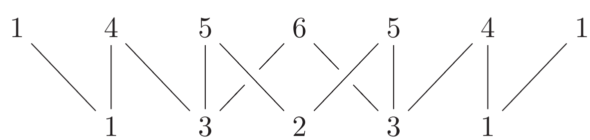


Figure 7.1 The Bratteli diagram for $L(\mathfrak{S}_4) \subseteq L(\mathfrak{S}_5)$.

(ii) Similarly, for the inclusion $L(\mathfrak{S}_3) \subseteq L(\mathfrak{S}_5)$ we have the diagram in Figure 5.2

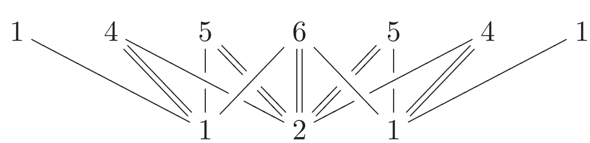


Figure 7.2 The Bratteli diagram for $L(\mathfrak{S}_3) \subseteq L(\mathfrak{S}_5)$.

When $\mathcal{A} = L(G)$ and $\mathcal{B} = L(H)$ are group algebras and $H \leq G$ (so that $\mathcal{B} \subseteq \mathcal{A}$) we also write $\mathfrak{M}(\widehat{G}, \widehat{H})$ to denote the inclusion matrix $\mathfrak{M}(\widehat{\mathcal{A}}, \widehat{\mathcal{B}})$.

We say that $\mathcal{B} \subseteq \mathcal{A}$ is a *multiplicity-free subalgebra* of $\mathcal{A}$ if $m_{\lambda,\rho} \in \{0, 1\}$ for all $\lambda \in \widehat{\mathcal{A}}$ and $\rho \in \widehat{\mathcal{B}}$. In other words, $\mathrm{Res}^{\mathcal{A}}_{\mathcal{B}} W$ decomposes without multiplicity for every irreducible representation W of $\mathcal{A}$. Clearly, the corresponding Bratteli diagram does not contain multiple edges. For instance, $L(\mathfrak{S}_{n-1})$ is a multiplicity-free subalgebra of $L(\mathfrak{S}_n)$, while the subalgebra $L(\mathfrak{S}_{n-k})$ of $L(\mathfrak{S}_n)$ has multiplicity if $k \geq 2$ (see Example 7.5.1).

Exercise 7.5.2 Let $\mathcal{B}$ be a subalgebra of $\mathcal{A}$. Let $\{I_{\lambda,\rho,j} : j = 1, 2, \ldots, m_{\lambda,\rho}\}$ be a basis for $\mathrm{Hom}_{\mathcal{B}}(Y_\rho, \mathrm{Res}^{\mathcal{A}}_{\mathcal{B}} W_\lambda)$, $\lambda \in \widehat{\mathcal{A}}$, $\rho \in \widehat{\mathcal{B}}$. Show that $S \in \mathcal{A}$ belongs to $\mathcal{B}$ if and only if for all $\rho \in \widehat{\mathcal{B}}$ there exists $S_\rho \in \mathrm{End}(Y_\rho)$ such that

$$SI_{\lambda,\rho,j} v = I_{\lambda,\rho,j} S_\rho v,$$

for all $\lambda \in \widehat{\mathcal{A}}$, $j = 1, 2, \ldots, m_{\lambda,\rho}$ and $v \in Y_\rho$.

Exercise 7.5.3 ([48]) Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ and $\mathcal{D}$ be finite dimensional $*$-algebras.

(1) Suppose that $\mathcal{B} \subseteq \mathcal{A}$, $\mathcal{C} \subseteq \mathcal{A}$ and that $\mathcal{B}$ and $\mathcal{C}$ are isomorphic, say $\mathcal{B} \cong \mathcal{C} \cong \oplus_{\rho \in R} \mathrm{End}(Y_\rho)$.
Show that $\mathfrak{M}(\mathcal{A}, \mathcal{B}) = \mathfrak{M}(\mathcal{A}, \mathcal{C})$, if and only if $\mathcal{B}$ and $\mathcal{C}$ are conjugate in $\mathcal{A}$, that is, there exists a unitary operator $T \in \mathcal{A}$ such that $T\mathcal{C}T^* = \mathcal{B}$.
(2) Suppose that $\mathcal{B} \subseteq \mathcal{A}$ and $\mathcal{D} \subseteq \mathcal{C}$ and that the inclusions have the same Bratteli diagrams. Show that there exists a $*$-isomorphism $\Phi : \mathcal{A} \to \mathcal{C}$ such that $\Phi(\mathcal{B}) = \mathcal{D}$.
(3) Suppose that $\mathcal{C} \subseteq \mathcal{B} \subseteq \mathcal{A}$. Show that $\mathfrak{M}(\mathcal{A}, \mathcal{C}) = \mathfrak{M}(\mathcal{A}, \mathcal{B})\mathfrak{M}(\mathcal{B}, \mathcal{C})$.

Hint. (1) Use Exercise 7.5.2 and choose the operators $I_{\lambda,\rho,j}$ in such a way that the decomposition $W_\lambda = \oplus_{\rho \in \widehat{\mathcal{B}}} \oplus_{j=1}^{m_{\lambda,\rho}} I_{\lambda,\rho,j} Y_\rho$ is orthogonal.

Example 7.5.4 Let $\mathcal{A} = M_{8,8}(\mathbb{C}) \oplus M_{8,8}(\mathbb{C})$, $\mathcal{B} = [M_{4,4}(\mathbb{C}) \oplus M_{4,4}(\mathbb{C})] \oplus M_{8,8}(\mathbb{C})$ and $\mathcal{C} = M_{8,8}(\mathbb{C}) \oplus [M_{4,4}(\mathbb{C}) \oplus M_{4,4}(\mathbb{C})]$. Note that $\mathcal{B}$ and $\mathcal{C}$ are isomorphic subalgebras of $\mathcal{A}$ with $\mathfrak{M}(\mathcal{A}, \mathcal{B}) \neq \mathfrak{M}(\mathcal{A}, \mathcal{C})$, so that they are not conjugate in $\mathcal{A}$. However, the Bratteli diagrams of the corresponding inclusions are isomorphic and the flip $\Phi : \mathcal{A} \to \mathcal{A}$ which exchanges the two blocks of $\mathcal{A}$ clearly satisfies $\Phi(\mathcal{B}) = \mathcal{C}$.

7.5.2 The centralizer of a subalgebra

Let $\mathcal{A}$ be a finite dimensional $*$-algebra and $\mathcal{B}$ a $*$-closed subalgebra. The *centralizer of* $\mathcal{B}$ *in* $\mathcal{A}$ is the $*$-subalgebra of $\mathcal{A}$ given by

$$\mathcal{C}(\mathcal{A}, \mathcal{B}) = \{A \in \mathcal{A} : AB = BA \text{ for all } B \in \mathcal{B}\}.$$

For instance, if $\mathcal{A} = \mathrm{End}(V)$ then $\mathcal{C}(\mathcal{A}, \mathcal{B}) \equiv \mathcal{B}'$. In general, if $\mathcal{A}$ is a $*$-algebra of operators on V, then $\mathcal{C}(\mathcal{A}, \mathcal{B}) \equiv \mathcal{A} \cap \mathcal{B}'$, where $\mathcal{B}'$ is the commutant of $\mathcal{B}$ in $\mathrm{End}(V)$. Let (σ, U) be an $\mathcal{A}$-representation and let (ρ, Y) be a $\mathcal{B}$-representation. Consider the space of intertwiners $\mathrm{Hom}_{\mathcal{B}}(Y, \mathrm{Res}^{\mathcal{A}}_{\mathcal{B}} U)$. There is a natural representation τ of $\mathcal{C}(\mathcal{A}, \mathcal{B})$ on this space: for $C \in \mathcal{C}(\mathcal{A}, \mathcal{B})$, $T \in \mathrm{Hom}_{\mathcal{B}}(Y, \mathrm{Res}^{\mathcal{A}}_{\mathcal{B}} U)$, the operator $\tau(C)T$ is defined by setting

$$[\tau(C)T](y) = \sigma(C)Ty,$$

for all $y \in Y$, that is, $\tau(C)T \equiv \sigma(C)T$ (the action of $\tau(C)$ on T is the product $\sigma(C)T$).

Lemma 7.5.5 τ *is a representation.*

Proof It suffices to prove that $\tau(C)T \in \mathrm{Hom}_{\mathcal{B}}(Y, \mathrm{Res}^{\mathcal{A}}_{\mathcal{B}} U)$ for all $C \in \mathcal{C}(\mathcal{A}, \mathcal{B})$ and $T \in \mathrm{Hom}_{\mathcal{B}}(Y, \mathrm{Res}^{\mathcal{A}}_{\mathcal{B}} U)$. For $B \in \mathcal{B}$ and $y \in Y$, we have:

$$\begin{aligned} \{[\tau(C)T]\rho(B)\}(y) &= \sigma(C)T\rho(B)y \\ (T \in \mathrm{Hom}_{\mathcal{B}}(Y, \mathrm{Res}^{\mathcal{A}}_{\mathcal{B}} U)) \quad &= \sigma(C)\sigma(B)Ty \\ (C \in \mathcal{C}(\mathcal{A}, \mathcal{B})) \quad &= \sigma(B)\sigma(C)Ty \\ &= \{\sigma(B)[\tau(C)T]\}(y) \end{aligned}$$

that is,

$$[\tau(C)T]\rho(B) = \sigma(B)[\tau(C)T]$$

and therefore $\tau(C)T \in \mathrm{Hom}_{\mathcal{B}}(Y, \mathrm{Res}^{\mathcal{A}}_{\mathcal{B}} U)$. □

The following theorem (and its corollaries) is a slight generalization of Proposition 1.4 in [100] (see also Lemma 1.0.1 [73]).

Theorem 7.5.6 *Let $\mathcal{A}$ and $\mathcal{B}$ be finite dimensional $*$-algebras and suppose that $\mathcal{B} \le \mathcal{A}$. Let $\mathcal{A} \cong \bigoplus_{\lambda \in \widehat{\mathcal{A}}} \mathrm{End}(W_\lambda)$ and $\mathcal{B} \cong \bigoplus_{\rho \in \widehat{\mathcal{B}}} \mathrm{End}(V_\rho)$ be their decompositions into a direct sum of simple algebras and suppose that $\mathrm{Res}^{\mathcal{A}}_{\mathcal{B}} W_\lambda = \bigoplus_{\rho \in \widehat{\mathcal{B}}} m_{\lambda,\rho} V_\rho$ for every $\lambda \in \widehat{\mathcal{A}}$. Then we have a $*$-isomorphism:*

$$\mathcal{C}(\mathcal{A}, \mathcal{B}) \cong \bigoplus_{\lambda \in \widehat{\mathcal{A}}} \bigoplus_{\rho \in \widehat{\mathcal{B}}} M_{m_{\lambda,\rho}, m_{\lambda,\rho}}(\mathbb{C}).$$

Proof From Proposition 7.4.9 it follows that, for every $C \in \mathcal{A}$,

$$C \in \mathcal{C}(\mathcal{A}, \mathcal{B}) \Leftrightarrow \sigma_\lambda(C) \in \mathcal{C}(\sigma_\lambda(\mathcal{A}), \sigma_\lambda(\mathcal{B}))$$

for all $\lambda \in \widehat{\mathcal{A}}$, and therefore

$$\mathcal{C}(\mathcal{A}, \mathcal{B}) \cong \bigoplus_{\lambda \in \widehat{\mathcal{A}}} \mathcal{C}(\sigma_\lambda(\mathcal{A}), \sigma_\lambda(\mathcal{B})).$$

Since $\sigma_\lambda(\mathcal{A}) \cong \operatorname{End}(W_\lambda)$, we may apply Theorem 7.2.8 and Theorem 7.3.4 to conclude that

$$\mathcal{C}(\sigma_\lambda(\mathcal{A}), \sigma_\lambda(\mathcal{B})) \cong \bigoplus_{\rho \in \widehat{\mathcal{B}}} M_{m_{\lambda,\rho}, m_{\lambda,\rho}}(\mathbb{C}). \qquad \square$$

Exercise 7.5.7 Use Exercise 7.5.2 to reformulate the proof of Theorem 7.5.6.

Corollary 7.5.8 *A subalgebra $\mathcal{B} \leq \mathcal{A}$ is multiplicity-free if and only if the centralizer $\mathcal{C}(\mathcal{A}, \mathcal{B})$ is commutative.*

Corollary 7.5.9 *In the notation of Lemma 7.5.5, if U is $\mathcal{A}$-irreducible and Y is $\mathcal{B}$-irreducible then $\operatorname{Hom}_{\mathcal{B}}(Y, \operatorname{Res}^{\mathcal{A}}_{\mathcal{B}} U)$ is $\mathcal{C}(\mathcal{A}, \mathcal{B})$-irreducible.*

Proof Combine Theorem 7.5.6 with Theorem 7.2.8 and Lemma 7.2.9. $\square$

We end this section by observing that given a subgroup $H \leq G$, then Corollary 7.5.8 with $L(H) = \mathcal{B}$ and $L(G) = \mathcal{A}$ yields another proof of the equivalence between (i) and (iii) in Theorem 2.1.10.

7.5.3 A reciprocity law for restriction

Let $\mathcal{B} \subseteq \mathcal{A}$ be an inclusion of $*$-algebras of operators on a complex vector space V. We have also the inclusion $\mathcal{A}' \subseteq \mathcal{B}'$. Applying the structure theorem 7.3.4 we get a multiplicity-free decomposition

$$V \cong \bigoplus_{\lambda \in \widehat{\mathcal{A}}} (Z_\lambda \otimes W_\lambda) \tag{7.21}$$

with respect to the action θ of $\mathcal{A}' \otimes \mathcal{A}$ (where $Z_\lambda = \operatorname{Hom}_{\mathcal{A}}(W_\lambda, V)$). Similarly, under the action of $\mathcal{B}' \otimes \mathcal{B}$ we have

$$V \cong \bigoplus_{\rho \in \widehat{\mathcal{B}}} (U_\rho \otimes Y_\rho) \tag{7.22}$$

where $U_\rho = \operatorname{Hom}_{\mathcal{B}}(Y_\rho, V)$. We recall that we use the same index set $\widehat{\mathcal{A}}$ (resp. $\widehat{\mathcal{B}}$) for the irreducible representations of $\mathcal{A}$ and $\mathcal{A}'$ (resp. $\mathcal{B}$ and $\mathcal{B}'$); see the comments after the proof of Proposition 7.4.8.

Following [49], we say that $\mathcal{B} \subseteq \mathcal{A}$ and $\mathcal{A}' \subseteq \mathcal{B}'$ form a *seesaw pair*.

Theorem 7.5.10 (Seesaw reciprocity [49], [48]) *The inclusion matrix $\mathfrak{M}(\mathcal{B}', \mathcal{A}')$ for the pair $\mathcal{A}' \subseteq \mathcal{B}'$ is the transpose of the inclusion matrix $\mathfrak{M}(\mathcal{A}, \mathcal{B})$. In other words, for $\lambda \in \widehat{\mathcal{A}}$ and $\rho \in \widehat{\mathcal{B}}$, the multiplicity of Y_ρ in $\operatorname{Res}^{\mathcal{A}}_{\mathcal{B}} W_\lambda$ is equal to the multiplicity of Z_λ in $\operatorname{Res}^{\mathcal{B}'}_{\mathcal{A}'} U_\rho$.*

Proof First note that $\mathcal{A}' \otimes \mathcal{B}$ is a subalgebra of both $\mathcal{A}' \otimes \mathcal{A}$ and $\mathcal{B}' \otimes \mathcal{B}$. In particular, $\theta(\mathcal{A}' \otimes \mathcal{B})$ is a $*$-algebra of operators on V, that is, $\mathcal{A}' \otimes \mathcal{B}$ acts on V via $\mathrm{Res}^{\mathcal{A}'\otimes\mathcal{A}}_{\mathcal{A}'\otimes\mathcal{B}}\theta$ (note that $\mathrm{Res}^{\mathcal{B}'\otimes\mathcal{B}}_{\mathcal{A}'\otimes\mathcal{B}}$ leads to the same representation).

Suppose that $\mathrm{Res}^{\mathcal{A}}_{\mathcal{B}} W_\lambda = \bigoplus_{\rho\in\widehat{\mathcal{B}}} m_{\lambda,\rho} Y_\rho$ and that $\mathrm{Res}^{\mathcal{B}'}_{\mathcal{A}'} U_\rho = \bigoplus_{\lambda\in\widehat{\mathcal{A}}} n_{\rho,\lambda} Z_\lambda$, that is, $\mathfrak{M}(\mathcal{A},\mathcal{B}) = (m_{\lambda,\rho})_{\lambda\in\widehat{\mathcal{A}},\rho\in\widehat{\mathcal{B}}}$ and $\mathfrak{M}(\mathcal{B}',\mathcal{A}') = (n_{\rho,\lambda})_{\rho\in\widehat{\mathcal{B}},\lambda\in\widehat{\mathcal{A}}}$.

If the irreducible representation $Z_{\lambda_0} \otimes Y_{\rho_0}$, $\lambda_0 \in \widehat{\mathcal{A}}$ and $\rho_0 \in \widehat{\mathcal{B}}$ of $\mathcal{A}' \otimes \mathcal{B}$ appears in V, then it is necessarily contained both in $Z_{\lambda_0} \otimes W_{\lambda_0}$ and in $U_{\rho_0} \otimes Y_{\rho_0}$ (cf. Theorem 7.3.4 and (7.21), (7.22)). Then, we may compute the multiplicity of $Z_{\lambda_0} \otimes Y_{\rho_0}$ in V in two ways. First of all,

$$\begin{aligned}\mathrm{Res}^{\mathcal{A}'\otimes\mathcal{A}}_{\mathcal{A}'\otimes\mathcal{B}}(Z_{\lambda_0} \otimes W_{\lambda_0}) &= Z_{\lambda_0} \otimes \mathrm{Res}^{\mathcal{A}}_{\mathcal{B}} W_{\lambda_0} \\ &= \bigoplus_{\rho\in\widehat{\mathcal{B}}} m_{\lambda_0,\rho}(Z_{\lambda_0} \otimes Y_\rho)\end{aligned}$$

and therefore, the multiplicity is equal to m_{λ_0,ρ_0}.

On the other hand, from

$$\begin{aligned}\mathrm{Res}^{\mathcal{B}'\otimes\mathcal{B}}_{\mathcal{A}'\otimes\mathcal{B}}(U_{\rho_0} \otimes Y_{\rho_0}) &= \left(\mathrm{Res}^{\mathcal{B}'}_{\mathcal{A}'} U_{\rho_0}\right) \otimes Y_{\rho_0} \\ &= \bigoplus_{\lambda\in\widehat{\mathcal{A}}} n_{\rho_0,\lambda}(Z_\lambda \otimes Y_{\rho_0})\end{aligned}$$

we get that the multiplcity is also equal to n_{ρ_0,λ_0}.

We have shown that $m_{\lambda_0,\rho_0} = n_{\rho_0,\lambda_0}$ and this ends the proof. □

In the following exercise, we connect Theorem 7.5.10 with the results in Subsection 7.5.2. $\mathcal{C}(\mathcal{A},\mathcal{B})$ denotes the centralizer of $\mathcal{B}$ in $\mathcal{A}$ and θ is the representation of $\mathcal{A}' \otimes \mathcal{A}$ on V.

Exercise 7.5.11

(1) Show that

$$\mathcal{C}(\mathcal{A},\mathcal{B}) = \mathcal{C}(\mathcal{B}',\mathcal{A}') = [\theta(\mathcal{A}' \otimes \mathcal{B})]'.$$

(2) Show that the spaces $\mathrm{Hom}_{\mathcal{B}}(Y_\rho, \mathrm{Res}^{\mathcal{A}}_{\mathcal{B}} W_\lambda)$, $\mathrm{Hom}_{\mathcal{A}'}(Z_\lambda, \mathrm{Res}^{\mathcal{B}'}_{\mathcal{A}'} U_\rho)$ and $\mathrm{Hom}_{\mathcal{A}'\otimes\mathcal{B}}(Z_\lambda \otimes Y_\rho, V)$ are isomorphic as irreducible $\mathcal{C}(\mathcal{A},\mathcal{B})$-representations. Construct explicit isomorphisms.

(3) Set $X_{\lambda,\rho} = \mathrm{Hom}_{\mathcal{B}}(Y_\rho, \mathrm{Res}^{\mathcal{A}}_{\mathcal{B}} W_\lambda)$. Show that

$$V = \bigoplus_{\lambda\in\widehat{\mathcal{A}}} \bigoplus_{\rho\in\widehat{\mathcal{B}}} [X_{\lambda,\rho} \otimes Z_\lambda \otimes Y_\rho]$$

is the (multiplicity-free) decomposition of V into irreducible $(\mathcal{C}(\mathcal{A},\mathcal{B}) \otimes \mathcal{A}' \otimes \mathcal{B})$-representations.

Corollary 7.5.12 *Let G be a finite group and $H \leq G$ a subgroup. Let $\mathfrak{M}(\widehat{G}, \widehat{H}) = (m_{\lambda,\rho})_{\lambda\in\widehat{G},\rho\in\widehat{H}}$ be the corresponding inclusion matrix. Let (σ, V) be a unitary representation of G. Then the inclusion matrix of the pair $\mathrm{End}_G(V) \subseteq \mathrm{End}_H[\mathrm{Res}^G_H V]$ is the transpose of $\mathfrak{M}(\widehat{G}, \widehat{H})$.*

Exercise 7.5.13 (Jones' basic constuction) Let $\mathcal{A}$ and $\mathcal{B}$ be two finite dimensional $*$-algebras on the same space V, and suppose that $\mathcal{B} \subseteq \mathcal{A}$. Consider the representation θ of $\mathcal{B}$ on $\mathcal{A}$ defined by setting

$$\theta(T)(S) = TS$$

for all $T \in \mathcal{B}$ and $S \in \mathcal{A}$. Note that, in the notation of Section 7.4.5, we have $\theta = \mathrm{Res}^{\mathcal{A}}_{\mathcal{B}}\lambda$, where λ is the left regular representation of $\mathcal{A}$. Set $\mathcal{C} = \mathrm{End}_{\mathcal{B}}(\mathcal{A})$ and consider $\mathcal{A}$ as a subalgebra of $\mathrm{End}_{\mathcal{B}}(\mathcal{A})$ by identifying $A \in \mathcal{A}$ with $\rho(A) \in \mathrm{End}_{\mathcal{B}}(\mathcal{A})$, where ρ is the right regular representation of $\mathcal{A}$ (use Proposition 7.4.13). Observe that $\mathcal{C}$ is the commutant of $\theta(\mathcal{B})$ in $\mathrm{End}(\mathcal{A})$, so that there is a natural bijection between $\widehat{\mathcal{C}}$ and $\widehat{\mathcal{B}}$. Show that

$$\mathfrak{M}(\mathcal{C}, \mathcal{A}) = \mathfrak{M}(\mathcal{A}, \mathcal{B}).$$

Hint. Use Proposition 7.4.13 to show that $\rho(\mathcal{A}) \subseteq \mathcal{C}$ and $\theta(\mathcal{B}) \subseteq \lambda(\mathcal{A})$ is a seesaw pair.

7.5.4 A reciprocity law for induction

In this section, we present a reciprocity law for induction of representations of finite groups.

Let G be a finite group and $H \leq G$ a subgroup. If (θ, N) is a (unitary) representation of H, we show that there is a natural inclusion $\mathrm{End}_H(N) \hookrightarrow \mathrm{End}_G[\mathrm{Ind}^G_H N]$ and we show that the inclusion matrix for these algebras is a submatrix of the inclusion matrix of $L(H) \subseteq L(G)$.

We first need a preliminary result. Let $G = \coprod_{s\in S} sH$ be the decomposition of G into right cosets of H (we may suppose $1_G \in S$). Let (θ_i, N_i), $i = 1, 2$, be two (unitary) representations of H and set $V_i = \mathrm{Ind}^G_H N_i$ and $\sigma_i = \mathrm{Ind}^G_H \theta_i$, $i = 1, 2$. In view of (1.48) (see also Lemma 1.6.2) we may suppose that N_i is a subspace of V_i,

$$V_i = \bigoplus_{s\in S} \sigma_i(s)N_i, \tag{7.23}$$

and that $\sigma_i(h)u = \theta_i(h)u$, for all $h \in H$, $u \in N_i$, $i = 1, 2$.

Given $T \in \mathrm{Hom}_H(N_1, N_2)$, we define an operator $\widetilde{T} : V_1 \to V_2$ by setting

$$\widetilde{T}\sigma_1(s)u = \sigma_2(s)Tu$$

for all $u \in N_1$, $s \in S$, and using (7.23) to extend $\widetilde{T}$ to the whole V_1.

Proposition 7.5.14
(i) *The definition of the operator $\widetilde{T}$ does not depend on the choice of the representative set S.*
(ii) *For every $T \in \mathrm{Hom}_H(N_1, N_2)$ we have $\widetilde{T} \in \mathrm{Hom}_G(V_1, V_2)$ and the map $T \mapsto \widetilde{T}$ is linear and injective.*
(iii) *If $N_1 = N_2$, and therefore $V_1 = V_2$, then $\{\widetilde{T} : T \in \mathrm{End}_H(N_1)\}$ is a subalgebra of $\mathrm{End}_G(V_1)$ isomorphic to $\mathrm{End}_H(N_1)$.*

Proof (i) For $s \in S$, $h \in H$ and $u \in N_1$ we have

$$\begin{aligned}\widetilde{T}\sigma_1(sh)u &= \widetilde{T}\sigma_1(s)[\sigma_1(h)u]\\ &= \sigma_2(s)T[\theta_1(h)u]\\ &= \sigma_2(s)\theta_2(h)Tu\\ &= \sigma_2(sh)Tu,\end{aligned}$$

and this shows that using sh as a representative for sH, we get the same operator on $\sigma_1(s)N_1$.

(ii) Let $g \in G$, $s \in S$ and $u \in N_1$. Let $t \in S$ and $h \in H$ be such that $gs = th$. We then have

$$\begin{aligned}\widetilde{T}\sigma_1(g)[\sigma_1(s)u] &= \widetilde{T}\sigma_1(t)[\sigma_1(h)u]\\ &= \sigma_2(t)T\theta_1(h)u\\ &= \sigma_2(t)\theta_2(h)Tu\\ &= \sigma_2(g)\sigma_2(s)Tu\\ &= \sigma_2(g)\widetilde{T}[\sigma_1(s)u]\end{aligned}$$

and this shows that $\widetilde{T} \in \mathrm{Hom}_G(V_1, V_2)$.

(iii) This is obvious. □

Note that in [19] we denoted the operator $\widetilde{T}$ by $\mathrm{Ind}_H^G T$. It was called the operator obtained by inducing T from H to G (or, simply, the *induced operator*).

Let now (θ, N) be a (unitary) representation of H and set $V = \mathrm{Ind}_H^G N$. We introduce the following notation: $\mathcal{A} = \mathrm{End}_G(V)$ and $\mathcal{B} = \{\widetilde{T} : T \in \mathrm{End}_H(N)\} \subseteq \mathcal{A}$.

The following proposition is obvious.

Proposition 7.5.15 *We have that $\mathcal{B} = \{A \in \mathcal{A} : AN \subseteq N\}$.*

Let

$$N = \bigoplus_{\rho \in R} a_\rho Y_\rho \tag{7.24}$$

be the decomposition of N into irreducible representations, where $R \subseteq \widehat{H}$ and $a_\rho > 0$ for all $\rho \in R$. By Theorem 7.4.14, we also have the decomposition of N into irreducible $\mathrm{End}_H(N) \otimes L(H)$ representations

$$N = \bigoplus_{\rho \in R} (U_\rho \otimes Y_\rho),$$

where $U_\rho = \mathrm{Hom}_H(Y_\rho, N)$.

Set $\widetilde{U}_\rho = \{\widetilde{T} : T \in \mathrm{Hom}_H(Y_\rho, N)\}$. Then, by Proposition 7.5.14, we have $\widetilde{U}_\rho \subseteq \mathrm{Hom}_G(\mathrm{Ind}_H^G(Y_\rho), \mathrm{Ind}_H^G(N))$. The algebra $\mathcal{B}$ acts on $\widetilde{U}_\rho$ in the obvious way: if $\widetilde{B} \in \mathcal{B}$ (so that $B \in \mathrm{End}_H(N)$) and $\widetilde{T} \in \widetilde{U}_\rho$ (so that $T \in \mathrm{Hom}_H(Y_\rho, N)$), then

$$\widetilde{B}\widetilde{T} = \widetilde{BT}$$

yields a representation of $\mathcal{B}$ on $\widetilde{U}_\rho$ which is a translation, by means of the isomorphism $\mathcal{B} \cong \mathrm{End}_H(N)$, of the irreducible representation U_ρ (compare with (7.7)). In other words, $\{\widetilde{U}_\rho : \rho \in R\}$ is a complete set of irreducible and pairwise inequivalent representations of $\mathcal{B}$ and $\mathcal{B} \cong \bigoplus_{\rho \in R} \mathrm{End}(\widetilde{U}_\rho)$.

Let now

$$V = \bigoplus_{\lambda \in \Lambda} b_\lambda W_\lambda \tag{7.25}$$

be the decomposition of V into irreducible G-representations, with $\Lambda \subseteq \widehat{G}$ and $b_\lambda > 0$ for all $\lambda \in \Lambda$. Moreover,

$$V = \bigoplus_{\lambda \in \Lambda} (Z_\lambda \otimes W_\lambda)$$

as in Theorem 7.4.14.

Theorem 7.5.16 (Reciprocity law for induction) *The multiplicity of* U_ρ *in* $\mathrm{Res}^{\mathcal{A}}_{\mathcal{B}} Z_\lambda$ *is equal to the multiplicity of* W_λ *in* $\mathrm{Ind}_H^G Y_\rho$ *(and therefore, by Frobenius reciprocity Theorem 1.6.11, to the multiplicity of* Y_ρ *in* $\mathrm{Res}^G_H W_\lambda$*). In other words, the inclusion matrix* $\mathfrak{M}(\Lambda, R)$ *(where* $\Lambda \equiv \widehat{\mathcal{A}}$ *and* $R \equiv \widehat{\mathcal{B}}$*) is a submatrix of* $\mathfrak{M}(\widehat{G}, \widehat{H})$ *(taking* $\Lambda \subseteq \widehat{G}$, $R \subseteq \widehat{H}$*).*

Proof Let $\{I_{\rho,1}, I_{\rho,2}, \dots I_{\rho,a_\rho}\}$ be a basis for U_ρ, $\rho \in R$ so that (7.24) becomes

$$N = \bigoplus_{\rho \in R} \bigoplus_{j=1}^{a_\rho} I_{\rho,j} Y_\rho \tag{7.26}$$

(compare with (1.9) and (7.4); we may also suppose that the decomposition is orthogonal).

Inducing up to G, (7.26) becomes

$$V = \mathrm{Ind}_H^G N = \bigoplus_{\rho\in R}\bigoplus_{j=1}^{a_\rho} \widetilde{I}_{\rho,j}\mathrm{Ind}_H^G Y_\rho \tag{7.27}$$

and, in particular, $\{\widetilde{I}_{\rho,1}, \widetilde{I}_{\rho,2}, \dots \widetilde{I}_{\rho,a_\rho}\}$ is a basis for $\widetilde{U}_\rho$, for all $\rho \in R$.

For $\lambda \in \Lambda$ and $\rho \in R$, let $m_{\lambda,\rho}$ be the multiplicity of W_λ in $\mathrm{Ind}_H^G Y_\rho$ and take a basis $\{J_{\lambda,\rho,1}, J_{\lambda,\rho,2}, \dots J_{\lambda,\rho,m_{\lambda,\rho}}\}$ of $\mathrm{Hom}_G(W_\lambda, \mathrm{Ind}_H^G(Y_\rho))$, so that

$$\mathrm{Ind}_H^G Y_\rho = \bigoplus_{\lambda\in\Lambda}\bigoplus_{i=1}^{m_{\lambda,\rho}} J_{\lambda,\rho,i}W_\lambda \tag{7.28}$$

is a decomposition of $\mathrm{Ind}_H^G Y_\rho$ into irreducible subspaces (and we may again suppose that it is orthogonal). From (7.27) and (7.28) we get

$$V = \bigoplus_{\lambda\in\Lambda}\bigoplus_{\rho\in R}\bigoplus_{i=1}^{m_{\lambda,\rho}}\bigoplus_{j=1}^{a_\rho} \widetilde{I}_{\rho,j}J_{\lambda,\rho,i}W_\lambda$$

which is a more explicit and structured version of (7.25). Note that, in particular, the compatibility equations

$$\sum_{\rho\in R} m_{\lambda,\rho}a_\rho = b_\lambda$$

for all $\lambda \in \Lambda$ must be satisfied. Moreover, for every $\lambda \in \Lambda$, the set

$$\{\widetilde{I}_{\rho,j}J_{\lambda,\rho,i} : \rho \in R,\ j = 1, 2, \dots, a_\rho, i = 1, 2, \dots, m_{\lambda,\rho}\}$$

is a basis for Z_λ.

But for each $\rho \in R$ and for all $i = 1, 2, \dots, m_{\lambda,\rho}$, the subspace

$$\langle \widetilde{I}_{\rho,j}J_{\lambda,\rho,i} : j = 1, 2, \dots, a_\rho\rangle$$

is $\mathcal{B}$-invariant and clearly equivalent to $\widetilde{U}_\rho$ as a $\mathcal{B}$-representation. Therefore

$$\mathrm{Res}_{\mathcal{B}}^{\mathcal{A}} Z_\lambda = \bigoplus_{\rho\in R} m_{\lambda,\rho}\widetilde{U}_\rho$$

and the theorem is proved. □

Note that we have proved that the map

$$\mathcal{T} : \mathrm{Hom}_G(W_\lambda, \mathrm{Ind}_H^G Y_\rho) \to \mathrm{Hom}_{\mathcal{B}}(\widetilde{U}_\rho, \mathrm{Res}_{\mathcal{B}}^{\mathcal{A}} Z_\lambda),$$

given by $\mathcal{T}(J)\widetilde{T} = \widetilde{T}J$, for all $\widetilde{T} \in \widetilde{U}_\rho$ and $J \in \mathrm{Hom}_G(W_\lambda, \mathrm{Ind}_H^G Y_\rho)$, is a linear isomorphism.

Example 7.5.17 We now present an example which constitutes the main application of Theorem 7.5.16. It will be used both in the next subsection and in the next chapter.

Let (σ, V) be a (unitary) representation of G. Set $X = G/H$ and let $x_0 \in X$ be a point stabilized by H. Let us apply the construction that has led to Theorem 7.5.16 to $N = \mathrm{Res}^G_H V$. By Corollary 1.6.10 we have that

$$\mathrm{Ind}^G_H N \equiv \mathrm{Ind}^G_H(\mathrm{Res}^G_H V) \cong V \otimes L(X).$$

Now (7.23) coincides simply with

$$V \otimes L(X) \cong \bigoplus_{x \in X} (V \otimes \delta_x).$$

Moreover, if $T \in \mathrm{End}_H(\mathrm{Res}^G_H V)$, $v \in V$, $g \in G$ and $x = gx_0 \in X$, then

$$\begin{aligned}\widetilde{T}(v \otimes \delta_x) &= \widetilde{T}\{\theta(g)[\sigma(g^{-1})v \otimes \delta_{x_0}]\}\\ &= \theta(g)[T\sigma(g^{-1})v \otimes \delta_{x_0}]\\ &= [\sigma(g)T\sigma(g^{-1})v] \otimes \delta_x\end{aligned}$$

where $\theta = \mathrm{Ind}^G_H[\mathrm{Res}^G_H \sigma]$, that is, $\theta(g)(v \otimes \delta_x) = \sigma(g)v \otimes \delta_{gx}$.

Note also that if $\mathcal{A} = \mathrm{End}_G[\mathrm{Ind}^G_H N]$ and $\mathcal{B} = \{\widetilde{T} : T \in \mathrm{End}_H(N)\} \subseteq \mathcal{A}$, then Proposition 7.5.15 yields the following characterization of $\mathcal{B}$:

$$\mathcal{B} = \{A \in \mathrm{End}_G[V \otimes L(X)] : A(V \otimes \delta_{x_0}) = V \otimes \delta_{x_0}\}. \tag{7.29}$$

7.5.5 Iterated tensor product of permutation representations

In this subsection, we anticipate, in an abstract form, a general construction that leads, as a particular case, to the Partition Algebras, which will be treated in the next chapter (see [58], [89] and [54]). Since this construction is an immediate application of the theory developed in this section, we found it natural to place it at the end of this chapter.

Let G be a finite group and $H \le G$ a subgroup. Denote by $X = G/H$ the corresponding homogeneus space and let $x_0 \in X$ be a point stabilized by H. Also denote by λ the permutation representation of G on X: $[\lambda(g) f](x) = f(g^{-1}x)$ for all $g \in G$ and $x \in X$. Set

$$L(X)^{\otimes k} = \underbrace{L(X) \otimes L(X) \otimes \cdots \otimes L(X)}_{k\text{-times}}$$

and $\lambda_k = \underbrace{\lambda \otimes \lambda \otimes \cdots \otimes \lambda}_{k\text{-times}}$.

The linear isomorphism

$$\begin{array}{ccc} L(X)^{\otimes k} & \longrightarrow & L(\underbrace{X \times X \times \cdots \times X}_{k\text{-times}}) \\ \delta_{x_1} \otimes \delta_{x_2} \otimes \cdots \otimes \delta_{x_k} & \mapsto & \delta_{(x_1, x_2, \ldots, x_k)} \end{array} \tag{7.30}$$

is also an isomorphism of G-representations, since

$$\lambda_k(g)[\delta_{x_1} \otimes \delta_{x_2} \otimes \cdots \otimes \delta_{x_k}] = \delta_{gx_1} \otimes \delta_{gx_2} \otimes \cdots \otimes \delta_{gx_k}.$$

We now make the assumption that H is a multiplicity-free subgroup of G (see Section 2.1.2). It is not essential but it rather simplifies the notation and the statements of the results. In Exercise 7.5.20 we leave to the reader the easy task of generalizing these results when the subgroup H is not multiplicity-free.

Using Corollary 1.6.10 (see also Example 7.5.17) we decompose $L(X)^{\otimes k}$ in the following way. First of all, note that

$$L(X)^{\otimes k} \cong L(X)^{\otimes (k-1)} \otimes L(X) = \mathrm{Ind}_H^G[\mathrm{Res}_H^G L(X)^{\otimes (k-1)}]$$

where the equality follows from Corollary 1.6.10. Therefore, to construct $L(X)^{\otimes k}$, we may start with the trivial representation W_0 of G and then, for k times, alternatively restricting from G to H and inducing from H to G. We thus write formally

$$L(X)^{\otimes k} \cong [\mathrm{Ind}_H^G \mathrm{Res}_H^G]^k W_0$$

where $[\mathrm{Ind}_H^G \mathrm{Res}_H^G]^k = \underbrace{\mathrm{Ind}_H^G \mathrm{Res}_H^G \mathrm{Ind}_H^G \mathrm{Res}_H^G \cdots \mathrm{Ind}_H^G \mathrm{Res}_H^G}_{k\text{-times}}.$

At any stage, we may use the fact that H is multiplicity-free: if W is an irreducible representation of G contained in $L(X)^{\otimes (k-1)}$, then $\mathrm{Res}_H^G W$ decomposes without multiplicities. On the other hand, if Y is an irreducible representation of H contained in $\mathrm{Res}_H^G W$, then $\mathrm{Ind}_H^G Y$ decomposes again without multiplicity. Starting from $L(X)$ which is multiplicity-free, say $L(X) \cong \oplus_{j \in J} W_j$, where $W_j \in \widehat{G}$ are pairwise inequivalent, we construct paths of representations as follows. Given $j_1 \in J$, we choose $Y_{i_1} \in \widehat{H}$ which is contained in $\mathrm{Res}_H^G W_{j_1}$. Then, we choose $j_2 \in J$ such that W_{j_2} is contained in $\mathrm{Ind}_H^G Y_{i_1}$. And so on, thus obtaining a path

$$p : W_0 \to Y_0 \to W_{j_1} \to Y_{i_1} \to W_{j_2} \to Y_{i_2} \to \cdots \to W_{j_{k-1}} \to Y_{i_{k-1}} \to W_{j_k} \tag{7.31}$$

where W_0 (resp. Y_0) is the trivial representation of G (resp. H) and Y_{i_ℓ} (resp. $W_{j_{\ell+1}}$) is an irreducible H-sub-representation (resp. G-sub-representation) of $\mathrm{Res}_H^G W_{j_\ell}$ (resp. $\mathrm{Ind}_H^G Y_{i_\ell}$), $\ell = 1, 2, \ldots, k-1$.

Denoting by W_p the last representation in the path p, (in other words, $W_p = W_{j_k}$ in (7.31)), we have the following orthogonal decomposition:

$$L(X)^{\otimes k} \cong \bigoplus_p W_p$$

where the sum runs over all paths p as above.

In order to formalize this procedure, we introduce a suitable *Bratteli diagram*. The levels of the diagram are parameterized by the numbers $\{0, \frac{1}{2}, 1, 1+\frac{1}{2}, 2, 2+\frac{1}{2}, \ldots\}$. For an integer $k \geq 0$, the kth level is formed by those irreducible representations of G that are contained in $L(X)^{\otimes k}$, while, the $(k+\frac{1}{2})$nd level is formed by those irreducible representations of H that are contained in $\mathrm{Res}^G_H[L(X)^{\otimes k}]$. If $W_j \in \widehat{G}$ belongs to the kth (resp. $(k+1)$st) level and $Y_i \in \widehat{H}$ belongs to the the $(k+\frac{1}{2})$nd level, then we draw an edge connecting W_j and Y_i if Y_i is contained in $\mathrm{Res}^G_H W_j$ (resp. if W_j is contained in $\mathrm{Ind}^G_H Y_i$). The 0th (resp. $\frac{1}{2}$st) level is formed only by the trivial representation $W_0 \in \widehat{G}$ (resp. $Y_0 \in \widehat{H}$). Clearly, a representation $W_j \in \widehat{G}$ (resp. $Y_i \in \widehat{H}$) belongs to the kth (resp. $(k+\frac{1}{2})$nd) level if and only if there exists a path p as in (7.31) starting with $W_0 \to Y_0$ and ending in W_j (resp. Y_i). Thus, a good parameterization of $\widehat{G}$ and $\widehat{H}$ and a clear description of the branching rule for Ind^G_H and Res^G_H, may lead to an explicit description of the Bratteli diagram (see Example 7.5.19 below).

Let now algebras come into the play. Set

$$\mathcal{A}_k = \mathrm{End}_G\left[L(X)^{\otimes k}\right]$$

and

$$\mathcal{A}_{k+\frac{1}{2}} = \mathrm{End}_H\left\{\mathrm{Res}^G_H\left[L(X)^{\otimes k}\right]\right\}.$$

Then, clearly $\mathcal{A}_k$ is a subalgebra of $\mathcal{A}_{k+\frac{1}{2}}$. On the other hand, by Proposition 7.5.14, we may identify $\mathcal{A}_{k+\frac{1}{2}}$ with a subalgebra of

$$\mathcal{A}_{k+1} = \mathrm{End}_G\left[L(X)^{\otimes(k+1)}\right] \equiv \mathrm{End}_G\left\{\mathrm{Ind}^G_H \mathrm{Res}^G_H\left[L(X)^{\otimes k}\right]\right\}.$$

Therefore, we have a *chain* (or *tower*) of algebras

$$\mathcal{A}_0 \subseteq \mathcal{A}_{\frac{1}{2}} \subseteq \mathcal{A}_1 \subseteq \mathcal{A}_{1+\frac{1}{2}} \subseteq \mathcal{A}_2 \subseteq \cdots \subseteq \mathcal{A}_k \subseteq \mathcal{A}_{k+\frac{1}{2}} \subseteq \mathcal{A}_{k+1} \subseteq \cdots \quad (7.32)$$

which is multiplicity-free. Indeed, $\mathrm{Res}^{\mathcal{A}_{k+\frac{1}{2}}}_{\mathcal{A}_k}$ (resp. $\mathrm{Res}^{\mathcal{A}_{k+1}}_{\mathcal{A}_{k+\frac{1}{2}}}$) is multiplicity-free by virtue of Theorem 7.5.10 (resp. Theorem 7.5.16). Note that, in order to apply these theorems, the kth level of the Bratteli diagram may be identified with $\widehat{\mathcal{A}_k}$

(cf. the comments after the proof of Proposition 7.4.8 and Theorem 7.4.14). Moreover, by Theorem 7.4.14,

$$L(X)^{\otimes k} \cong \bigoplus_{\lambda \in \widehat{\mathcal{A}}_k} (Z_\lambda \otimes W_\lambda)$$

and

$$\mathcal{A}_k \cong \bigoplus_{\lambda \in \widehat{\mathcal{A}}_k} \mathrm{End}(Z_\lambda)$$

(where $\{Z_\lambda : \lambda \in \widehat{\mathcal{A}}_k\}$ are the irreducible representations of $\widehat{\mathcal{A}}_k$ and $\{W_\lambda : \lambda \in \widehat{\mathcal{A}}_k\}$ are the irreducible representations of G contained in $L(X)^{\otimes k}$).

Similarly, the $(k + \frac{1}{2})$nd level may be identified with $\widehat{\mathcal{A}}_{k+\frac{1}{2}}$ and we have:

$$L(X)^{\otimes k} \cong \bigoplus_{\rho \in \widehat{\mathcal{A}}_{k+\frac{1}{2}}} (U_\rho \otimes Y_\rho)$$

and

$$\mathcal{A}_{k+\frac{1}{2}} \cong \bigoplus_{\rho \in \widehat{\mathcal{A}}_{k+\frac{1}{2}}} \mathrm{End}(U_\rho)$$

(where $\{U_\rho : \rho \in \widehat{\mathcal{A}}_{k+\frac{1}{2}}\}$ are the irreducible representations of $\widehat{\mathcal{A}}_{k+\frac{1}{2}}$ and $\{Y_\rho : \rho \in \widehat{\mathcal{A}}_{k+\frac{1}{2}}\}$ are the irreducible representations of H contained in $\mathrm{Res}^G_H[L(X)^{\otimes k}]$).

In the following theorem, we summarize and conclude the discussion above. We recall that a path is a sequence as in (7.31).

Theorem 7.5.18

(i) *The chain* (7.32) *is multiplicity-free.*

(ii) *For* $\lambda \in \widehat{\mathcal{A}}_k$*, the dimension of* Z_λ*, which is also the multiplicity of* W_λ *in* $L(X)^{\otimes k}$*, is equal to the number of paths starting from* $\widehat{\mathcal{A}}_0$ *and ending at* λ*.*

(iii) *For* $\rho \in \widehat{\mathcal{A}}_{k+\frac{1}{2}}$*, the dimension of* U_ρ*, which is also the multiplicity of* Y_ρ *in* $\mathrm{Res}^G_H[L(X)^{\otimes k}]$*, is equal to the number of paths starting from* $\widehat{\mathcal{A}}_0$ *and ending at* ρ*.*

(iv) *If* $\lambda \in \widehat{\mathcal{A}}_k$ *and* $\rho \in \widehat{\mathcal{A}}_{k+\frac{1}{2}}$ *then* Z_λ *is contained in* $\mathrm{Res}^{\widehat{\mathcal{A}}_{k+\frac{1}{2}}}_{\widehat{\mathcal{A}}_k} U_\rho$ *if and only if* Y_ρ *is contained in* $\mathrm{Res}^G_H W_\lambda$*.*

(v) *If* $\lambda \in \widehat{\mathcal{A}}_{k+1}$ *and* $\rho \in \widehat{\mathcal{A}}_{k+\frac{1}{2}}$ *then* U_ρ *is contained in* $\mathrm{Res}^{\widehat{\mathcal{A}}_{k+1}}_{\widehat{\mathcal{A}}_{k+\frac{1}{2}}} Z_\lambda$ *if and only if* W_λ *is contained in* $\mathrm{Ind}^G_H Y_\rho$*.*

Clearly, (iv) follows from Theorem 7.5.10, while (v) follows from Theorem 7.5.16. In view of (iv) and (v), the diagram constructed above will be called the *Bratteli diagram of the chain* (7.32).

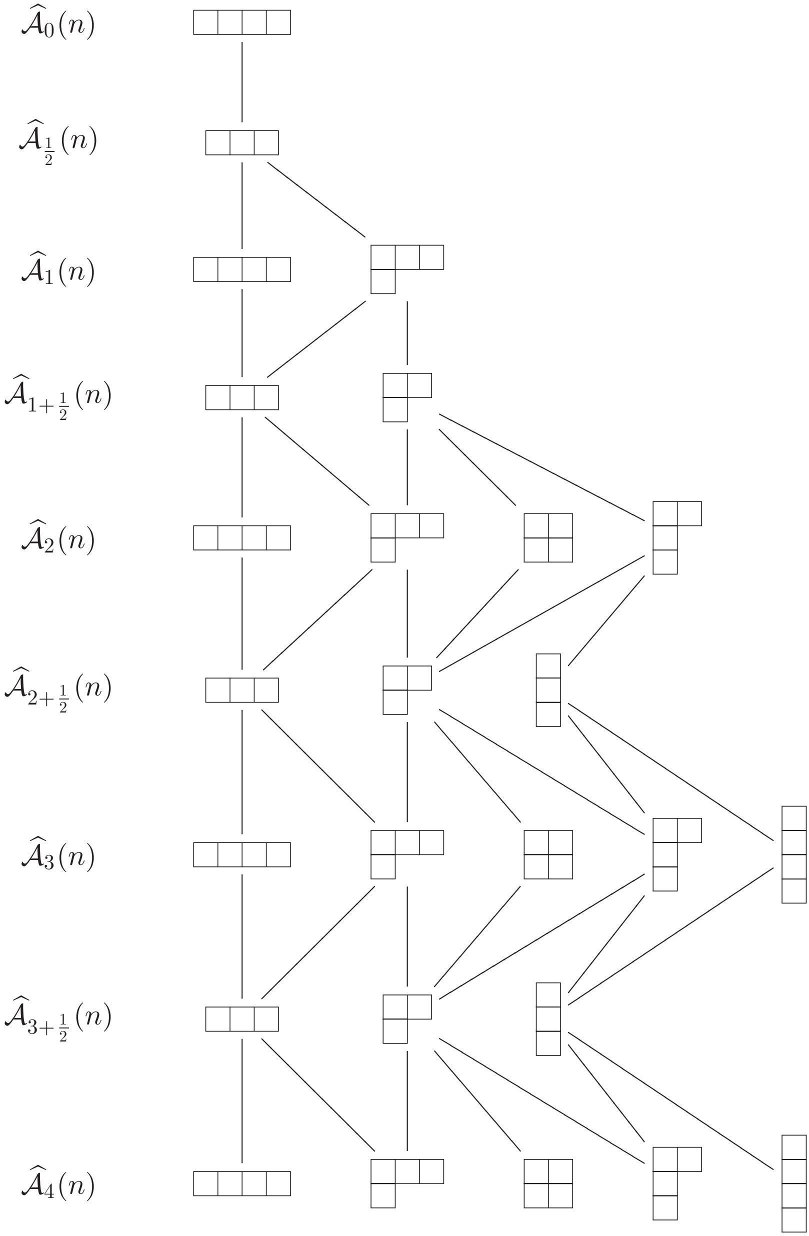


Figure 7.3 The Bratteli diagram for $n = 4$.

Example 7.5.19 The canonical example for the construction developed in the present subsection is the following. Take $G = \mathfrak{S}_n$, $H = \mathfrak{S}_{n-1}$, so that $X = \{1, 2, \ldots, n\}$ and (with the notation in Section 3.6.2) $L(X) = M^{n-1,1}$. Then

$$\mathcal{A}_k(n) = \mathrm{End}_{\mathfrak{S}_n}\left\{\left[M^{n-1,1}\right]^{\otimes k}\right\}$$

and

$$\mathcal{A}_{k+\frac{1}{2}}(n) = \mathrm{End}_{\mathfrak{S}_{n-1}}\left\{\mathrm{Res}^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}}\left[M^{n-1,1}\right]^{\otimes k}\right\}.$$

Moreover, from the branching rule for the pair $\mathfrak{S}_{n-1} \leq \mathfrak{S}_n$ (see Corollary 3.3.11) we immediately get

$$\widehat{\mathcal{A}}_k(n) = \{\lambda \vdash n : n - \lambda_1 \leq k\}$$

and

$$\widehat{\mathcal{A}}_{k+\frac{1}{2}}(n) = \{\lambda \vdash n - 1 : n - \lambda_1 \leq k\}.$$

Indeed, the corresponding Bratteli diagram may be constructed by means of the branching rule for $\mathfrak{S}_n$: $\widehat{\mathcal{A}}_{k+\frac{1}{2}}(n)$ is the set of all partitions that may be obtained by removing a box from some partition in $\widehat{\mathcal{A}}_k(n)$, and $\widehat{\mathcal{A}}_{k+1}(n)$ is the set of all partitions that may be obtained by adding a box to some partition in $\widehat{\mathcal{A}}_{k+\frac{1}{2}}(n)$. These rules also determine the edges in the diagram (see Corollary 3.3.11 for the branching rule for $\mathfrak{S}_n$). We illustrate this in the case $n = 4$ (see Figure 7.3).

Exercise 7.5.20 Generalize Theorem 7.5.18 to the case when the subgroup H is not multiplicity-free.

8

Schur–Weyl dualities and the partition algebra

The Schur–Weyl duality is a cornerstone of the representation theory. It establishes a connection between the representation theory of finite groups and the theory of (continuous) representations of classical Lie groups. We examine the commutant of two representations of the symmetric groups $\mathfrak{S}_n$ and $\mathfrak{S}_k$ on the iterated tensor product $\underbrace{V \otimes V \otimes \cdots \otimes V}_{k\text{-times}}$, where $V \cong \mathbb{C}^n$.

In the first case, the commutant is spanned by the operators of the natural representation of the general linear group $\mathrm{GL}(n, \mathbb{C})$ and this yields an important class of irreducible representations of this group. We shall not discuss the Lie-theoretical aspects, but we shall rather examine in detail a few topics such as the branching rule and the Littlewood–Richardson rule for the irreducible representations obtained in this way.

In the second case, the commutant is an algebra called the partition algebra because it is described in terms of partitions of the set $\{1, 2, \ldots, 2k\}$. We limit ourselves to presenting a brief (though detailed) introduction to the partition algebra, focusing again on the properties of the irreducible representations obtained by means of the Schur–Weyl construction.

8.1 Symmetric and antisymmetric tensors

In this section, V is a fixed n-dimensional vector space over $\mathbb{C}$ and $\{e_1, e_2, \ldots, e_n\}$ is a basis of V. We shall also suppose that V is endowed with a Hermitian scalar product $\langle \cdot, \cdot \rangle_V$ and that $\{e_1, e_2, \ldots, e_n\}$ is an orthonormal system, even though this requirement will not be always necessary. Actually, we may identify V with $\mathbb{C}^n$ and take $\{e_1, e_2, \ldots, e_n\}$ as the standard basis. Our treatment is based on the monographs by Goodman and Wallach [49], Sterberg [115], Simon [111], Fulton and Harris [43] and Clerc [21].

8.1.1 Iterated tensor product

In this section, we introduce the basic object of the present chapter: the iterated tensor product $V^{\otimes k} = \underbrace{V \otimes V \otimes \cdots \otimes V}_{k\text{-times}}$. We have already introduced tensor products in Section 1.1.7 (see also Section 7.5.5). Here, we treat more closely the space $V^{\otimes k}$ and we give three different (though equivalent) descriptions of it. We set $V^{\times k} = \underbrace{V \times V \times \cdots \times V}_{k\text{-times}}$ (direct product of k copies of V).

The first one, that we call *analytic* coincides with that one in Section 1.1.7 (which is taken from [111, Appendix A]): $V^{\otimes k}$ is the vector space of all bi-antilinear maps $B : V^{\times k} \to \mathbb{C}$. If $v_1, v_2, \ldots, v_k \in V$ the *simple* (or *decomposable*) tensor product $v_1 \otimes v_2 \otimes \cdots \otimes v_k$ is defined by setting

$$(v_1 \otimes v_2 \otimes \cdots \otimes v_k)(u_1, u_2, \ldots, u_k) = \prod_{i=1}^{k} \langle u_i, v_i \rangle_V,$$

for all $(u_1, u_2, \ldots, u_k) \in V^{\times k}$.

The space $V^{\otimes k}$ may be endowed with the Hermitian scalar product defined by setting

$$\langle u_1 \otimes u_2 \otimes \cdots \otimes u_k, v_1 \otimes v_2 \otimes \cdots \otimes v_k \rangle = \prod_{i=1}^{k} \langle u_i, v_i \rangle_V$$

and the set $\{e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_k} : i_1, i_2, \ldots, i_k \in \{1, 2, \ldots, n\}\}$ is an orthonormal basis for $V^{\otimes k}$.

The second definition is *algebraic* (or *categorical*). We recall that if U is another vector space, a map $\varphi : V^{\times k} \to U$ is multilinear if it is linear in each variable, that is,

$$V \ni v \mapsto \varphi(v_1, v_2, \ldots, v_{h-1}, v, v_{h+1}, \ldots, v_n) \in U$$

is a linear map for every $h \in \{1, 2, \ldots, k\}$ and all $v_1, v_2, \ldots, v_{h-1}, v_{h+1}, \ldots, v_n \in V$. Then, the tensor product $V^{\otimes k}$ may be also defined in the following way. It is a vector space equipped with a multilinear map

$$\Phi : V^{\times k} \to V^{\otimes k}$$

with the following *universal* property: if U is a vector space and $\Psi : V^{\times k} \to U$ is a multilinear map, then there exists a unique linear map $\theta : V^{\otimes k} \to U$ such that $\theta \circ \Phi = \Psi$, that is, such that the diagram

$$\begin{array}{ccc} V^{\times k} & \xrightarrow{\Phi} & V^{\otimes k} \\ & \searrow^{\Psi} & \downarrow{\scriptstyle\theta} \\ & & U \end{array}$$

is commutative. We leave it to the reader the easy task to verify that the tensor product defined in the first way satisfies the requirements of the second definition and that it proves the existence of $V^{\otimes k}$ (even if the use of a scalar product is not very natural in this approach) and to also prove the uniqueness of the tensor product defined in the second way. For more on the algebraic approach of the tensor product, we refer to [43], [76] and, especially, to [115, Appendix B] which is very detailed and friendly.

There is also a third and very elementary description of $V^{\otimes k}$, that we call *combinatorial*. We have already described it in Section 7.5.5. Set $[n] = \{1, 2, \dots, n\}$ and identify $V \cong \mathbb{C}^n \cong L([n])$. Set also $[n]^k = \underbrace{[n]\times[n]\times\cdots\times[n]}_{k\text{-times}}$. In other words, $[n]^k$ is the set of all ordered k-tuples $(i_1, i_2, \dots, i_k)$ of integers in $\{1, 2, \dots, n\}$. Then we have the natural isomorphisms

$$V^{\otimes k} \cong L([n])^{\otimes k} \cong L([n]^k) \tag{8.1}$$

(cf. (7.30)). The isomorphism between the left- and right-hand sides of (8.1) is given by the map

$$e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_k} \mapsto \delta_{(i_1,i_2,\dots,i_k)},$$

for all $(i_1, i_2, \dots, i_k) \in [n]^k$.

We also have

$$[\mathrm{End}(V)]^{\otimes k} \cong \mathrm{End}\left(V^{\otimes k}\right) \tag{8.2}$$

as in Section 1.1.7: given $A_1, A_2, \dots, A_k \in \mathrm{End}(V)$, the tensor product $A_1 \otimes A_2 \otimes \cdots \otimes A_k$ may be seen as the operator on $V^{\otimes k}$ defined by setting

$$(A_1 \otimes A_2 \otimes \cdots \otimes A_k)(v_1 \otimes v_2 \otimes \cdots \otimes v_k) = A_1 v_1 \otimes A_2 v_2 \otimes \cdots \otimes A_k v_k$$

for all decomposable tensors $v_1 \otimes v_2 \otimes \cdots \otimes v_k$, and then extended it by linearity.

In the combinatorial description of $V^{\otimes k}$, the isomorphism (8.2) may be derived by (1.22):

$$\mathrm{End}(L([n]^k)) \cong L([n]^k \times [n]^k) \cong L([n]^k) \otimes L([n]^k)$$

and this yields a matrix type description of $\mathrm{End}(V^{\otimes k})$. Indeed, with each $F \in L([n]^k \times [n]^k)$ we may associate the linear map $T_F \in \mathrm{End}(L([n]^k))$ defined by setting

$$T_F(e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_k}) = \sum_{j_1, j_2, \ldots, j_k = 1}^{n} F(j_1, j_2, \ldots, j_k; i_1, i_2, \ldots, i_k) e_{j_1} \otimes e_{j_2} \otimes \cdots \otimes e_{j_k}. \tag{8.3}$$

8.1.2 The action of $\mathfrak{S}_k$ on $V^{\otimes k}$

The symmetric group $\mathfrak{S}_k$ acts on $V^{\otimes k}$ in the obvious way: by *permuting the coordinates*. That is, we may define an action σ_k of $\mathfrak{S}_k$ on $V^{\otimes k}$ by setting

$$\sigma_k(\pi)(v_1 \otimes v_2 \otimes \cdots \otimes v_k) = v_{\pi^{-1}(1)} \otimes v_{\pi^{-1}(2)} \otimes \cdots \otimes v_{\pi^{-1}(k)}$$

for all $\pi \in \mathfrak{S}_k$ and all decomposable tensors $v_1 \otimes v_2 \otimes \cdots \otimes v_k$. It is easy to check that σ_k is indeed a (unitary) representation of $\mathfrak{S}_k$ on $V^{\otimes k}$.

From the combinatorial description point of view, the definition of σ_k can be given in the following clearer way. Consider the action of $\mathfrak{S}_k$ on $[n]^k$ given by

$$\pi(i_1, i_2, \ldots, i_k) = (i_{\pi^{-1}(1)}, i_{\pi^{-1}(2)}, \ldots, i_{\pi^{-1}(k)}). \tag{8.4}$$

Now, identifying $[n]^k$ with $[n]^{[k]} = \{\theta : \{1, 2, \ldots, k\} \to \{1, 2, \ldots, n\}\}$, by $[n]^{[k]} \ni \theta \leftrightarrow (\theta(1), \theta(2), \ldots, \theta(k)) \in [n]^k$, then (8.4) becomes $\pi(\theta) = \theta \circ \pi^{-1}$. Then, σ_k is the corresponding permutation representation of $\mathfrak{S}_k$ on $L([n]^k)$. Compare also with the construction in Section 3.7.4.

The combinatorial description of $V^{\otimes k}$ easily leads to the decomposition of $V^{\otimes k}$ into irreducible $\mathfrak{S}_k$-sub-representations. Denote by $C(k, n)$ the set of all compositions of k in n parts, where each part is ≥ 0 (cf. Section 3.5.3). With each $a = (a_1, a_2, \ldots, a_n) \in C(k, n)$ we associate the orbit $\Omega_a \subset [n]^k$ of $\mathfrak{S}_k$ consisting of all $(i_1, i_2, \ldots, i_k) \in [n]^k$ such that $|\{j : i_j = h\}| = a_h$, for all $h = 1, 2, \ldots, n$. Clearly,

$$[n]^k = \coprod_{a \in C(k,n)} \Omega_a \tag{8.5}$$

is the decomposition of $[n]^k$ into the $\mathfrak{S}_k$-orbits. Consequently, we have the following decomposition of $V^{\otimes k}$ as a direct sum of permutation representations:

$$V^{\otimes k} \cong \bigoplus_{a \in C(k,n)} M^a \tag{8.6}$$

where M^a is the Young module associated with the composition a (cf. Section 3.6.2).

In the tensor product notation, $\mathfrak{S}_k$ permutes the vectors of the basis $\{e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_k} : 1 \le i_j \le n, 1 \le j \le k\}$, and the stabilizer of the vector

$$\underbrace{e_1 \otimes e_1 \otimes \cdots \otimes e_1}_{a_1\text{-times}} \otimes \underbrace{e_2 \otimes e_2 \otimes \cdots \otimes e_2}_{a_2\text{-times}} \otimes \cdots \otimes \underbrace{e_k \otimes e_k \otimes \cdots \otimes e_k}_{a_k\text{-times}} \tag{8.7}$$

is the Young subgroup $S_{a_1} \times S_{a_2} \times \cdots \times S_{a_k}$. The $\mathfrak{S}_k$-translates of (8.7) span M^a.

Lemma 8.1.1 *We have*

$$|C(k,n)| = \binom{n+k-1}{n-1} \equiv \binom{n+k-1}{k}.$$

Proof Given integers $1 \le h \le m$, we denote by $\Omega(m,h)$ the set of all h-subsets of $[m]$. We sketch two proofs. In the first proof, with each $a = (a_1, a_2, \ldots, a_n) \in C(k,n)$ we associate the following $(n-1)$-subset of $[n+k-1]$:

$$Q_a = \{a_1 + 1, a_1 + a_2 + 2, \ldots, a_1 + a_2 + \cdots + a_{n-1} + n - 1\}.$$

It is easy to see that the map $C(k,n) \ni a \to Q_a \in \Omega(n+k-1, n-1)$ is a bijection. In the second proof, with the sequence

$$(\underbrace{1,1,\ldots,1}_{a_1\text{-times}}, \underbrace{2,2,\ldots,2}_{a_2\text{-times}}, \ldots, \underbrace{n,n,\ldots,n}_{a_n\text{-times}}) = (i_1, i_2, \ldots, i_k)$$

we associate the k-subset of $[n+k-1]$

$$R_a = \{i_1, i_2 + 1, \ldots, i_k + k - 1\}.$$

Again, the map $C(k,n) \ni a \to R_a \in \Omega(n+k-1, k)$ is a bijection. □

Note that, in the second proof above, we have counted the number of k-*multisets* of an n-set: see the monographs by Aigner [2], Cameron [16] and Stanley [112].

8.1.3 Symmetric tensors

The *symmetrizing operator* on $V^{\otimes k}$ is the linear map $\mathrm{Sym} : V^{\otimes k} \to V^{\otimes k}$ defined by setting

$$\mathrm{Sym} := \frac{1}{k!} \sum_{\pi \in \mathfrak{S}_k} \sigma_k(\pi),$$

where σ_k is the representation of $\mathfrak{S}_k$ on $V^{\otimes k}$ defined in Section 8.1.2. In other words,

$$\mathrm{Sym}(v_1 \otimes_2 \otimes \cdots \otimes v_k) = \frac{1}{k!} \sum_{\pi \in \mathfrak{S}_k} v_{\pi^{-1}(1)} \otimes v_{\pi^{-1}(2)} \otimes \cdots \otimes v_{\pi^{-1}(k)},$$

for every decomposable tensor $v_1 \otimes v_2 \otimes \cdots \otimes v_k$. Lemma 1.3.5 ensures that Sym is the orthogonal projection on $V^{\otimes k}$ onto the isotypic component of the trivial representation of $\mathfrak{S}_k$ (see also Corollary 1.3.15). The image of Sym, that we simply denote by $\mathrm{Sym}(V^{\otimes k})$, is called the space of *symmetric k-tensors* over V or the kth *symmetric power* of V. In other monographs, it is denoted by $S^k(V)$. We also set $\mathrm{Sym}(V^{\otimes 0}) \equiv \mathbb{C}$. In other words, a tensor $T \in V^{\otimes k}$ belong to $\mathrm{Sym}(V^{\otimes k})$ if and only if $\sigma_k(\pi)T = T$ for all $\pi \in \mathfrak{S}_k$. For instance, $v \otimes w + w \otimes v$ is a tensor in $\mathrm{Sym}(V^{\otimes 2})$ for all $v, w \in V$. From (8.6) and Lemma 8.1.1, it follows immediately that

$$\mathrm{dim}\mathrm{Sym}(V^{\otimes k}) = \binom{n+k-1}{k}. \tag{8.8}$$

Given $v_1, v_2, \ldots, v_k \in V$, following the monograph by Fulton and Harris [43], we set

$$\begin{aligned} v_1 \cdot v_2 \cdot \ldots \cdot v_k &:= k!\, \mathrm{Sym}(v_1 \otimes_2 \otimes \cdots \otimes v_k) \\ &\equiv \sum_{\pi \in \mathfrak{S}_k} v_{\pi(1)} \otimes v_{\pi(2)} \otimes \cdots \otimes v_{\pi(k)}. \end{aligned}$$

Clearly, the set $\{e_{i_1} \cdot e_{i_2} \cdot \ldots \cdot e_{i_k} : 1 \le i_1 \le i_2 \le \cdots \le i_k \le n\}$ is an orthogonal (not normalized) basis for $\mathrm{Sym}(V^{\otimes k})$.

An alternative basis is the following: $\{e_a = \sum_{(i_1, i_2, \ldots, i_k) \in \Omega_a} e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_k} : a \in C(k, n)\}$, where Ω_a is as in (8.5). Clearly, if $(i_1, i_2, \ldots, i_k) \in \Omega_a$ then

$$e_{i_1} \cdot e_{i_2} \cdot \ldots \cdot e_{i_k} = a_1! a_2! \cdots a_k! e_a. \tag{8.9}$$

Consider the multilinear map

$$\begin{aligned} \varphi : \quad V^{\times k} &\to \mathrm{Sym}(V^{\otimes k}) \\ (v_1, v_2, \ldots, v_k) &\mapsto v_1 \cdot v_2 \cdot \ldots \cdot v_k. \end{aligned}$$

Note that φ is *symmetric*, that is, $\varphi(v_{\pi(1)}, v_{\pi(2)}, \ldots, v_{\pi(k)}) = \varphi(v_1, v_2, \ldots, v_k)$ for all $\pi \in \mathfrak{S}_k$ and $v_1, v_2, \ldots, v_k \in V$. This map is universal in the following sense. If W is another vector space and $\Psi : V^{\times k} \to W$ is a symmetric multilinear map, then there exists a unique linear map $\psi : \mathrm{Sym}(V^{\otimes k}) \to W$ such that the diagram

$$\begin{array}{ccc} V^{\times k} & \xrightarrow{\varphi} & \mathrm{Sym}(V^{\otimes k}) \\ & \searrow^{\Psi} & \downarrow \psi \\ & & W \end{array} \tag{8.10}$$

commutes (that is, $\Psi = \psi \circ \varphi$). Indeed, we may set

$$\psi(v_1 \cdot v_2 \cdot \ldots \cdot v_k) = \Psi(v_1, v_2, \ldots, v_k).$$

Note that ψ is well defined (because Ψ is symmetric) and it is the unique map that makes the diagram in (8.10) commutative. It is easy to check that this universal property characterizes $\mathrm{Sym}(V^{\otimes k})$.

Suppose now that $V = V_1 \oplus V_2$, with $\dim V_1 = m$, and choose the basis $\{e_1, e_2, \ldots, e_n\}$ of V in such a way that the first m vectors are in V_1 and the remaining $n - m$ vectors are in V_2. For $0 \leq h \leq k$ consider the linear map

$$\begin{array}{rcl} \mathrm{Sym}(V_1^{\otimes h}) \bigotimes \mathrm{Sym}(V_2^{\otimes(k-h)}) & \to & \mathrm{Sym}\left[(V_1 \oplus V_2)^{\otimes k}\right] \\ (v_1 \cdot v_2 \cdot \ldots \cdot v_h) \otimes (v_{h+1} \cdot v_{h+2} \cdot \ldots \cdot v_k) & \mapsto & v_1 \cdot v_2 \cdot \ldots \cdot v_k \end{array} \tag{8.11}$$

where $v_1, v_2, \ldots, v_h \in V_1$ and $v_{h+1}, v_{h+2}, \ldots, v_k \in V_2$.

Proposition 8.1.2 *The map* (8.11) *determines an isomorphism*

$$\mathrm{Sym}\left[(V_1 \oplus V_2)^{\otimes k}\right] \cong \bigoplus_{h=0}^{k} \left[\mathrm{Sym}(V_1^{\otimes h}) \bigotimes \mathrm{Sym}(V_2^{\otimes(k-h)})\right].$$

Proof Just note that if $1 \leq i_1 \leq i_2 \leq \cdots \leq i_h \leq m$ and $m + 1 \leq i_{m+1} \leq i_{m+2} \leq \cdots \leq i_k \leq n$, then we have

$$(e_{i_1} \cdot e_{i_2} \cdot \ldots \cdot e_{i_h}) \otimes (e_{i_{h+1}} \cdot e_{i_{h+2}} \cdot \ldots \cdot e_{i_k}) \mapsto e_{i_1} \cdot e_{i_2} \cdot \ldots \cdot e_{i_k}. \qquad \square$$

Note that from Proposition 8.1.2 and (8.8) we get an algebraic proof of the combinatorial identity

$$\sum_{h=0}^{k} \binom{m+h+1}{m-1} \binom{n-m+k-h+1}{n-m-1} = \binom{n+k-1}{n-1},$$

which is a variation of the Vandermonde identity (see the book by Graham, Knuth and Patashnik [51]).

Now we state and prove a technical but fundamental result on $\mathrm{Sym}(V^{\otimes k})$ which constitutes the essential tool for development of the duality between the symmetric group $\mathfrak{S}_k$ and the general linear group $\mathrm{GL}(n, \mathbb{C})$. For $v \in V$, we set

$$v^{\otimes k} = \underbrace{v \otimes v \otimes \cdots \otimes v}_{k\text{-times}}.$$

We suppose that $V \cong \mathbb{C}^n$ is endowed with the standard topology.

Theorem 8.1.3 *Let $U \subseteq V$ be a nontrivial open subset of V. Then the set*

$$\{u^{\otimes k} : u \in U\}$$

spans the whole of $\mathrm{Sym}(V^{\otimes k})$.

Proof Let $\varphi : \mathrm{Sym}(V^{\otimes k}) \to \mathbb{C}$ be a linear map such that

$$\varphi(u^{\otimes k}) = 0 \tag{8.12}$$

for all $u \in U$. If $u = \sum_{j=1}^n x_j e_j$, then we have

$$\varphi(u^{\otimes k}) = \sum_{a \in C(k,n)} x^a \varphi(e_a)$$

where e_a is as in (8.9) and $x^a = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$. In other words, $\varphi(u^{\otimes k})$ is a homogeneous polynomial of degree k in the variables $x_1, x_2, \ldots, x_n$. Since by (8.12) such a polynomial vanishes on a nontrivial open set of $\mathbb{C}^n$, then necessarily it is the zero polynomial; equivalently its coefficients must be all equal to zero. Thus $\varphi(e_a) = 0$ for all $a \in C(k, n)$, that is $\varphi = 0$, and this shows that the set $\{u^{\otimes k} : u \in U\}$ spans the whole of $\mathrm{Sym}(V^{\otimes k})$. □

As an example, we have

$$v_1 \otimes v_2 + v_2 \otimes v_1 = \frac{1}{2}[(v_1 + v_2) \otimes (v_1 + v_2) - (v_1 - v_2) \otimes (v_1 - v_2)]$$

(see also (8.14) in the exercise below).

In the following exercise, we sketch an alternative more constructive proof of a weaker version of Theorem 8.1.3. It is taken from the book by Goodman and Wallach [49, Lemma B.25]; see also the monograph by Bump [15, Proposition 38.1].

Exercise 8.1.4 (1) Let $C_2^{k-1} = \{(\gamma_2, \gamma_3, \ldots, \gamma_k) : \gamma_i \in \{-1, 1\}, i = 2, 3, \ldots, k\}$ be the product of $(k-1)$ copies of the (multiplicative) cyclic group $C_2 = \{-1, 1\}$. Identify the dual group $\widehat{C_2^{k-1}}$ with the collection of all subsets of $\{2, 3, \ldots, k\}$ by setting, for $J \subseteq \{2, 3, \ldots, k\}$,

$$\chi^J(\gamma) = \prod_{j \in J} \gamma_j$$

for all $\gamma = (\gamma_2, \gamma_3, \ldots, \gamma_k) \in C_2^{k-1}$. Show that the Fourier transform on C_2^{k-1} and its inversion formula are given by

$$\widehat{f}(J) = \sum_{\gamma \in C_2^{k-1}} f(\gamma)\chi^J(\gamma)$$

and

$$f(\gamma) = \frac{1}{2^{k-1}} \sum_{J \in \widehat{C_2^{k-1}}} \widehat{f}(J)\chi^J(\gamma),$$

for all $f \in L(C_2^{k-1})$.

(2) Let $\Phi : \mathrm{Sym}(V^{\otimes k}) \to \mathbb{C}$ be a linear map. Given $v_1, v_2, \ldots v_h \in V$ set

$$f(\gamma) = \Phi\left[(v_1 + \gamma_2 v_2 + \cdots + \gamma_k v_k)^{\otimes k}\right] \tag{8.13}$$

for all $\gamma \in C_2^{k-1}$. Set $J_0 = \{2, 3, \ldots, k\}$. Show that

$$\widehat{f}(J_0) = 2^{k-1}k!\Phi\left[\mathrm{Sym}(v_1 \otimes v_2 \otimes \cdots \otimes v_k)\right].$$

(3) Deduce from (2) the following *polarization identity*:

$$\begin{aligned}&\mathrm{Sym}(v_1 \otimes v_2 \otimes \cdots \otimes v_k)\\ &\quad = \frac{1}{2^{k-1}k!} \sum_{\gamma \in C_2^{k-1}} \left(\prod_{j=2}^{k} \gamma_j\right) (v_1 + \gamma_2 v_2 + \cdots + \gamma_k v_k)^{\otimes k}.\end{aligned} \tag{8.14}$$

(4) Deduce from (3) that if $U \subseteq V$ is a dense subset, then $\{u^{\otimes k} : u \in U\}$ spans the whole of $\mathrm{Sym}(V^{\otimes k})$.

Hint. (2) Expand the tensor product in (8.13) and consider the simple tensors of the form $v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_k}$, with $i_1, i_2, \ldots, i_k$ distinct.

(3) Write down the expression for $\widehat{f}(J_0)$ and use the fact that the map Φ is arbitrary.

8.1.4 Antisymmetric tensors

Let σ_k be again the representation of $\mathfrak{S}_k$ on $V^{\otimes k}$ defined in Section 8.1.2. The *alternating operator* on $V^{\otimes k}$ is the linear map $\mathrm{Alt} : V^{\otimes k} \to V^{\otimes k}$ defined by setting

$$\mathrm{Alt} := \frac{1}{k!} \sum_{\pi \in \mathfrak{S}_k} \varepsilon(\pi)\sigma_k(\pi)$$

where ε is the alternating representation of $\mathfrak{S}_k$. In other words,

$$\mathrm{Alt}(v_1 \otimes_2 \otimes \cdots \otimes v_k) = \frac{1}{k!} \sum_{\pi \in \mathfrak{S}_k} \varepsilon(\pi) v_{\pi^{-1}(1)} \otimes v_{\pi^{-1}(2)} \otimes \cdots \otimes v_{\pi^{-1}(k)}$$

for every decomposable tensor $v_1 \otimes v_2 \otimes \cdots \otimes v_k$.

From Corollary 1.3.15 it follows that Alt is the orthogonal projection of $V^{\otimes k}$ onto the isotypic component of ϵ in $V^{\otimes k}$. The image of Alt, namely $\mathrm{Alt}(V^{\otimes k})$, is called the space of *antisymmetric* (or *alternating*) k-tensors over V. It is also called the kth *exterior power* of V and it is often denoted by $\bigwedge^k V$. Finally, we set $\mathrm{Alt}(V^{\otimes 0}) = \mathbb{C}$.

Clearly, a tensor $T \in V^{\otimes k}$ belongs to $\mathrm{Alt}(V^{\otimes k})$ if and only if $\sigma_k(\pi)T = \varepsilon(\pi)T$, for all $\pi \in \mathfrak{S}_k$.

Given $v_1, v_2, \ldots, v_k \in V$, we set

$$\begin{aligned} v_1 \wedge v_2 \wedge \cdots \wedge v_k &:= k!\, \mathrm{Alt}(v_1 \otimes_2 \otimes \cdots \otimes v_k) \\ &\equiv \sum_{\pi \in \mathfrak{S}_k} \varepsilon(\pi) v_{\pi(1)} \otimes v_{\pi(2)} \otimes \cdots \otimes v_{\pi(k)} \end{aligned}$$

(here we follow the monograph by Fulton and Harris [43]; in the book by Simon [111] there is a $\sqrt{k!}$ in place of $k!$, while in the book by Goodman and Wallach [49] there is no normalizing factor in front of Alt).

Proposition 8.1.5 *The set*

$$\{e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k} : 1 \leq i_1 < i_2 < \cdots < i_k \leq n\}$$

is an orthogonal basis for $\mathrm{Alt}(V^{\otimes k})$. *In particular,* $\dim \mathrm{Alt}(V^{\otimes k}) = \binom{n}{k}$ *for* $0 \leq k \leq n$ *and* $\mathrm{Alt}(V^{\otimes k}) = \{0\}$ *for* $k > n$.

Proof Let $a = (a_1, a_2, \ldots, a_n) \in C(k, n)$ and suppose that $a_i \in \{0, 1\}$ for all $i = 1, 2, \ldots, n$. Then there exist $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ such that $a_{i_1} = a_{i_2} = \cdots = a_{i_k} = 1$ and $a_i = 0$ if $i \notin \{i_1, i_2, \ldots, i_k\}$. Moreover, the vectors $e_{i_{\pi(1)}} \otimes e_{i_{\pi(2)}} \otimes \cdots \otimes e_{i_{\pi(k)}}$ with $\pi \in \mathfrak{S}_k$ form a basis for the corresponding permutation module M^a in (8.6). But $\mathrm{Alt}(e_{i_{\pi(1)}} \otimes e_{i_{\pi(2)}} \otimes \cdots \otimes e_{i_{\pi(k)}}) = \varepsilon(\pi) e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k}$. Therefore $M^a \cong M^{1^k}$ contains the alternating representation of $\mathfrak{S}_k$ with multiplicity one and the corresponding space is spanned by $e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k}$.

Now suppose that $a_j \geq 2$ for some index j. If $(i_1, i_2, \ldots, i_k) \in \Omega_a$, then $i_s = i_t = j$ for a pair of distinct indeces s and t, and therefore

$$\mathrm{Alt}(e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_k}) = 0$$

Since the set $\{e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_k} : (i_1, i_2, \ldots, i_k) \in \Omega_a\}$ is a basis of M^a, this shows that M^a does not contain the alternating representation of $\mathfrak{S}_k$ and this ends the proof. $\square$

Note that in the above proof we have derived again a particular case of the Young rule (Corollary 3.7.11), namely the case $\mu = (1, 1, \ldots, 1)$.

Let W be another vector space. A multilinear map

$$\Phi : V^{\times k} \to W \tag{8.15}$$

is *alternating* (or *skew-symmetric*, or *antisymmetric*) if

$$\Phi(v_{\pi(1)}, v_{\pi(2)}, \ldots, v_{\pi(k)}) = \varepsilon(\pi)\Phi(v_1, v_2, \ldots, v_k),$$

for all $\pi \in \mathfrak{S}_k$ and $v_1, v_2, \ldots, v_k \in V$. By virtue of the elementary polarization identity

$$\alpha(v_1 + v_2, v_1 + v_2) - \alpha(v_1, v_1) - \alpha(v_2, v_2) = \alpha(v_1, v_2) + \alpha(v_2, v_1)$$

valid for all bilinear maps $\alpha : V^{\times 2} \to W$, we have that the map Φ is alternating if and only if $\Phi(v_1, v_2, \ldots, v_k) = 0$ whenever $v_i = v_j$ for some $1 \leq i < j \leq k$.

We define an alternating multilinear map

$$\begin{array}{rccc} \psi : & V^{\times k} & \to & \mathrm{Alt}(V^{\otimes k}) \\ & (v_1, v_2, \ldots, v_k) & \mapsto & v_1 \wedge v_2 \wedge \cdots \wedge v_k. \end{array}$$

This map is universal in the following sense. If Φ is an alternating multilinear map as in (8.15), then there exists a unique linear map $\phi : \mathrm{Alt}(V^{\otimes k}) \to W$ such that $\Phi = \phi \circ \psi$). Indeed, we may set

$$\phi(v_1 \wedge v_2 \wedge \cdots \wedge v_k) = \Phi(v_1, v_2, \ldots, v_k).$$

It is easy to check (exercise) that this universal property characterizes $\mathrm{Alt}(V^{\otimes k})$.

Suppose now that $V = V_1 \oplus V_2$, with $\dim V_1 = m$, and choose the basis $\{e_1, e_2, \ldots, e_n\}$ of V in such a way that the first m vectors are in V_1 and the remaining $n - m$ vectors are in V_2. For $0 \leq h \leq k$ consider the linear map

$$\begin{array}{ccc} \mathrm{Alt}(V_1^{\otimes h}) \bigotimes \mathrm{Alt}(V_2^{\otimes(k-h)}) & \to & \mathrm{Alt}\left[(V_1 \oplus V_2)^{\otimes k}\right] \\ (v_1 \wedge v_2 \wedge \cdots \wedge v_k) \otimes (v_{h+1} \wedge v_{h+2} \wedge \cdots \wedge v_k) & \mapsto & v_1 \wedge v_2 \wedge \cdots \wedge v_k \end{array} \tag{8.16}$$

where $v_1, v_2, \ldots, v_h \in V_1$ and $v_{h+1}, v_{h+2}, \ldots, v_k \in V_2$. In particular, if $1 \leq i_1 < i_2 < \cdots < i_h \leq m$ and $m+1 \leq i_{m+1} < i_{m+2} < \cdots < i_k \leq n$, we have

$$(e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_h}) \otimes (e_{i_{h+1}} \wedge e_{i_{h+2}} \wedge \cdots \wedge e_{i_k}) \mapsto e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k}.$$

We then get the following analogous of Proposition 8.1.2.

Proposition 8.1.6 *The map* (8.16) *determines an isomorphism*

$$\mathrm{Alt}\left[(V_1 \oplus V_2)^{\otimes k}\right] \cong \bigoplus_{h=0}^{k} \left[\mathrm{Alt}(V_1^{\otimes h}) \bigotimes \mathrm{Alt}(V_2^{\otimes(k-h)})\right].$$

From the algebraic identity above the usual Vandermonde identity for the binomial coefficients follows (see (5.6)).

8.2 Classical Schur–Weyl duality

In this section we examine the classical Schur–Weyl duality between $\mathfrak{S}_k$ and $\mathrm{GL}(n, \mathbb{C})$. However, we do not go deeply into the representation theory of $\mathrm{GL}(n, \mathbb{C})$, nor do we discuss the representation theory of the unitary group $U(n)$ and $SU(n)$. For a thorough dicussion of these topics, that requires tools like Lie algebras and representation theory of compact, locally compact and algebraic groups, we refer to the books by Simon [111], Goodman and Wallach [49], Sternberg [115], Bump [15], Clerc [21], Fulton and Harris [43] and the recent book by Procesi [103]. These monographs are the sources of most of the material presented in this section.

Our aim is mainly the study of the action of $\mathfrak{S}_k$ on $V^{\otimes k}$ and the representation theory of its commutant, which is, in any way, the milestone in the study of the Schur–Weyl duality. Then we limit ourselves to show that the irreducible representations of the commutant of $\mathfrak{S}_k$ yields a family of irreducible representations of $\mathrm{GL}(n, \mathbb{C})$. We also show how to use the machinery developed in Chapter 2 to describe these irreducible representations of $\mathrm{GL}(n, \mathbb{C})$ and introduce a related Young poset.

8.2.1 The general linear group $\mathrm{GL}(n, \mathbb{C})$

The *general linear group* $\mathrm{GL}(n, \mathbb{C})$ is the group of all invertible $n \times n$, complex matrices. We identify it with $\mathrm{GL}(V)$ the group of all invertible linear transformation on $V \cong \mathbb{C}^n$. If $\{e_1, e_2, \ldots, e_n\}$ is a basis of V as in Section 8.1, the isomorphism $\mathrm{GL}(n, \mathbb{C}) \cong \mathrm{GL}(V)$ associates with the invertible matrix

$g = (g_{i,j})_{i,j=1,2,\dots,n} \in \mathrm{GL}(n, \mathbb{C})$ the linear transformation given by

$$e_j \mapsto \sum_{i=1}^{n} g_{i,j} e_i. \tag{8.17}$$

For $g \in \mathrm{GL}(V)$ and $v \in V$, we denote by gv the g-image of v so that we also write (8.17) in the form

$$ge_j = \sum_{i=1}^{n} g_{i,j} e_i. \tag{8.18}$$

A *linear representation* (σ, W) of $\mathrm{GL}(V)$ is a group homomorphism $\sigma : \mathrm{GL}(V) \to \mathrm{GL}(W)$. A $*$-representation is a linear representation σ such that $\sigma(g^*) = \sigma(g)^*$ for all $g \in \mathrm{GL}(V)$. Two representations (σ, W) and (ρ, U) of $\mathrm{GL}(V)$ are *equivalent* if there exists a linear bijection $T : W \to U$ such that $T\sigma(g) = \rho(g)T$ for all $g \in \mathrm{GL}(V)$. A subspace $U \le W$ is σ*-invariant* if $\sigma(g)w \in W$ for all $w \in W$, $g \in \mathrm{GL}(V)$. We say that (σ, W) is *irreducible* when W has no nontrivial invariant subspaces.

Example 8.2.1 For $k \ge 1$ we define a $*$-representation ρ_k of $\mathrm{GL}(V)$ on $V^{\otimes k}$ by setting

$$\rho_k(g)(v_1 \otimes v_2 \otimes \cdots \otimes v_k) = gv_1 \otimes gv_2 \otimes \cdots \otimes gv_k,$$

for all $g \in \mathrm{GL}(V)$ and all decomposable tensors $v_1 \otimes v_2 \otimes \cdots \otimes v_k \in V^{\otimes k}$.

In terms of (8.2) we have $\rho_k(g) \equiv g \otimes g \otimes \cdots \otimes g$, that is $\rho_k \cong \rho_1^{\otimes k}$ if the k-iterated internal tensor product of representations of $\mathrm{GL}(V)$ is defined as in Section 1.1.7. The representation ρ_1 is also called the *defining* representation of $\mathrm{GL}(V)$.

From (8.18) we immediately get the following matrix representation of $\rho_k(g)$, $g = (g_{i,j})_{i,j=1,2,\dots,n} \in \mathrm{GL}(n, \mathbb{C}) \cong \mathrm{GL}(V)$:

$$\begin{aligned}\rho_k(g)&(e_{j_1} \otimes e_{j_2} \otimes \cdots \otimes e_{j_k}) \\ &= \sum_{i_1, i_2, \dots, i_k = 1}^{n} g_{i_1,j_1} g_{i_2,j_2} \cdots g_{i_k,j_k} e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_k}.\end{aligned}$$

The representations ρ_k of $\mathrm{GL}(V)$ and σ_k of $\mathfrak{S}_k$ (defined in Section 8.1.2) commute:

$$\rho_k(g)\sigma_k(\pi) = \sigma_k(\pi)\rho_k(g), \tag{8.19}$$

for all $g \in \mathrm{GL}(V)$ and $\pi \in \mathfrak{S}_k$.

Example 8.2.2 The subspace $\mathrm{Sym}(V^{\otimes k})$ (see Section 8.1.3) of $V^{\otimes k}$ is ρ_k-invariant. Indeed, for all $1 \leq j_1 \leq j_2 \leq \cdots \leq j_k \leq n$ we have,

$$\begin{aligned}\rho_k(g)(e_{j_1} \cdot e_{j_2} \cdot \ldots \cdot e_{j_k}) &= \rho_k(g) k!\, \mathrm{Sym}(e_{j_1} \otimes e_{j_2} \otimes \cdots \otimes e_{j_k}) \\ \text{(by (8.19))} \quad &= k!\, \mathrm{Sym}[\rho_k(g)(e_{j_1} \otimes e_{j_2} \otimes \cdots \otimes e_{j_k})] \\ &= k!\, \mathrm{Sym}[\sum_{i_1,i_2,\ldots,i_k=1}^{n} g_{i_1,j_1} g_{i_2,j_2} \cdots g_{i_k,j_k} \cdot \\ &\qquad \cdot e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_k}] \\ &= \sum_{i_1,i_2,\ldots,i_k=1}^{n} g_{i_1,j_1} g_{i_2,j_2} \cdots g_{i_k,j_k} e_{i_1} \cdot e_{i_2} \cdot \ldots \cdot e_{i_k}.\end{aligned} \tag{8.20}$$

In the next subsection, we show that the restriction of the representation ρ_k of $\mathrm{GL}(V)$ to the invariant subspace $\mathrm{Sym}(V^{\otimes k})$ is irreducible. For the moment, we present an isomorphic representation.

Let W_n^k denote the space of all homogeneous polynomials of degree k with complex coefficients in the variables $x_1, x_2, \ldots, x_n$ (see Section 4.1.2). There is an obvious linear isomorphism of vector spaces between $\mathrm{Sym}(V^{\otimes k})$ and W_n^k, given by the map

$$e_{j_1} \cdot e_{j_2} \cdot \ldots \cdot e_{j_k} \mapsto x_{j_1} x_{j_2} \cdots x_{j_k}, \tag{8.21}$$

for all $1 \leq j_1 \leq j_2 \leq \cdots \leq j_k \leq n$. We define a representation θ_k of $\mathrm{GL}(V)$ on W_n^k by setting

$$(\theta_k(g)p)(x_1, x_2, \ldots, x_n) = p(\sum_{i=1}^{n} g_{i,1} x_i, \sum_{i=1}^{n} g_{i,2} x_i, \ldots, \sum_{i=1}^{n} g_{i,n} x_i)$$

for all $g \in \mathrm{GL}(V)$ and $p \in W_n^k$. It is easy to see (exercise) that θ_k is a representation of $\mathrm{GL}(V)$. Note that in the jth coordinate of p there is $\sum_{i=1}^{n} g_{i,j} x_i$ and not $\sum_{i=1}^{n} g_{j,i} x_i$, that is, by identifying g with the matrix $g = (g_{i,j})_{i,j=1,2,\ldots,n}$ and the polynomial p as a function on the row vector $(x_1, x_2, \ldots, x_n)$, we have

$$(\theta_k(g)p)(x_1, x_2, \ldots, x_n) = p[(x_1, x_2, \ldots, x_n) g].$$

Then, θ_k is isomorphic to the restriction of ρ_k to $\mathrm{Sym}(V^{\otimes k})$: if $x_{j_1} x_{j_2} \cdots x_{j_k}$ is as in (8.21), then

$$\begin{aligned}\theta_k(g)(x_{j_1} x_{j_2} \cdots x_{j_k}) &= (\sum_{i_1=1}^{n} g_{i_1,j_1} x_{i_1}) \cdot (\sum_{i_2=1}^{n} g_{i_2,j_2} x_{i_2}) \cdots (\sum_{i_k=1}^{n} g_{i_k,j_k} x_{i_k}) \\ &= \sum_{i_1,i_2,\ldots,i_k=1}^{n} g_{i_1,j_1} g_{i_2,j_2} \cdots g_{i_k,j_k} x_{i_1} x_{i_2} \cdots x_{i_k}\end{aligned}$$

that coincides with (8.20), modulo (8.21).

Example 8.2.3 The subspace $\mathrm{Alt}(V^{\otimes k})$ (see Section 8.1.4) is also ρ_k-invariant. Indeed, for all $1 \leq j_1 < j_2 < \cdots < j_k \leq n$ we have,

$$\begin{aligned}\rho_k(g)(e_{j_1} \wedge e_{j_2} \wedge \cdots \wedge e_{j_k}) &= \rho_k(g)k!\ \mathrm{Alt}(e_{j_1} \otimes e_{j_2} \otimes \cdots \otimes e_{j_k})\\ \text{(by (8.19))} \quad &= k!\ \mathrm{Alt}[\rho_k(g)(e_{j_1} \otimes e_{j_2} \otimes \cdots \otimes e_{j_k})]\\ &= \sum_{i_1,i_2,\ldots,i_k=1}^{n} g_{i_1,j_1} g_{i_2,j_2} \cdots g_{i_k,j_k} e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k}\end{aligned}$$

which equals

$$\sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} \left(\sum_{\pi \in \mathfrak{S}_k} \varepsilon(\pi) g_{i_{\pi(1)},j_1} g_{i_{\pi(2)},j_2} \cdots g_{i_{\pi(k)},j_k} \right) e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k}.$$

In the next subsection, we show that the restriction of the representation ρ_k of $\mathrm{GL}(V)$ to the invariant subspace $\mathrm{Alt}(V^{\otimes k})$ is irreducible.

The *character* of a representation (σ, W) of $\mathrm{GL}(V)$ is defined in the obvious way: $\chi^\sigma(g) = tr[\sigma(g)]$. Clearly, equivalent representations have the same character. In particular, suppose $x_1, x_2, \ldots, x_n$ are the solutions of the characteristic equation of $\det(g - \lambda I) = 0$, that is, the eigenvalues of g (where each eigenvalue is counted according to its multiplicity as a solution of the characteristic equation). Then

$$\chi^{\rho_1}(g) = x_1 + x_2 + \cdots + x_n \equiv tr(g)$$

and, more generally, as it immediately follows from (1.18),

$$\chi^{\rho_k}(g) = (x_1 + x_2 + \cdots + x_n)^k \equiv tr(g)^k$$

for all $g \in \mathrm{GL}(V)$.

Following the monograph by Simon [111, Lemma A.1] we now give an important generalization of these identities. We start with an elementary lemma which is a particular case of the so called Spectral mapping principle (see, for instance, [34, Theorem 20.13]).

Lemma 8.2.4 *Let $A \in M_{n,n}(\mathbb{C})$ and denote by $x_1, x_2, \ldots, x_n$ the eigenvalues of A. Then, for every integer $h \geq 1$, the numbers $x_1^h, x_2^h, \ldots, x_n^h$ are the eigenvalues of A^h. In particular,*

$$tr(A^h) = x_1^h + x_2^h + \cdots + x_n^h.$$

Proof Let $\omega = \exp(2\pi i/h)$ be a primitive hth root of 1. Then we have

$$\lambda^h I - A^h = (\lambda I - A)(\lambda I - \omega A) \cdots (\lambda I - \omega^{h-1} A)$$

and therefore

$$\begin{aligned}\det(\lambda^h I - A^h) &= \det\left[\prod_{j=0}^{h-1}(\lambda I - \omega^j A)\right]\\ &= \prod_{j=0}^{h-1}\det(\lambda I - \omega^j A)\\ &= \prod_{j=0}^{h-1}\prod_{i=1}^{n}(\lambda - \omega^j x_i)\\ &= \prod_{i=1}^{n}(\lambda^h - x_i^h).\end{aligned}$$ □

Consider the representation η of $\mathrm{GL}(V) \times \mathfrak{S}_k$ on $V^{\otimes k}$ defined by setting $\eta(g, \pi) = \rho_k(g)\sigma_k(\pi)$ for all $g \in \mathrm{GL}(V)$ and $\pi \in \mathfrak{S}_k$. In other words,

$$\eta(g,\pi)(v_1 \otimes v_2 \otimes \cdots \otimes v_k) = gv_{\pi^{-1}(1)} \otimes gv_{\pi^{-1}(2)} \otimes \cdots \otimes gv_{\pi^{-1}(k)}, \quad (8.22)$$

for all $g \in \mathrm{GL}(V)$, $\pi \in \mathfrak{S}_k$ and $v_1, v_2, \ldots, v_k \in V$. Moreover, η is a representation because ρ_k and σ_k commute. We now compute the character χ^η of η.

Proposition 8.2.5 *Let $g \in \mathrm{GL}(V)$ and $\pi \in \mathfrak{S}_k$. Suppose that π belongs to the $\mathfrak{S}_k$-conjugacy class $\mathcal{C}_\lambda$, where $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_h) \vdash k$ (cf. Proposition 3.1.3), and let $x_1, x_2, \ldots, x_n$ be the eigenvalues of g. Then*

$$\chi^\eta(g,\pi) = p_\lambda(x_1, x_2, \ldots, x_n), \quad (8.23)$$

where p_λ is the power sum symmetric polynomial associated with the partition λ (cf. Section 4.1.3).

Proof Let $\sigma \in \mathfrak{S}_k$. Since

$$\chi^\eta(g, \sigma\pi\sigma^{-1}) = \chi^\eta[(I_n,\sigma)(g,\pi)(I_n,\sigma^{-1})] = \chi^\eta(g,\pi),$$

we may suppose that

$$\pi = (1 \to \lambda_1 \to \lambda_1 - 1 \to \cdots \to 2 \to 1)(\lambda_1 + 1 \to \lambda_1 + \lambda_2 \to \cdots \to \lambda_1 + 2 \to \lambda_1 + 1)\cdots(k - \lambda_h + 1 \to k \to k-1 \to \cdots \to k - \lambda_h + 2 \to k - \lambda_h + 1).$$

We then have:

$$
\begin{aligned}
tr[\eta(\pi, g)] &= \sum_{j_1, j_2, \dots, j_k = 1}^{n} \langle \eta(\pi, g)(e_{j_1} \otimes e_{j_2} \otimes \cdots \otimes e_{j_k}), e_{j_1} \otimes e_{j_2} \otimes \cdots \otimes e_{j_k} \rangle_{V^{\otimes k}} \\
&= \sum_{j_1, j_2, \dots, j_k = 1}^{n} \langle g e_{j_{\pi^{-1}(1)}}, e_{j_1} \rangle_V \langle g e_{j_{\pi^{-1}(2)}}, e_{j_2} \rangle_V \cdots \langle g e_{j_{\pi^{-1}(k)}}, e_{j_k} \rangle_V \\
&= \sum_{r=1}^{h} \sum_{t_1, t_2, \dots, t_{\lambda_r} = 1}^{n} \langle g e_{t_2}, e_{t_1} \rangle_V \langle g e_{t_3}, e_{t_2} \rangle_V \cdots \langle g e_{t_1}, e_{t_{\lambda_r}} \rangle_V \\
&= \sum_{r=1}^{h} \sum_{t_1, t_2, \dots, t_{\lambda_r} = 1}^{n} g_{t_1, t_2} g_{t_2, t_3} \cdots g_{t_{\lambda_r}, t_1} \\
&= \sum_{r=1}^{h} tr[g^{\lambda_r}] \\
&=_* \sum_{r=1}^{h} (x_1^{\lambda_r} + x_2^{\lambda_r} + \cdots + x_n^{\lambda_r}) \\
&= p_\lambda(x_1, x_2, \dots, x_r),
\end{aligned}
$$

where $=_*$ follows from Lemma 8.2.4. □

Corollary 8.2.6 *Denote by χ^S (resp. χ^A) the character of the restriction of the representation ρ_k of* $\mathrm{GL}(V)$ *on* $\mathrm{Sym}(V^{\otimes k})$ *(resp.* $\mathrm{Alt}(V^{\otimes k})$*). Then we have*

$$\chi^S(g) = h_k(x_1, x_2, \dots, x_n)$$

and

$$\chi^A(g) = e_k(x_1, x_2, \dots, x_n),$$

where $x_1, x_2, \dots, x_n$ are the eigenvalues of g, and h_k and e_k are the complete and elementary symmetric polynomials (cf. Section 4.1.3).

Proof We have

$$
\begin{aligned}
\chi^S(g) &= tr\left[\rho_k(g)\mathrm{Sym}\right] \\
&= \frac{1}{k!} tr[\sum_{\pi \in \mathfrak{S}_k} \sigma_k(\pi)\rho_k(g)] \\
&= \frac{1}{k!} \sum_{\pi \in \mathfrak{S}_k} tr[\eta(\pi, g)]
\end{aligned}
$$

$$\begin{aligned}\text{(by Propositions 4.1.1 and 8.2.5)} &= \sum_{\lambda \vdash k} \frac{1}{z_\lambda} p_\lambda(x_1, x_2, \ldots, x_n)\\ \text{(by Lemma 4.1.10)} &= h_k(x_1, x_2, \ldots, x_n).\end{aligned}$$

The proof of the second identity is similar: we replace Lemma 4.1.10 with Remark 4.1.14 (cf., in particular, (4.15)). □

The *group algebra* of $\mathrm{GL}(V)$, denoted by $\mathbb{C}[\mathrm{GL}(V)]$, is the space of all functions $f : \mathrm{GL}(V) \to \mathbb{C}$ whose *support* $\mathrm{supp}(f) := \{g \in \mathrm{GL}(V) : f(g) \neq 0\}$ is finite. It is an associative algebra where the multiplication of two elements $f_1, f_2 \in \mathbb{C}[\mathrm{GL}(V)]$ is their convolution $f_1 * f_2$ defined by

$$f_1 * f_2(g) = \sum_{h \in \mathrm{supp}(f_1)} f_1(h) f_2(h^{-1}g)$$

for all $g \in \mathrm{GL}(V)$. Equivalently, we may identify an element $f \in \mathbb{C}[\mathrm{GL}(V)]$ with the (finite) formal sum $\sum_{g \in \mathrm{GL}(V)} f(g)g$. Then the convolution becomes formal multiplication:

$$\begin{aligned}f_1 * f_2 &= \left(\sum_{g_1 \in \mathrm{GL}(V)} f_1(g_1)g_1\right)\left(\sum_{g_2 \in \mathrm{GL}(V)} f_2(g_2)g_2\right)\\ &= \sum_{g_1, g_2 \in \mathrm{GL}(V)} f_1(g_1)f_2(g_2)g_1g_2\\ &= \sum_{g \in \mathrm{GL}(V)} \left[\sum_{h \in \mathrm{supp}(f_1)} f_1(h)f_2(h^{-1}g)\right] g,\end{aligned}$$

where the last equality follows by setting $g = g_1g_2$ and $h = g_1$.

Clearly, $\mathbb{C}[\mathrm{GL}(V)]$ is infinite dimensional.

Let (σ, W) be a $*$-representation of $\mathrm{GL}(V)$ and $f \in \mathbb{C}[\mathrm{GL}(V)]$. We set

$$\sigma(f) = \sum_{h \in \mathrm{supp}(f)} f(h)\sigma(h) \in \mathrm{End}(W).$$

Then $\sigma(\mathbb{C}[\mathrm{GL}(V)])$ is a finite dimensional $*$-algebra of operators on W. Moreover, (σ, W) is $\mathrm{GL}(V)$-irreducible if and only if it is $\sigma(\mathbb{C}[\mathrm{GL}(V)])$-irreducible.

8.2.2 Duality between $\mathrm{GL}(n, \mathbb{C})$ and $\mathfrak{S}_k$

We recall that we have identified the algebras $\mathrm{End}(V^{\otimes k})$ and $\mathrm{End}(V)^{\otimes k}$ by means of the isomorphism (8.2).

Define a representations θ_k of $\mathfrak{S}_k$ on $\mathrm{End}(V^{\otimes k}) \cong \mathrm{End}(V)^{\otimes k}$ by setting

$$\theta_k(\pi)(A_1 \otimes A_2 \otimes \cdots \otimes A_k) = A_{\pi^{-1}(1)} \otimes A_{\pi^{-1}(2)} \otimes \cdots A_{\pi^{-1}(k)}$$

for all $\pi \in \mathfrak{S}_k$ and $A_1, A_2, \ldots, A_k \in \mathrm{End}(V)$. It is clearly the analogue of σ_k with V replaced by $\mathrm{End}(V)$.

Proposition 8.2.7 *For every $\mathcal{T} \in \mathrm{End}(V^{\otimes k})$ and $\pi \in \mathfrak{S}_k$, we have*

$$\theta_k(\pi)\mathcal{T} = \sigma_k(\pi)\mathcal{T}\sigma_k(\pi^{-1}). \tag{8.24}$$

Proof Given the decomposable tensors $\mathcal{T} = A_1 \otimes A_2 \otimes \cdots \otimes A_k \in \mathrm{End}(V)^{\otimes k} \cong \mathrm{End}(V^{\otimes k})$ and $v_1 \otimes v_2 \otimes \cdots \otimes v_k \in V^{\otimes k}$ we have

$$\begin{aligned}
&[\theta_k(\pi)(A_1 \otimes A_2 \otimes \cdots \otimes A_k)](v_1 \otimes v_2 \otimes \cdots \otimes v_k) \\
&\quad = A_{\pi^{-1}(1)}v_1 \otimes A_{\pi^{-1}(2)}v_2 \otimes \cdots A_{\pi^{-1}(k)}v_k \\
&\quad = \sigma_k(\pi)(A_1 v_{\pi(1)} \otimes A_2 v_{\pi(2)} \otimes \cdots \otimes A_k v_{\pi(k)}) \\
&\quad = [\sigma_k(\pi)(A_1 \otimes A_2 \otimes \cdots \otimes A_k)\sigma_k(\pi^{-1})](v_1 \otimes v_2 \otimes \cdots \otimes v_k).
\end{aligned}$$

□

We are now in position to prove one of the main ingredients of the classical Schur–Weyl duality: the characterization of the commutant $\mathrm{End}_{\mathfrak{S}_k}(V^{\otimes k})$ of $V^{\otimes k}$ with respect to the representation σ_k.

Theorem 8.2.8 *With respect to the isomorphism* (8.2)*, we have*

$$\mathrm{End}_{\mathfrak{S}_k}(V^{\otimes k}) \equiv \mathrm{Sym}[\mathrm{End}(V)^{\otimes k}] \equiv \rho_k(\mathbb{C}[\mathrm{GL}(V)]),$$

where $\mathrm{Sym} = \frac{1}{k!}\sum_{\pi \in \mathfrak{S}_k} \theta_k(\pi)$ *is the symmetrizing operator on* $\mathrm{End}(V)^{\otimes k}$ *with respect to the $\mathfrak{S}_k$-representation θ_k.*

Proof The first identity is an immediate consequence of Corollary 1.2.22, Lemma 1.3.5 and Proposition 8.2.7: $\mathrm{End}_{\mathfrak{S}_k}(V^{\otimes k})$ coincides with the isotypic component of the trivial representation with respect to the representation θ_k of $\mathfrak{S}_k$ on $\mathrm{End}(V^{\otimes k})$ (see (8.24)) and Sym is the projection operator onto the trivial representation.

The second identity is a consequence of Theorem 8.1.3: $\mathrm{GL}(V)$ is a nontrivial open set in $\mathrm{End}(V)$ ($J \in \mathrm{End}(V)$ belongs to $\mathrm{GL}(V)$ if and only if $\det(J) \neq 0$, and det is a continuous function: it is a polynomial in the coefficients of the operator J). Therefore, the set

$$\left\{ \sum_{g \in \mathrm{GL}(V)} f(g) g^{\otimes k} : f \in \mathbb{C}[\mathrm{GL}(V)] \right\}$$

coincides with $\mathrm{Sym}[\mathrm{End}(V)^{\otimes k}]$, and $\sum_{g \in \mathrm{GL}(V)} f(g) g^{\otimes k} \equiv \rho_k(f)$. □

Remark 8.2.9 Theorem 8.2.8 may be formulated in the following equivalent way: the commutant of $\sigma_k[L(\mathfrak{S}_k)]$ in $\mathrm{End}(V^{\otimes k})$ coincides with $\rho_k\{\mathbb{C}[\mathrm{GL}(V)]\}$.

Therefore we can apply the double commutant theory (Theorem 7.3.4). However, we shall state the results in terms of representations of the groups $\mathfrak{S}_k$ and $\mathrm{GL}(V)$.

For $\lambda \vdash k$, set

$$U^\lambda = \mathrm{Hom}_{\mathfrak{S}_k}(S^\lambda, V^{\otimes k}), \tag{8.25}$$

where S^λ is the irreducible representation of $\mathfrak{S}_k$ canonically associated with λ (see Definition 3.3.8). As in Theorem 7.3.4, we may consider the representation ρ_λ of $\mathrm{GL}(V)$ on U^λ defined by setting

$$[\rho_\lambda(g)](T) = \rho_k(g)T$$

for all $g \in \mathrm{GL}(V)$ and $T \in U^\lambda$ (the right-hand side is the composition of $T : S^\lambda \to V^{\otimes k}$ with $\rho_k(g) : V^{\otimes k} \to V^{\otimes k}$). We also recall that χ^λ denotes the character of S^λ, d_λ the dimension of S^λ and s_λ is the Schur polynomial (in the variables $x_1, x_2, \dots, x_n$) associated with λ (see Section 4.3.1). The character of $(\rho_\lambda, U^\lambda)$ will be denoted by ϕ^λ and its dimension by m_λ. We will also use the Young seminormal units $\{e_T : T \in \mathrm{Tab}(k)\}$ introduced in Section 3.4.4.

Theorem 8.2.10 (Classical Schur–Weyl duality)

(i) *The set* $\{U^\lambda : \lambda \vdash k,\ \ell(\lambda) \le n, k = 1, 2, 3, \dots\}$ *is a set of irreducible, pairwise inequivalent representations of* $\mathrm{GL}(V)$.

(ii) *Setting* $\lambda_{\ell(\lambda)+1} = \lambda_{\ell(\lambda)+2} = \cdots = \lambda_n = 0$, *we have*

$$m_\lambda := \dim U^\lambda = \frac{d_\lambda}{k!} \prod_{i=1}^{\ell(\lambda)} \prod_{j=1}^{\lambda_i} (j - i + n) = \prod_{1 \le i < j \le n} \frac{\lambda_i - \lambda_j + j - i}{j - i}.$$

(iii) *The character* φ^λ *of* $(\rho_\lambda, U^\lambda)$ *is given by*

$$\varphi^\lambda(g) = s_\lambda(x_1, x_2, \dots, x_n),$$

where $x_1, x_2, \dots, x_n$ *are the eigenvalues of* $g \in \mathrm{GL}(V)$ *(as in Lemma 8.2.4).*

(iv) *We have that*

$$V^{\otimes k} \cong \bigoplus_{\substack{\lambda \vdash k: \\ \ell(\lambda) \le n}} (U^\lambda \otimes S^\lambda)$$

is the (multiplicity-free) decomposition of $(\eta, V^{\otimes k})$ *into irreducible* $(\mathrm{GL}(V) \times \mathfrak{S}_k)$*-representations (cf. (8.22)).*

(v) *We have that*

$$V^{\otimes k} \cong \bigoplus_{\substack{\lambda \vdash k:\\ \ell(\lambda) \le n}} m_\lambda S^\lambda$$

is the decomposition of $(\sigma_k, V^{\otimes k})$ *into irreducible* $\mathfrak{S}_k$*-representations.*

(vi) *We have that*

$$V^{\otimes k} \cong \bigoplus_{\substack{\lambda \vdash k:\\ \ell(\lambda) \le n}} d_\lambda U^\lambda$$

is the decomposition of $(\rho_k, V^{\otimes k})$ *into irreducible* $\mathrm{GL}(V)$*-representations. In particular,*

$$V^{\otimes k} \cong \bigoplus_{\substack{\lambda \vdash k:\\ \ell(\lambda) \le n}} \bigoplus_{T \in \mathrm{Tab}(\lambda)} \sigma_k(e_T) V^{\otimes k} \tag{8.26}$$

is an orthogonal decomposition of $(\rho_k, V^{\otimes k})$ *into irreducible* $\mathrm{GL}(V)$*-representations, with* $\sigma_k(e_T)V^{\otimes k} \cong U^\lambda$ *for all* $T \in Tab(\lambda)$ *and* $d_\lambda U^\lambda \equiv \bigoplus_{T \in \mathrm{Tab}(\lambda)} \sigma_k(e_T) V^{\otimes k}$.

Proof First of all, recall the decomposition (8.6). We may then apply the Young rule (Corollary 3.7.11; actually, the weaker version in Theorem 3.6.11 suffices and applies in an easier way): the $\mathfrak{S}_k$-representation S^λ appears in $V^{\otimes k}$ if and only if it appears in M^a for some $a \in C(k, n)$ and therefore if and only if $\ell(\lambda) \le n$. With this remark in mind, and taking into account Theorem 8.2.8 and Remark 8.2.9, we may apply the double commutant theory (Theorems 7.3.4 and 7.4.14) to the representations ρ_k and σ_k, yielding (iv) and the irreducibility and pairwise inequivalence of the U^λ's.

From (iv) and an obvious generalization of Proposition 1.3.4 (iv) and (vii), we get the following expression for the character of η:

$$\chi^\eta(g, \pi) = \sum_{\substack{\lambda \vdash k:\\ \ell(\lambda) \le n}} \varphi^\lambda(g) \chi^\lambda(\pi) \tag{8.27}$$

for all $g \in \mathrm{GL}(V)$ and $\pi \in \mathfrak{S}_k$. Comparing (8.27) and (8.23) we get that if $x_1, x_2, \ldots, x_n$ are the eigenvalues of g and $\pi \in \mathcal{C}_\mu$, where $\mu \vdash k$, then

$$p_\mu(x_1, x_2, \ldots, x_n) = \sum_{\substack{\lambda \vdash k:\\ \ell(\lambda) \le n}} \varphi^\lambda(g) \chi^\lambda(\pi).$$

Since $\{s_\lambda : \lambda \vdash k, \ell(\lambda) \le n\}$ and $\{p_\mu : \mu \vdash k, \ell(\mu) \le k\}$ are bases of the vector space Λ_n^k (cf. Theorem 4.3.1 and Theorem 4.1.12, respectively), Theorem 4.3.9 and the orthogonality relations in Corollary 1.3.7 ensure that $\varphi^\lambda(g) = s_\lambda(x_1, x_2, \ldots, x_n)$, and this proves (iii).

Then (ii) follows from Theorem 4.3.3 and (4.39).

Moreover, if $\lambda \vdash k$ and $\mu \vdash h$ with $k \neq h$, then $\rho_\lambda \not\sim \rho_\mu$ as the corresponding characters are different and this gives (i).

Finally, (v) and the first part of (vi) follow from Theorem 7.3.4, while the second part of (vi) follows from Theorem 7.4.14 and the definition of e_T (see (3.58)). □

Observe that, since Sym is the projection onto the isotyopic component of the trivial representation (see also Exercise 3.4.13.(1)), we have

$$U^{(k)} \cong \mathrm{Sym}(V^{\otimes k}) \tag{8.28}$$

as $\mathrm{GL}(V)$-representations, and the computation of the character χ^S in Corollary 8.2.6 agrees with that in Theorem 8.2.10.(iii)); indeed, from the Jacobi–Trudi identity (Corollary 4.3.18), it follows that $s_{(k)} \equiv h_k$. Similarly, as $\mathrm{GL}(V)$-representations,

$$U^{(1^k)} \cong \mathrm{Alt}(V^{\otimes k})$$

and the two computations of the characters agree; the details are left to the reader. Finally, it is clear that $U^{(1)} \equiv V$.

8.2.3 Clebsch–Gordan decomposition and branching formulas

We recall the definition of the Littlewood–Richardson coefficients: for $\lambda \vdash k$ and $1 \leq h \leq k-1$, these are the nonnegative integers $c^\lambda_{\nu,\mu}$ involved in the decomposition

$$\mathrm{Res}^{\mathfrak{S}_k}_{\mathfrak{S}_h \times \mathfrak{S}_{k-h}} S^\lambda \cong \bigoplus_{\substack{\nu \vdash h, \\ \mu \vdash k-h}} c^\lambda_{\nu,\mu} [S^\nu \boxtimes S^\mu] \tag{8.29}$$

(cf. Definition 6.1.25.(3) and Corollary 6.1.35 (the Littlewood–Richardson rule)).

We have devoted the entire Chapter 6 to the combinatorial description of the coefficients $c^\lambda_{\nu,\mu}$ and to a deep analysis of (8.29). In the present section, we only use (8.29) because we have to show that the same coefficients govern two different rules involving the irreducible representation U^λ of $\mathrm{GL}(V)$ (cf., (8.25)), that is, we do not need the explicit results in Chapter 6.

In what follows, we regard $\mathfrak{S}_h \times \mathfrak{S}_{k-h}$ as the stabilizer of $\{1, 2, \ldots, h\}$. Let $\nu \vdash h$, $\mu \vdash k-h$ and $\ell(\nu), \ell(\mu) \leq n$. We consider the external tensor product of U^ν and U^μ, that we denote by $U^\nu \boxtimes U^\mu$ (with a slight modification of our convention in Section 1.1.7), which is a representation of $\mathrm{GL}(V) \times \mathrm{GL}(V)$. We also consider the internal tensor product $U^\nu \otimes U^\mu = \mathrm{Res}^{\mathrm{GL}(V)\times\mathrm{GL}(V)}_{\widetilde{\mathrm{GL}}(V)}(U^\nu \boxtimes U^\mu)$, where $\widetilde{\mathrm{GL}}(V)$ is the diagonal subgroup $\{(g,g) : g \in \mathrm{GL}(V)\} \le \mathrm{GL}(V) \times \mathrm{GL}(V)$.

Theorem 8.2.11 (Clebsch–Gordan decomposition) *For $\nu \vdash h$, $\mu \vdash k-h$ and $\ell(\nu), \ell(\mu) \le n$ the $\mathrm{GL}(V)$-representation $U^\nu \otimes U^\mu$ has the following decomposition as a direct sum of irreducible representations:*

$$U^\nu \otimes U^\mu \cong \bigoplus_{\substack{\lambda \vdash k:\\ \ell(\lambda)\le n}} c^\lambda_{\nu,\mu} U^\lambda.$$

Proof Consider the representation $\sigma_h \otimes \sigma_{k-h}$ of $\mathfrak{S}_h \times \mathfrak{S}_{k-h}$ on $V^{\otimes k} = V^{\otimes h} \otimes V^{\otimes(k-h)}$. We set

$$\mathcal{B} = \sigma_h \otimes \sigma_{k-h}[L(\mathfrak{S}_h \times \mathfrak{S}_{k-h})] \cong \sigma_h[L(\mathfrak{S}_h)] \otimes \sigma_{k-h}[L(\mathfrak{S}_{k-h})]$$

that is, $\mathcal{B}$ is the subalgebra of $\mathrm{End}(V^{\otimes k}) \cong \mathrm{End}(V^{\otimes h}) \otimes \mathrm{End}(V^{\otimes(k-h)})$ spanned by all operators $\sigma_h(\pi) \otimes \sigma_{k-h}(\tau)$, $\pi \in \mathfrak{S}_h$ and $\tau \in \mathfrak{S}_{k-h}$. From Theorem 8.2.8 it follows that the commutant of $\mathcal{B}$ is given by:

$$\mathcal{B}' \equiv (\rho_h \otimes \rho_{k-h})\{\mathbb{C}[\mathrm{GL}(V) \times \mathrm{GL}(V)]\} \cong \rho_h\{\mathbb{C}[\mathrm{GL}(V)]\} \otimes \rho_{k-h}\{\mathbb{C}[\mathrm{GL}(V)]\}$$

which is spanned by all operators of the form $\rho_h(g_1) \otimes \rho_{k-h}(g_2)$, $g_1, g_2 \in \mathrm{GL}(V)$. Clearly, $\mathcal{B} \subseteq \mathcal{A} := \sigma_k[L(\mathfrak{S}_k)]$ and $\mathcal{B}' \supseteq \mathcal{A}' := \rho_k\{\mathbb{C}[\mathrm{GL}(V)]\}$, where $\mathcal{A}$ and $\mathcal{A}'$ are the algebras used in the proof of Theorem 8.2.10. Note also that, applying Theorem 8.2.10 both to $V^{\otimes h}$ and to $V^{\otimes(k-h)}$ and taking into account Proposition 7.3.1, we deduce that the decomposition of $V^{\otimes k}$ into $\mathcal{B} \otimes \mathcal{B}'$-irreducible representations is:

$$V^{\otimes k} \cong \bigoplus_{\substack{\mu \vdash h:\\ \ell(\mu)\le n}} \bigoplus_{\substack{\nu \vdash k-h:\\ \ell(\nu)\le n}} [(U^\nu \boxtimes U^\mu) \boxtimes (S^\nu \boxtimes S^\mu)]. \tag{8.30}$$

This is also the decomposition into irreducible $(\mathrm{GL}(V) \times \mathrm{GL}(V) \times \mathfrak{S}_h \times \mathfrak{S}_{k-h})$-representations, and this also proves that $U^\nu \boxtimes U^\mu$ is $(\mathrm{GL}(V) \times \mathrm{GL}(V))$-irreducible. We now apply the seesaw reciprocity theorem (Theorem 7.5.10): the multiplicity of U^λ in $\mathrm{Res}^{\mathrm{GL}(V)\times\mathrm{GL}(V)}_{\widetilde{\mathrm{GL}}(V)}(U^\nu \boxtimes U^\mu)$ is equal to the multiplicity of $S^\nu \boxtimes S^\mu$ in $\mathrm{Res}^{\mathfrak{S}_k}_{\mathfrak{S}_h \times \mathfrak{S}_{k-h}} S^\lambda$. It is also clear that $\mathrm{Res}^{\mathrm{GL}(V)\times\mathrm{GL}(V)}_{\widetilde{\mathrm{GL}}(V)} \equiv \mathrm{Res}^{\mathcal{B}'}_{\mathcal{A}'}$. □

As a particular case of Theorem 8.2.11 we get the following remarkable rule (recall that $V \cong U^{(1)}$).

Corollary 8.2.12 *For $\mu \vdash k-1$, $\ell(\mu) \le n$, we have:*

$$U^\mu \otimes V = \bigoplus_{\substack{\lambda \vdash k: \\ \ell(\lambda) \le n \\ \lambda \to \mu}} U^\lambda,$$

where $\lambda \to \mu$ is the notation for λ covers μ introduced in Section 3.1.6.

Proof For $h = k-1$ the Littlewood–Richardson rule reduces to the branching rule (Corollary 3.3.11). □

It is worthwhile to examine more closely Corollary 8.2.12 because it leads to a remarkable connection with the Okounkov–Vershik theory developed in Chapter 2. For example, if $k = 2$ we have the decomposition:

$$V \otimes V \cong U^{(2)} \oplus U^{1,1} \cong \mathrm{Sym}(V^{\otimes 2}) \oplus \mathrm{Alt}(V^{\otimes 2}). \tag{8.31}$$

Starting from (8.31) and iteratively applying Corollary 8.2.12 we can easily get an (orthogonal) decomposition of $V^{\otimes k}$ into irreducible $\mathrm{GL}(V)$-representations (and we will show that it coincides with (8.26) in Theorem 8.2.10). For instance, for $k = 3$, we have

$$\begin{aligned} V \otimes V \otimes V &\cong [U^{(2)} \oplus U^{1,1}] \otimes V \\ &\cong [U^{(3)} \oplus U^{2,1}] \oplus [U^{2,1} \oplus U^{1,1,1}], \end{aligned} \tag{8.32}$$

and we may continue this way. In order to formalize this, we need some more considerations. Note that, by the branching rule for $\mathfrak{S}_k$, we have, for $\lambda \vdash k$, $\ell(\lambda) \le n$,

$$\mathrm{Res}^{\mathfrak{S}_k \times \mathrm{GL}(V)}_{\mathfrak{S}_{k-1} \times \mathrm{GL}(V)} (S^\lambda \boxtimes U^\lambda) \cong \bigoplus_{\substack{\mu \vdash k-1: \\ \lambda \to \mu}} (S^\mu \boxtimes U^\lambda)$$

and, applying this decomposition to Theorem 8.2.10.(iv), we get the following (multiplicity-free!) decomposition of $V^{\otimes k}$ under $\mathfrak{S}_{k-1} \times \mathrm{GL}(V)$:

$$V^{\otimes k} \cong \bigoplus_{\substack{\lambda \vdash k: \\ \ell(\lambda) \le n}} \bigoplus_{\substack{\mu \vdash k-1: \\ \lambda \to \mu}} (S^\mu \boxtimes U^\lambda). \tag{8.33}$$

We can get (8.33) in the following alternative way. Consider the representation of $\mathfrak{S}_{k-1} \times \mathrm{GL}(V) \times \mathrm{GL}(V)$ on $V^{\otimes k} \cong V^{\otimes(k-1)} \otimes V$, where $\mathfrak{S}_{k-1}$ permutes the first $k-1$ coordinates and $\mathrm{GL}(V) \times \mathrm{GL}(V)$ acts via $\rho_{k-1} \times \rho_1$. Then, decomposing $V^{\otimes(k-1)} \otimes V$ as in (8.30) and applying Corollary 8.2.12, that is considering $\mathrm{Res}^{\mathfrak{S}_{k-1}\times \mathrm{GL}(V)\times \mathrm{GL}(V)}_{\mathfrak{S}_{k-1}\times \widetilde{\mathrm{GL}}(V)}$, we get:

$$\begin{aligned} V^{\otimes k} \cong V^{\otimes(k-1)} \otimes V &\cong \bigoplus_{\substack{\mu\vdash k-1:\\ \ell(\mu)\le n}} (S^\mu \boxtimes U^\mu \boxtimes V) \\ &\cong \bigoplus_{\substack{\mu\vdash k-1:\\ \ell(\mu)\le n}} \bigoplus_{\substack{\lambda\vdash k:\\ \ell(\lambda)\le n\\ \lambda\to\mu}} (S^\mu \boxtimes U^\lambda). \end{aligned} \tag{8.34}$$

Note that we have just reproduced, in a particular case, the arguments in the proof of Theorem 8.2.11 (and those in the proof of Theorem 7.5.10 on which the former is based). Let us apply these considerations to get a different description of the decomposition (8.26) in Theorem 8.2.10).

We first need to modify the Young poset introduced in Section 3.1.6. Given a positive integer n, the *n-truncated Young poset* $\mathbb{Y}_n$ is obtained from the usual Young poset $\mathbb{Y}$ by considering only the partitions λ such that $\ell(\lambda) \le n$. For instance, the bottom of the 3-truncated Young poset $\mathbb{Y}_3$ is shown in Figure 8.1 (cf. Figure 3.12):

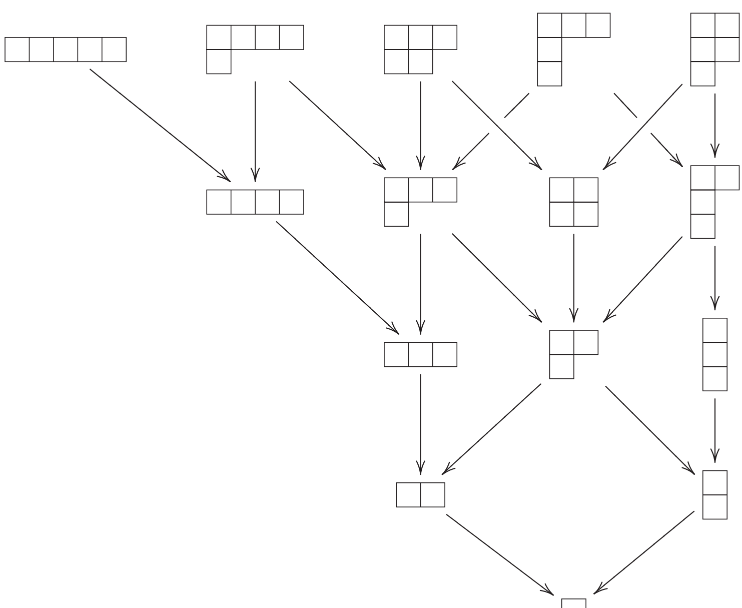


Figure 8.1 The bottom of the 3-truncated Young poset $\mathbb{Y}_3$.

Let $\lambda \vdash k$ and $\lambda = \lambda^{(k)} \to \lambda^{(k-1)} \to \lambda^{(k-2)} \to \cdots \to \lambda^{(1)} \equiv (1)$ be a path in $\mathbb{Y}_n$, let $T \in \mathrm{Tab}(\lambda)$ be the corresponding standard tableau and w_T the associated

GZ-vector (cf. Section 3.4.2). Clearly, in (8.26) in Theorem 8.2.10, we have

$$\sigma_k(e_T)V^{\otimes k} \equiv \mathbb{C}w_T \boxtimes U^\lambda.$$

The subspace $\mathbb{C}w_T \boxtimes U^\lambda$ may be reached by an iterative use of the branching rule for $\mathfrak{S}_n$, by means of the following chain of subspaces:

$$S^\lambda \boxtimes U^\lambda \supseteq S^{\lambda^{(k-1)}} \boxtimes U^\lambda \supseteq S^{\lambda^{(k-2)}} \boxtimes U^\lambda \supseteq \cdot \supseteq S^{\lambda^{(2)}} \boxtimes U^\lambda \supseteq w_T \boxtimes U^\lambda. \tag{8.35}$$

We now formalize the construction in (8.31) and (8.32). Let $\lambda = \lambda^{(k)} \to \lambda^{(k-1)} \to \lambda^{(k-2)} \to \cdots \to \lambda^{(1)} \equiv (1)$ be the same path that has led to (8.35). Consider the following chain of subspaces: $W^{\lambda^{(1)}} := V^{\otimes k}$; $W^{\lambda^{(2)}}$ is chosen in $V^{\otimes k} \cong V^{\otimes 2} \otimes V^{\otimes(k-2)} \cong (U^{(2)} \oplus U^{1,1}) \otimes V^{\otimes(k-2)}$ as follows:

$$W^{\lambda^{(2)}} := \begin{cases} U^{(2)} \otimes V^{\otimes(k-2)} & \text{if } \lambda^{(2)} = (2) \\ U^{1,1} \otimes V^{\otimes(k-2)} & \text{if } \lambda^{(2)} = (1,1). \end{cases}$$

Iteratively, $W^{\lambda^{(h)}}$ is the subspace $U^{\lambda^{(h)}} \otimes V^{\otimes(k-h)}$ of

$$\begin{aligned} W^{\lambda^{(h-1)}} &\cong U^{\lambda^{(h-1)}} \otimes V^{\otimes(k-h+1)} \\ &\cong (U^{\lambda^{(h-1)}} \otimes V) \otimes V^{\otimes(k-h)} \\ &\bigoplus_{\substack{\mu \vdash h: \\ \ell(\mu)\le n \\ \mu\to\lambda^{(h-1)}}} (U^\mu \otimes V^{\otimes(k-h)}). \end{aligned}$$

The chain ends with a subspace isomorphic to U^λ:

$$V^{\otimes k} = W^{\lambda^{(1)}} \supseteq W^{\lambda^{(2)}} \supseteq \cdots \supseteq W^{\lambda^{(k)}} \cong U^\lambda.$$

We set $W^T := W^{\lambda^{(k)}}$, and clearly, $V^{\otimes k} \cong \bigotimes_{T\in \mathrm{Tab}(k)} W^T$ is an (orthogonal) decomposition of $V^{\otimes k}$ into irreducible $\mathrm{GL}(V)$-representations. We show that it coincides with the decomposition (8.26) in Theorem 8.2.10.

Proposition 8.2.13 *We have* $W^T \equiv \sigma_k(e_T)V^{\otimes k}$.

Proof We proceed by induction on k. The case $k = 1$ is obvious. We have $W^{\lambda^{(k-1)}} \cong U^{\lambda^{(k-1)}} \boxtimes V$, and, by the inductive hypothesis, in $V^{\otimes(k-1)}$ the subspace $U^{\lambda^{(k-1)}}$ is given by the chain

$$\begin{aligned} U^{\lambda^{(k-1)}} = \mathbb{C}w_{\overline{T}} \boxtimes U^{\lambda^{(k-1)}} &\cong S^{(1)} \boxtimes U^{\lambda^{(k-1)}} \\ &\subseteq S^{\lambda^{(2)}} \boxtimes U^{\lambda^{(k-1)}} \subseteq \cdots \subseteq S^{\lambda^{(k-1)}} \boxtimes U^{\lambda^{(k-1)}} \end{aligned} \tag{8.36}$$

as in (8.35) ($\overline{T}$ is the tableau obtained by removing from T the box containing k, as in Section 3.4.4). Tensoring (8.36) on the right by V, and decomposing $U^{\lambda^{(k-1)}} \boxtimes V$ by means of Corollary 8.2.12 we get the desired result. □

Exercise 8.2.14 Give an alternative proof of Proposition 8.2.13 by

(i) using the fact that $e_{\overline{T}}$ is a factor of e_T (cf. (3.59));
(ii) using the spectral analysis of the YJM elements applied to $V^{\otimes k}$.

We end this section by giving the Littlewood–Richardson rule for $\mathrm{GL}(V)$. We still use the symbol $\boxtimes$ for the external tensor products even for vector spaces.

Theorem 8.2.15 *Let $V = V_1 \oplus V_2$ with $\dim V_1 = m$, $1 \leq m \leq n-1$. Consider the subgroup $\mathrm{GL}(V_1) \times \mathrm{GL}(V_2)$ of $\mathrm{GL}(V)$ associated with this decomposition. Then, for every $\lambda \vdash k$, $\ell(\lambda) \leq n$, we have*

$$\mathrm{Res}^{\mathrm{GL}(V)}_{\mathrm{GL}(V_1)\times\mathrm{GL}(V_2)} U^\lambda = \bigoplus_{\substack{\mu,\nu \vdash k \\ \ell(\nu)\leq m \\ \ell(\nu)\leq n-m}} c^\lambda_{\mu,\nu}[U^\mu \boxtimes U^\nu],$$

where the coefficients $c^\lambda_{\mu,\nu}$ coincide with those in (8.29) (for the symmetric group $\mathfrak{S}_k$).

Proof We follow the indications for the solution of [43, Exercise 6.11]. We first need some notation. Denote by Ω_h the space of all h-subsets of $\{1, 2, \ldots, k\}$ and, for each $A \in \Omega_h$, choose $\sigma_A \in \mathfrak{S}_k$ such that $\sigma_A(\{1, 2, \ldots \ldots, h\}) = A$. This way,

$$\mathfrak{S}_k = \coprod_{A\in\Omega_h} \sigma_A(\mathfrak{S}_h \times \mathfrak{S}_{k-h})$$

is the decomposition of $\mathfrak{S}_k$ into left $(\mathfrak{S}_h \times \mathfrak{S}_{k-h})$-cosets.

Consider the basis $\{e_1, e_2, \ldots, e_n\}$ of $V = V_1 \oplus V_2$ in such a way that $\{e_1, e_2, \ldots, e_m\}$ spans V_1 and $\{e_{m+1}, e_{m+2}, \ldots, e_n\}$ spans V_2. For $0 \leq h \leq k$, consider the subspace $\mathcal{W}_h$ of $V^{\otimes k} \equiv (V_1 \oplus V_2)^{\otimes k}$ spanned by the (basis)-vectors $e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_k}$ with $|\{t : i_t \in \{1, 2, \ldots, m\}| = h$. Clearly, we have (see Lemma 1.6.2)

$$\begin{aligned}\mathcal{W}_h &= \bigoplus_{A\in\Omega_h} \sigma_A(V_1^{\otimes h} \otimes V_2^{\otimes(k-h)}) \\ &\cong \mathrm{Ind}^{\mathfrak{S}_k}_{\mathfrak{S}_h\times\mathfrak{S}_{k-h}}[V_1^{\otimes h} \boxtimes V_2^{\otimes(k-h)}].\end{aligned}$$

Therefore,

$$
\begin{aligned}
V^{\otimes k} = (V_1 \oplus V_2)^{\otimes k} &= \bigoplus_{h=0}^{k} \mathcal{W}_h \\
&\cong \bigoplus_{h=0}^{k} \mathrm{Ind}_{\mathfrak{S}_h \times \mathfrak{S}_{k-h}}^{\mathfrak{S}_k} [V_1^{\otimes h} \boxtimes V_2^{\otimes (k-h)}] \\
&\cong_* \bigoplus_{h=0}^{k} \mathrm{Ind}_{\mathfrak{S}_h \times \mathfrak{S}_{k-h}}^{\mathfrak{S}_k} \left(\left[\bigoplus_{\substack{\mu \vdash h: \\ \ell(\mu) \le m}} (U^\mu \boxtimes S^\mu) \right] \boxtimes \left[\bigoplus_{\substack{\nu \vdash k-h: \\ \ell(\nu) \le n-m}} (U^\nu \boxtimes S^\nu) \right] \right) \\
&\cong \bigoplus_{h=0}^{k} \bigoplus_{\substack{\mu \vdash h: \\ \ell(\mu) \le m}} \bigoplus_{\substack{\nu \vdash k-h: \\ \ell(\nu) \le n-m}} \left[U^\mu \boxtimes U^\nu \boxtimes \mathrm{Ind}_{\mathfrak{S}_h \times \mathfrak{S}_{k-h}}^{\mathfrak{S}_k} (S^\mu \boxtimes S^\nu) \right] \\
&\cong_{**} \bigoplus_{h=0}^{k} \bigoplus_{\substack{\mu \vdash h: \\ \ell(\mu) \le m}} \bigoplus_{\substack{\nu \vdash k-h: \\ \ell(\nu) \le n-m}} \bigoplus_{\substack{\lambda \vdash k: \\ \ell(\lambda) \le n}} c_{\mu,\nu}^{\lambda} [U^\mu \boxtimes U^\nu \boxtimes S^\lambda]
\end{aligned}
$$

where $\cong_*$ follows from Theorem 8.2.10.(iv) and $\cong_{**}$ follows from (8.29) and Corollary 1.6.12. This is the decomposition of $V^{\otimes k}$ into irreducible $(\mathrm{GL}(V_1) \times \mathrm{GL}(V_2) \times \mathfrak{S}_k)$- representations.

On the other hand, we may start from Theorem 8.2.10.(iv) and apply $\mathrm{Res}_{\mathrm{GL}(V_1) \times \mathrm{GL}(V_2) \times \mathfrak{S}_k}^{\mathrm{GL}(V) \times \mathfrak{S}_k}$. Comparing the two results, the statement follows. □

Exercise 8.2.16 Show that the decomposition in Proposition 8.1.2 and Proposition 8.1.6 are particular cases of Theorem 8.2.15.

In [49, Theorem 9.2.3] it is shown that using seesaw reciprocity one may connect Theorem 8.2.11 and Theorem 8.2.15. This is one of the results in the remarkable paper [60].

8.3 The partition algebra

This section is devoted to another kind of Schur–Weyl duality. In the preceding section we studied the commutant of the action of $\mathfrak{S}_k$ on $V^{\otimes k}$ obtained by permuting tensors; we showed that $\mathrm{GL}(n, \mathbb{C})$ spans this commutant and we deduced several properties of a remarkable class of representations of $\mathrm{GL}(n, \mathbb{C})$. In the present section, we consider the action of $\mathfrak{S}_n$ on $V^{\otimes k}$ obtained by resticting the representation of $\mathrm{GL}(n, \mathbb{C})$ to the subgroup formed by all permutation matrices, which is clearly isomorphic to $\mathfrak{S}_n$. From another point of view, $V^{\otimes k} \cong L([n]^k)$

and $\mathfrak{S}_n$ acts on $L([n]^k)$ by tensoring k-times the permutation representations on the Young module $L([n]) \cong M^{n-1,1}$. We show that the commutant $\mathrm{End}_{\mathfrak{S}_n}(V^{\otimes k})$ is an algebra (called the *partition algebra*) described in terms of partitions of $\{1, 2, \ldots, 2k\}$ in a nice pictorial way (actually, for $n < 2k$ the commutant of $\mathrm{End}_{\mathfrak{S}_n}(V^{\otimes k})$ is a quotient of the partition algebra, while for $n \geq 2k$ they are isomorphic). The partition algebra was introduced, indipendently, by V. F. R. Jones [67] and P. Martin [86, 87]. Our main source is the paper by Halverson and Ram [58]. We have also benefited from the introductory part of the paper [8]. However, our treatment is more elementary and in some sense we are more interested in $\mathrm{End}_{\mathfrak{S}_n}(V^{\otimes k})$ than in the structure of the partition algebras, so that we have omitted several results such as semisemplicity of the partition algebras, the role of the basic constructions, the presentations of the partition monoids and the explicit expression of the YJM elements. For them we refer to [58]. Other important references on the partition algebras and related constructions are: [9, 13, 14, 30, 37, 38, 48, 56, 57, 88, 89, 90, 91, 92, 126].

8.3.1 The partition monoid

A *monoid* is a set with a binary operation which is associative and has an identity. In this subsection, for each positive integer $k \geq 1$, we construct a finite monoid P_k, that we call the *partition monoid*. The elements of P_k are the partitions of the set $\{1, 2, \ldots, k, 1', 2', \ldots, k'\}$, that we consider as the disjoint union of two copies of $\{1, 2, \ldots, k\}$. Given $d \in P_k$, $d = \{A_1, A_2, \ldots, A_m\}$, a *diagram representing* d is any (simple, undirected) graph (without loops) with vertex set $\{1, 2, \ldots, k, 1', 2', \ldots, k'\}$ and whose connected components are precisely the *parts* (or *blocks*) $A_1, A_2, \ldots, A_m$ of the partition d. This means that two vertices of the graph are connected by a path if and only if they belong to the same part of d. In most cases, such a graph is not unique, but our constructions do not depend on the particular chosen graph. A diagram representing an element of P_k will be drawn in the form of a two-row array: in the top row we put the elements $1, 2, \ldots, k$ and in the bottom row we put $1', 2', \ldots, k'$, in both cases respecting the natural order.

For example, for $k = 9$ the diagram in Figure 8.2

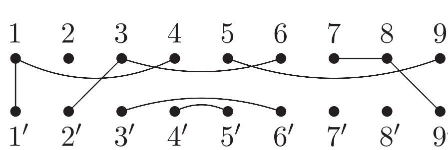


Figure 8.2

represents the partition

$$\{\{1, 1', 4\}, \{2\}, \{3, 6, 2'\}, \{5, 9\}, \{7, 8, 9'\}, \{3', 6'\}, \{4', 5'\}, \{7'\}, \{8'\}\}.$$

In general, we will define a partition $d \in P_k$ giving a diagram representing it. Note also that d induces a partion of $\{1, 2, \ldots, k\}$ whose parts are called *top blocks* and a partition of $\{1', 2', \ldots, k'\}$ whose parts are called *bottom blocks*. Morever, a top/bottom block may be *isolated* or connected respectively with exactly *one* bottom/top block.

An edge connecting two vertices i, j (or i', j') may be drawn as a straight line if $j = i + 1$ ($j' = i' + 1$) or, in general, as an arc which is curved inside the box delimited by the vertices; the latter representation is useful when the arc belongs to the middle row of a three-row diagram (see below) and we want to indicate the graph (the upper or the lower) to which it belongs.

Suppose that $d_1, d_2 \in P_k$. We define their product $d_1 \circ d_2$ by showing how to construct a diagram representing it. The construction is in three steps (see Figure 8.3).

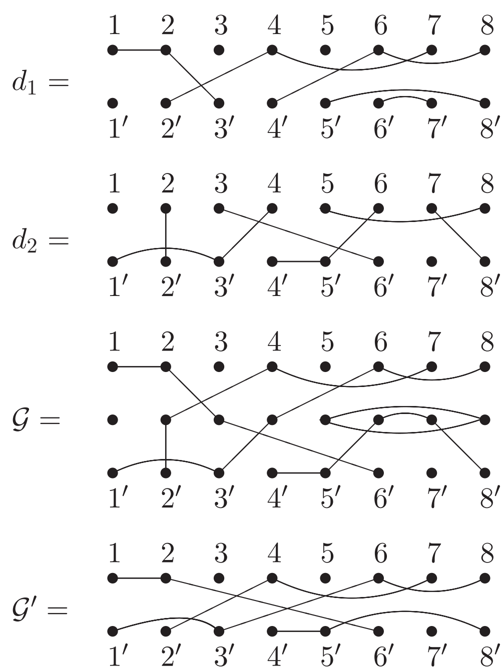


Figure 8.3

- First of all, we draw a diagram of d_1 above a diagram of d_2, and we identify the bottom row of d_1 with the top row of d_2. In such a way we get a graph $\mathcal{G}$ with three rows, that we call the *intermediate diagram*.

- Then we delete all the connected components of $\mathcal{G}$ that are entirely contained in the middle row.
- Finally, we eliminate all the other vertices in the middle row and we draw a graph $\mathcal{G}'$ whose vertex set consists of the top and bottom rows and that satisfies the following property: two vertices of $\mathcal{G}'$ are connected by a path if and only if they were connected by a path in $\mathcal{G}$. Then $d_1 \circ d_2$ is the partition represented by $\mathcal{G}'$.

It is easy to check that this product is associative: when we compute $d_1 \circ (d_2 \circ d_3)$ or $(d_1 \circ d_2) \circ d_3$, in both cases we can start constructing a four-row diagram, placing d_1 above d_2 and d_2 above d_3. Then we can follow the above procedure, deleting the second and the third row simultaneously. It is also clear that the identity element is the partition identified by the diagram in Figure 8.4:

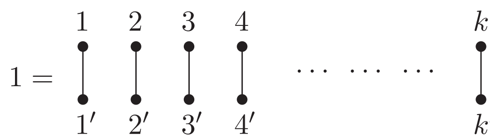


Figure 8.4

The *propagating number* $\mathrm{pd}(d)$ of a partition $d = \{A_1, A_2, \cdots, A_m\} \in P_k$ is the number of parts A_i such that both $A_i \cap \{1, 2, \ldots, k\}$ and $A_i \cap \{1', 2', \ldots, k'\}$ are nontrivial. The set of all $d \in P_k$ such that $\mathrm{pd}(d) = k$ coincides with the symmetric group on k elements, and thus it will be denoted by $\mathfrak{S}_k$. Indeed, if $\mathrm{pd}(d) = k$ then d must of the form $\{\{i_1, 1'\}, \{i_2, 2'\}, \ldots, \{i_k, k'\}\}$, where $i_1, i_2, \ldots, i_k$ is a permutation of $1, 2, \ldots, k$, and therefore we can associate it to the element $\pi \in \mathfrak{S}_k$ defined by setting: $\pi(j) = i_j$. It is also clear that in this identification the product $\circ$ coincides with the composition of permutations, that is the symmetric group $\mathfrak{S}_k$ is a *submonoid* of P_k.

For $i = 1, 2, \ldots, k-1$ and $j = 1, 2, \ldots, k$, we define the special elements of P_k (see Figure 8.5).

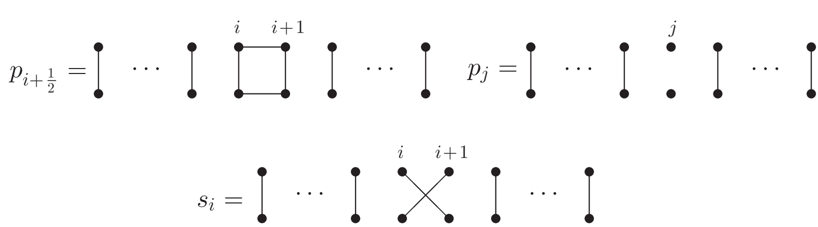


Figure 8.5

We also set $p_0 = p_{\frac{1}{2}} = 1$. Now we give a set of relations satisfied by these elements.

Proposition 8.3.1 *The following relations hold (for simplicity we omit the product symbol* $\circ$*):*

(i) *For* $i, j = \frac{1}{2}, 1, 1 + \frac{1}{2}, 2, \dots, k - \frac{1}{2}, k$ *we have:*

$$p_i^2 = p_i, \quad p_i p_{i \pm \frac{1}{2}} p_i = p_i$$

and, when $|i - j| > \frac{1}{2}$,

$$p_i p_j = p_j p_i.$$

(ii) *For* $i = 1, 2, \dots, k - 1$ *we have:*

$$s_i^2 = 1, \quad s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$$

and, when $|i - j| > 1$,

$$s_i s_j = s_j s_i.$$

(iii) *For* $i = 1, 2, \dots, k - 1$ *we have:*

$$\begin{aligned} s_i p_i p_{i+1} &= p_i p_{i+1} s_i = p_i p_{i+1}, \\ s_i p_{i+\frac{1}{2}} &= p_{i+\frac{1}{2}} s_i = p_{i+\frac{1}{2}}, \\ s_i s_{i+1} p_{i+\frac{1}{2}} s_{i+1} s_i &= p_{i+\frac{3}{2}}, \\ s_i p_i s_i &= p_{i+1} \end{aligned}$$

and, when $j \neq i - \frac{1}{2}, i, i + \frac{1}{2}, i + 1, i + \frac{3}{2}$,

$$s_i p_j = p_j s_i.$$

Proof These relations are easy to verify; in particular, those in (ii) correspond to the usual relations for the adjacent transpositions in $\mathfrak{S}_n$, (see (3.33)). As an example, we draw the diagram (Figure 8.6) that proves the relation $s_i s_{i+1} p_{i+\frac{1}{2}} s_{i+1} s_i = p_{i+\frac{3}{2}}$:

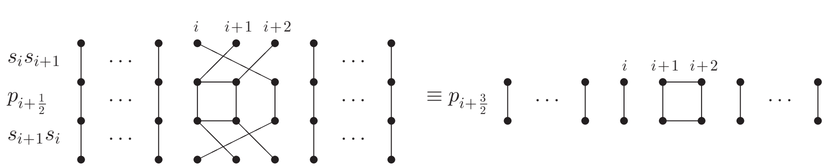


Figure 8.6

□

Proposition 8.3.2 *The elements* $p_1, p_{1+\frac{1}{2}}, p_2, \dots, p_{k-\frac{1}{2}}, p_k, s_1, s_2, \dots, s_{k-1}$ *generate the partition monoid* P_k.

Proof We begin by introducing a definition: a *special planar partition* is a partition $d \in P_k$ of the form:

$$\begin{aligned} d = \Big\{ &\{1, 2, \dots, i_1, 1', 2', \dots, j_1'\}, \\ &\{i_1+1, i_1+2, \dots, i_2, (j_1+1)', (j_1+2)', \dots, j_2'\}, \\ &\dots, \{i_{t-1}+1, i_{t-1}+2, \dots, i_t, (j_{t-1}+1)', (j_{t-1}+2)', \dots, j_t'\}, \\ &\{i_t+1, i_t+2, \dots, i_{t+1}\}, \{i_{t+1}+1, i_{t+1}+2, \dots, i_{t+2}\}, \\ &\dots, \{i_{s-1}+1, i_{s-1}+2, \dots, i_s\}, \\ &\{(j_t+1)', (j_t+2)', \dots, (j_{t+1})'\}, \{(j_{t+1}+1)', (j_{t+1}+2)', \dots, (j_{t+2})'\}, \\ &\dots, \{(j_{r-1}+1)', (j_{r-1}+2)', \dots, (j_r)'\} \Big\} \end{aligned}$$

where $i_1 < i_2 < \cdots i_t < i_{t+1} < \cdots < i_s = k$ and $j_1' < j_2' < \cdots j_t' < j_{t+1}' < \cdots < j_r' = k'$. It will be represented by a diagram which is a sequence of trapezoids followed by a sequence of horizontal lines, in the top and bottom row. We give an example: for $t = 2$, $s = 4$, $r = 6$, $i_1 = 4$, $i_2 = 8$, $i_3 = 10$, $i_4 = 13$, $j_1' = 3'$, $j_2' = 6'$, $j_3' = 9'$, $j_4' = 11'$, $j_5' = 12'$ and $j_6' = 13'$ the diagram is shown in Figure 8.7.

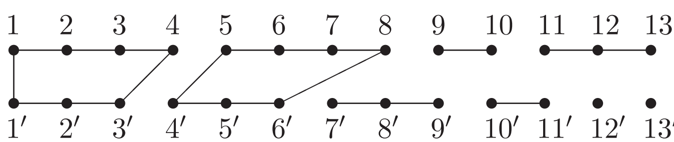


Figure 8.7

We claim that every special planar partition may be written as a product of the elements $p_1, p_{1+\frac{1}{2}}, p_2, \dots, p_{k-\frac{1}{2}}, p_k$. It suffices to analyze the above example: its expression as a product of $p_1, p_{\frac{3}{2}}, \dots, p_k$ may be obtained step by step as indicated in Figure 8.8 (again, we omit the symbol $\circ$).

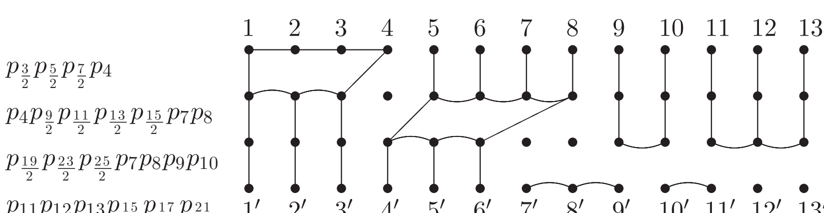


Figure 8.8

All the other special planar partitions may be handled in the same way. To end the proof of the proposition, it suffices to note that every $d \in P_k$ may be written in the form: $d = d_1 \circ d_2 \circ d_3$, where $d_1, d_2 \in \mathfrak{S}_k$ and d_2 is a special planar partition. Then we may simply invoke the fact that the elements $s_1, \ldots, s_{k-1}$ generate $\mathfrak{S}_k$. □

Remark 8.3.3 In [58] it is shown that the generators in Proposition 8.3.2 together with the relations in Proposition 8.3.1 form a *presentation* of the partiton monoid.

Remark 8.3.4 The submonoid generated by the elements $p_{\frac{1}{2}}, p_1, p_{1+\frac{1}{2}}, p_2, \ldots, p_{k-\frac{1}{2}}, p_k$ is called the *planar partition monoid*. It is formed by all partitions that may be represented by a diagram which is *planar*, that is, no two edges intersect. There are planar diagrams that are not special (in the definition introduced in the proof of Proposition 8.3.2): Figure 8.9 is a simple example

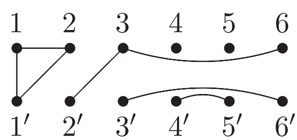


Figure 8.9

that may be obtained as a product of the p's as indicated in Figure 8.10 (as usual, for simplicity, we omit the symbol ∘):

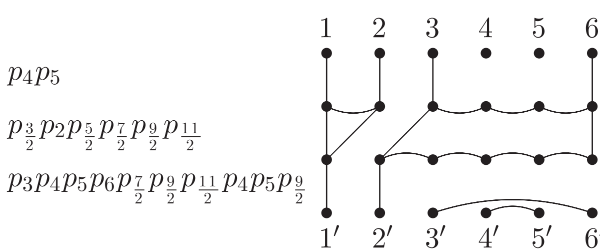


Figure 8.10

The planar partition algebra is isomorphic to the famous *Temperley–Lieb algebra*; we refer to [48] for more details on the latter algebra and to [58] for details on the isomorphism.

It is convenient to introduce a particular submonoid of P_{k+1}. If k is a positive integer, $P_{k+\frac{1}{2}}$ is the submonoid of P_{k+1} formed by all partitions d such that $k+1$ and $(k+1)'$ belong to the same part. We also set $P_0 \equiv P_{\frac{1}{2}} = \{1\}$.

Exercise 8.3.5 Show that $P_{k+\frac{1}{2}}$ is generated by the elements $p_1, p_{1+\frac{1}{2}}, p_2, \ldots$ $\ldots, p_{k-\frac{1}{2}}, p_k, p_{k+\frac{1}{2}}, s_1, s_2, \ldots, s_{k-1}$.

We end this subsection considering again the Stirling numbers of the second kind (see Section 4.4.3).

Proposition 8.3.6 *For a couple of integers satifying* $0 \leq h \leq m$*, the number of partitions of* $\{1, 2, \ldots, m\}$ *into* h *parts is equal to the Stirling number of the second kind* $S(m, h)$*.*

Proof Let $\widetilde{S}(m, h)$ be the number of partitions of $\{1, 2, \ldots, m\}$ into h parts. It suffices to check that $\widetilde{S}(m, 0) = 0$, $\widetilde{S}(m, 1) = 1$ (both obvious) and that

$$\widetilde{S}(m + 1, h) = \widetilde{S}(m, h - 1) + h\widetilde{S}(m, h); \tag{8.37}$$

see Exercise 4.4.11. The identity (8.37) has a natural combinatorial proof: a partition of $\{1, 2, \ldots, m + 1\}$ into h parts may be obtained adding the part $\{m + 1\}$ to a partition of $\{1, 2, \ldots, m\}$ into $h - 1$ parts or adding $m + 1$ to a part of a partition of $\{1, 2, \ldots, m\}$ into h parts. □

The *Bell numbers* are the positive integers $B(m)$ defined by setting

$$B(m) = \sum_{h=1}^{m} S(m, h).$$

From Proposition 8.3.6 it follows that $B(m)$ is equal to the number of partitions of $\{1, 2, \ldots, m\}$. In particular, the cardinality of the partition monoid P_k is given by:

$$|P_k| = B(2k) \qquad \text{for} \qquad k = 1, 1 + \frac{1}{2}, 2, 2 + \frac{1}{2}, \ldots .$$

Exercise 8.3.7 Show that the Bell numbers have the following exponential generating function:

$$\sum_{h=0}^{\infty} B(h)\frac{z^h}{h!} = \exp(e^z - 1).$$

Hint. Use 4. in Exercise 4.4.11.

8.3.2 The partition algebra

For each $k \in \{0, \frac{1}{2}, 1, 1 + \frac{1}{2}, \ldots\}$ and $n \in \mathbb{N}$, we define an (abstract) associative algebra $\mathcal{P}_k(n)$, called the *partition algebra* of parameters k and n. As a vector space, it is formed by all formal linear combinations of elements of the partition monoid P_k, that is, an element of $\mathcal{P}_k(n)$ is of the form $\sum_{d\in P_k} f(d)d$, where $f : P_k \to \mathbb{C}$. In other words, P_k may be identified with a vector space basis of $\mathcal{P}_k(n)$. In order to complete the definition of $\mathcal{P}_k(n)$, it suffices to define an associative product on the basis elements. It is a slight modification of the product $d_1 \circ d_2$ in the partition monoid, and it is denoted simply by d_1d_2 (juxtaposition). For $d_1, d_2 \in P_k$ let $\mathcal{G}$ be the intermediate diagram constructed to define $d_1 \circ d_2$ (see Section 8.3.1). Then we set

$$d_1d_2 = n^c d_1 \circ d_2,$$

where c is the number of connected components of $\mathcal{G}$ that are entirely contained in the middle row. For example, if d_1 and d_2 are as in Figure 8.3(a), then

$$d_1d_2 = n^2 d_1 \circ d_2.$$

For $k = 0, \frac{1}{2}$ we have: $\mathcal{P}_0(n) \cong \mathcal{P}_{\frac{1}{2}}(n) \cong \mathbb{C}$, since $P_0 \cong P_{\frac{1}{2}} \equiv \{1\}$.

Example 8.3.8 For $k = 1$, $P_1 = \{1, p_1\}$, 1 is the identity and we have:

$$p_1p_1 = np_1.$$

Given two partitions $d_1, d_2 \in P_k$, we say that d_1 is *coarser* than d_2 (or that d_2 is a *refinement* of d_1) when each block of d_2 is contained in a block of d_1 (equivalently, each block of d_1 is the union of some blocks of d_2). For instance, $d_1 = \{\{1, 2\}, \{3, 4, 1', 2'\}, \{3', 4'\}\}$ is coarser than $d_2 = \{\{1\}, \{2\}, \{4\}, \{3, 1', 2'\}, \{3', 4'\}\}$. We write $d_1 \le d_2$ when d_1 is coarser than d_2. Note that $\le$ is a partial order on the set P_k. It is convenient to extend $\le$ to a total order. It is simple: put the $B(2k)$ partitions in P_k into a sequence

$$d_1 \prec d_2 \prec d_3 \prec \cdots \prec d_{B(2k)-1} \prec d_{B(2k)} \tag{8.38}$$

in which the number of parts of d_1 is less than or equal to the number of parts of d_{i+1}, $i = 1, 2, \dots, B(2k) - 1$. Clearly $d_1 = \{1, 2, \dots, k, 1', 2', \dots, k'\}$ and

$$d_{B(2k)} = \{\{1\}, \{2\}, \dots, \{k\}, \{1'\}, \{2'\}, \dots, \{k'\}\}.$$

Then we get a total order $\prec$ such that

$$d, d' \in P_k,\ d < d' \Rightarrow d \prec d'.$$

We can use this order to define a set of elements $\{x_d : d \in P_k\}$ of $\mathcal{P}_k(n)$ by means of the following relations:

$$d = \sum_{d'\le d} x_{d'}. \tag{8.39}$$

Indeed, clearly $x_{d_1} \equiv d_1$ (where $d_1 = \{1, 2, \dots, k, 1', 2', \dots, k'\}$) and then we can compute $x_{d_2}, x_{d_3}, \dots, x_{d_{B(2k)}}$, that are uniquely defined $(d_2, d_3, \dots, d_{B(2k)}$

are as in (8.38)). From the relation $d_i = x_{d_i} + \sum_{d'<d_i} x_{d'}$ and induction on i, it follows that the matrix that represents the x_{d_i}'s in terms of the d_i's is lower triangular with 1's on the diagonal. Therefore we have proved the following proposition.

Proposition 8.3.9 *The set $\{x_d : d \in P_k\}$ is a basis of $\mathcal{P}_k(n)$.*

Example 8.3.10 For $k = 1$ (see Example 8.3.8) we have $d_1 \equiv 1$, $d_2 \equiv p_1$ and $x_{d_1} = 1$ and $x_{d_2} = p_1 - 1$, so that

$$(x_{d_2})^2 = (n-1)x_{d_2} + nx_{d_1}.$$

For $k \in \mathbb{N}$, there is a natural inclusion $\mathcal{P}_{k+\frac{1}{2}}(n) \subseteq \mathcal{P}_{k+1}(n)$, because $P_{k+\frac{1}{2}}$ is defined as a submonoid of P_{k+1}. But we can also define an (injective) inclusion

$$\begin{array}{ccc} P_k & \hookrightarrow & P_{k+\frac{1}{2}} \\ d & \mapsto & d' \end{array} \tag{8.40}$$

where the partition d' is obtained from $d \in P_k$ by adding the isolated part $\{k+1, (k+1)'\}$. This leads to an inclusion of $\mathcal{P}_k(n)$ in $\mathcal{P}_{k+\frac{1}{2}}(n)$, that is $\mathcal{P}_{k+\frac{1}{2}}(n)$ contains a subalgebra isomorphic to $\mathcal{P}_k(n)$ and we identify $\mathcal{P}_k(n)$ with this subalgebra. This way, we get a *chain* (or *tower*) of algebras:

$$\begin{aligned} &\mathcal{P}_0(n) \subseteq \mathcal{P}_{\frac{1}{2}}(n) \subseteq \mathcal{P}_1(n) \subseteq \mathcal{P}_{1+\frac{1}{2}}(n) \subseteq \cdots \\ &\qquad\qquad \cdots \subseteq \mathcal{P}_k(n) \subseteq \mathcal{P}_{k+\frac{1}{2}}(n) \subseteq \mathcal{P}_{k+1}(n) \subseteq \cdots \end{aligned} \tag{8.41}$$

where $\mathcal{P}_0(n) \cong \mathcal{P}_{\frac{1}{2}}(n) \cong \mathbb{C}$ by definition.

8.3.3 Schur–Weyl duality for the partition algebra

Let V and $V^{\otimes k}$ be as in Sections 8.1 and 8.2. In particular, $n = \dim V$ and $\{e_1, e_2, \ldots, e_n\}$ is an orthonormal basis of V. We define a representation ρ_k of the symmetric group $\mathfrak{S}_n$ on $V^{\otimes k}$ by setting

$$\rho_k(\pi)(e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_k}) = e_{\pi(i_1)} \otimes e_{\pi(i_2)} \otimes \cdots \otimes e_{\pi(i_k)}$$

for all $\pi \in \mathfrak{S}_n$ and all basis vectors $e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_k}$. Note that for $k = 1$, it is just the permutation representation in Example 1.4.5 (and in the notation of Section 3.6.2 it is the Young module $M^{n-1,1}$). Note also that ρ_k may be considered as the restriction of the representation (denoted again by ρ_k) of $\mathrm{GL}(V)$ on $V^{\otimes k}$ presented in Section 8.2.1 to the subgroup of $\mathrm{GL}(V)$ consisting of the permutation matrices (in the basis $\{e_1, e_2, \ldots, e_n\}$). For instance, the matrix $\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ represents the permutation

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \to \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \to \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \to \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},$$

that is $(1 \to 2 \to 3 \to 1)$. If we identify $V^{\otimes k}$ with $L([n]^k)$ (cf. (8.1)) then ρ_k coincides with the tensor product of permutation representations considered in Example (7.5.19). As in that example, we set $V \cong M^{n-1,1}$, $V^{\otimes 0} \cong M^{(n)}$, the trivial representation of $\mathfrak{S}_n$ and

$$\mathcal{A}_k(n) = \mathrm{End}_{\mathfrak{S}_n}(V^{\otimes k}) \quad k = 0, 1, 2, \dots,$$

$$\mathcal{A}_{k+\frac{1}{2}}(n) = \mathrm{End}_{\mathfrak{S}_{n-1}}(\mathrm{Res}^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}} V^{\otimes k}) \quad k = 0, 1, 2, \dots,$$

The aim of the present section is to show that $\mathcal{A}_k(n)$, $k \in \{0, \frac{1}{2}, 1, 1 + \frac{1}{2}, \dots, \}$ is isomorphic to $\mathcal{P}_k(n)$ for certain values of k and n, or to a quotient on $\mathcal{P}_k(n)$ for the other values. First of all, we introduce a notation for $\mathrm{End}(V^{\otimes k})$. An operator $A \in \mathrm{End}(V^{\otimes k})$ is determined by a matrix $A^{i_1, i_2, \dots, i_k}_{i_{1'}, i_{2'}, \dots, i_{k'}}$ where the indices $(i_1, i_2, \dots, i_k)$ and $(i_{1'}, i_{2'}, \dots, i_{k'})$ range in $[n]^k$. More precisely, we set

$$A(e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_k}) = \sum_{(i_{1'}, i_{2'}, \dots, i_{k'}) \in [n]^k} A^{i_1, i_2, \dots, i_k}_{i_{1'}, i_{2'}, \dots, i_{k'}} e_{i_{1'}} \otimes e_{i_{2'}} \otimes \cdots \otimes e_{i_{k'}} \tag{8.42}$$

for all basis vector $e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_k}$. In the notation of (8.3),

$$A^{i_1, i_2, \dots, i_k}_{i_{1'}, i_{2'}, \dots, i_{k'}} = F(i_{1'}, i_{2'}, \dots, i_{k'}, i_1, i_2, \dots, i_k)$$

and $A \equiv T_F$. We can define a representation Φ_k of $\mathcal{P}_k(n)$ on $V^{\otimes k}$ by specifying, for each $d \in P_k$, the matrix representing the operator $\Phi_k(d)$. We need a notation: for $d \in P_k$, we write $s \overset{d}{\sim} t$ to denote that $s, t \in \{1, 2, \dots, k, 1', 2', \dots, k'\}$ belong to the same part of d (they are equivalent for the relation associated with d). Then we set

$$[\Phi_k(d)]^{i_1, i_2, \dots, i_k}_{i_{1'}, i_{2'}, \dots, i_{k'}} = \begin{cases} 1 & \text{if } s \overset{d}{\sim} t \Rightarrow i_s = i_t \\ 0 & \text{otherwise} \end{cases} \tag{8.43}$$

for all $d \in P_k$. This is another description of $[\Phi_k(d)]^{i_1, i_2, \dots, i_k}_{i_{1'}, i_{2'}, \dots, i_{k'}}$. Taken $(i_1, i_2, \dots, i_k, i_{1'}, i_{2'}, \dots, i_{k'}) \in [n]^k \times [n]^k$, define a partition d' by taking the subsets on which the function

$$\{1, 2, \dots, k, 1', 2', \dots, k'\} \ni r \mapsto i_r$$

is constant. Then $[\Phi_k(d)]^{i_1,i_2,\dots,i_k}_{i_{1'},i_{2'},\dots,i_{k'}} = 1$ if d' is coarser then d and it is equal to 0 otherwise.

A third description of $\Phi_k(d)$ is the following. Let $\mathcal{G}$ be a diagram representing d. Then

$$[\Phi_k(d)]^{i_1,i_2,\dots,i_k}_{i_{1'},i_{2'},\dots,i_{k'}} = \prod \delta_{i_s,i_t},$$

where the product is over all $s, t \in \{1, 2, \dots, k, 1', 2', \dots, k'\}$ that are connected by an edge in $\mathcal{G}$. For instance, if d is represented by the diagram of Figure 8.2, then

$$[\Phi_k(d)]^{i_1,i_2,\dots,i_9}_{i_{1'},i_{2'},\dots,i_{9'}} = \delta_{i_1,i_{1'}}\delta_{i_1,i_4}\delta_{i_{2'},i_3}\delta_{i_3,i_6}\delta_{i_{3'},i_{6'}}\delta_{i_{4'},i_{5'}}\delta_{i_5,i_9}\delta_{i_7,i_8}\delta_{i_8,i_{9'}}.$$

We also give the expression of Φ_k on the generators in Proposition 8.3.2: on each basis element $e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_k}$ we have:

$$\begin{aligned}
&\Phi_k(s_j)(e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_k})\\
&\quad = e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_{j-1}} \otimes e_{i_{j+1}} \otimes e_{i_j} \otimes e_{i_{j+2}} \otimes \cdots \otimes e_{i_k}
\end{aligned}$$

$$\begin{aligned}
&\Phi_k(p_j)(e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_k})\\
&\quad = \sum_{\ell=1}^{n} e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_{j-1}} \otimes e_{i_\ell} \otimes e_{i_{j+1}} \otimes \cdots \otimes e_{i_k}
\end{aligned}$$

$$\begin{aligned}
&\Phi_k(p_{j+\frac{1}{2}})(e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_k})\\
&\quad = \delta_{i_j,i_{j+1}} e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_j} \otimes e_{i_{j+1}} \otimes \cdots \otimes e_{i_k}.
\end{aligned}$$

Lemma 8.3.11 *Φ_k is indeed a representation of $\mathcal{P}_k(n)$ on $V^{\otimes k}$.*

Proof It suffices to check that, for $d, d' \in P_k$, we have $\Phi_k(dd') = \Phi_k(d)\Phi_k(d')$. For

$$(i_1, i_2, \dots, i_k), (i_{1''}, i_{2''}, \dots, i_{k''}) \in [n]^k$$

we have:

$$[\Phi_k(d)\Phi_k(d')]^{i_1,i_2,\dots,i_k}_{i_{1''},i_{2''},\dots,i_{k''}} = \sum_{i_{1'},i_{2'},\dots,i_{k'}=1}^{n} [\Phi_k(d)]^{i_1,i_2,\dots,i_k}_{i_{1'},i_{2'},\dots,i_{k'}} [\Phi_k(d')]^{i_{1'},i_{2'},\dots,i_{k'}}_{i_{1''},i_{2''},\dots,i_{k''}} \tag{8.44}$$

Suppose that $dd' = n^c d''$ and let $\mathcal{G}$ be the intermediate diagram that is constructed to compute $d'' \equiv d \circ d'$ (see Section 8.3.1). When we compute the right-hand side of (8.44), in each isolated block in the middle row, the corresponding $i_{\ell'}$ are all equal and they can take all the values $1, 2, \ldots, n$. This gives a factor n^c (c = number of those isolated components). The values of those $i_{\ell'}$ that do not correspond to an isolated part are determined by the values of those $i_1, i_2, \ldots, i_k, i_{1''}, i_{2''}, \ldots, i_{k''}$ to which are connected in $\mathcal{G}$. Moreover, the values of those $i_1, i_2, \ldots, i_k, i_{1''}, i_{2''}, \ldots, i_{k''}$ that correspond to a block of $d'' = d \circ d'$ must be equal (for instance, if $r \overset{d}{\sim} \ell'$ and $\ell' \overset{d'}{\sim} s''$ then $i_r = i_{\ell'}$ and $i_{\ell'} = i_{s''}$ force $i_r = i_{s''}$). The above considerations lead to the conclusion that the right-hand side of (8.44) is equal to $[\Phi_k(d'')]^{i_1, i_2, \ldots, i_k}_{i_{1''}, i_{2''}, \ldots, i_{k''}}$ and this ends the proof. □

We now compute $\Phi_k(x_d)$, $d \in P_k$.

Lemma 8.3.12 *For each $d \in P_k$ we have:*

(i)

$$\Phi_k(x_d) = \begin{cases} 1 & \text{if } s \overset{d}{\sim} t \Leftrightarrow i_s = i_t \\ 0 & \text{otherwise} \end{cases}$$

when the number of parts of d is $\leq n$.

(ii) $\Phi_k(x_d) = 0$ *when the number of parts of d is $> n$.*

Proof Compare the definition of $\Phi_k(d)$ (see (8.43)) with (i): $s \overset{d}{\sim} t \Rightarrow i_s = i_t$ has been replaced by $s \overset{d}{\sim} t \Leftrightarrow i_s = i_t$. This means that $[\Phi_k(x_d)]^{i_1, i_2, \ldots, i_k}_{i_{1'}, i_{2'}, \ldots, i_{k'}} = 1$ if and only if the subsets on which the function $\{1, 2, \ldots, k, 1', 2', \ldots, k'\} \ni r \mapsto i_r$ is constant are precisely the blocks of d. Therefore (i) is an immediate consequence of the definition of x_d (see (8.39)) and of $\Phi_k(d)$, with an obvious inductive argument (using the order in (8.38)). Note also that (i) implies (ii): if the number of parts of d is $> n$, then we cannot construct a function $\{1, 2, \ldots, k, 1', 2', \ldots, k'\} \ni r \mapsto i_r \in \{1, 2, \ldots, n\}$ such that the subsets on which it is constant are exactly the blocks of d. Alternatively, the inductive argument, used in the first part of the proof, leads easily to the conclusion that $\Phi_k(x_d) = 0$. □

This is an alternative and fundamental description of $\Phi_k(x_d)$. Consider the following action of $\mathfrak{S}_n$ on $[n]^k \times [n]^k$:

$$\pi(i_1, i_2, \ldots, i_k, i_{1'}, i_{2'}, \ldots, i_{k'}) = (\pi(i_1), \pi(i_2), \ldots, \pi(i_k), \pi(i_{1'}), \pi(i_{2'}), \ldots, \pi(i_{k'})). \quad (8.45)$$

Denote by $P_k(\leq n)$ the set of all $d \in P_k$ that have at most n blocks. With each $d \in P_k(\leq n)$ we associate the set Ω_d of all $(i_1, i_2, \ldots, i_k, i_{1'}, i_{2'}, \ldots, i_{k'}) \in$

$[n]^k \times [n]^k$ such that the subsets on which the functions $\{1, 2, \ldots, k, 1, 2, \ldots \ldots, k'\} \ni r \mapsto i_r$ is constant are exactly the blocks of d. Then

$$[n]^k \times [n]^k = \coprod_{d \in P_k(\leq n)} \Omega_d \tag{8.46}$$

is the decomposition of $[n]^k \times [n]^k$ into $\mathfrak{S}_n$-orbits. Indeed, the orbit of $\mathfrak{S}_n$ containing $(i_1, i_2, \ldots, i_k, i_{1'}, i_{2'}, \ldots, i_{k'})$ is determined by the equivalence relation $s \overset{d}{\sim} t \Leftrightarrow i_s = i_t$. Moreover $\Phi_k(x_d)$ is exactly the operator associated with the characteristic function of the orbit Ω_d (see (8.3) or Section 1.4.1). These considerations, together with Lemma 8.3.12, lead to the Schur–Weyl duality for $\mathcal{P}_k(n)$, $k = 0, 1, 2, \ldots$.

Theorem 8.3.13 (Schur–Weyl duality for $\mathcal{P}_k(n)$) *For $k \in \mathbb{N}$, the map Φ_k is a surjective homomorphism of $\mathcal{P}_k(n)$ onto $\mathcal{A}_k(n)$. Moreover, for $n \geq 2k$ it is also an isomorphism: $\mathcal{P}_k(n) \cong \mathcal{A}_k(n)$, while for $n < 2k$ we have*

$$\mathrm{Ker}\Phi_k = \langle x_d : d \in P_k \text{ and it has more than } n \text{ parts}\rangle.$$

Proof Indeed, by Proposition 1.4.1, Lemma 8.3.11, Lemma 8.3.12 and (8.46) the set $\{\Phi_k(x_d) : d \in P_k(\leq n)\}$ is a basis for $\mathcal{A}_k(n)$, while $\Phi_k(x_d) = 0$ if d has more than n parts. □

Now we examine the Schur–Weyl duality for $\mathcal{P}_{k+\frac{1}{2}}(n)$, $k = 0, 1, 2, \ldots$ We define a representation $\Phi_{k+\frac{1}{2}}$ of $\mathcal{P}_{k+\frac{1}{2}}(n)$ on $V^{\otimes k}$ by setting, for $d \in P_{k+\frac{1}{2}} (\subseteq P_{k+1})$ and

$$(i_1, i_2, \ldots, i_k, i_{1'}, i_{2'}, \ldots, i_{k'}) \in [n]^k \times [n]^k,$$

$$[\Phi_{k+\frac{1}{2}}(d)]^{i_1, i_2, \ldots, i_k}_{i_{1'}, i_{2'}, \ldots, i_{k'}} = [\Phi_{k+1}(d)]^{i_1, i_2, \ldots, i_k, n}_{i_{1'}, i_{2'}, \ldots, i_{k'}, n}. \tag{8.47}$$

In other words, the value of $\Phi_{k+\frac{1}{2}}(d)$ for the indices $(i_1, i_2, \ldots, i_k, i_{1'}, i_{2'}, \ldots \ldots, i_{k'})$ is equal to the value of $\Phi_{k+1}(d)$ for the indices $(i_1, i_2, \ldots, i_k, n, i_{1'}, i_{2'}, \ldots, i_{k'}, n)$, that is, $i_{k+1} = i_{(k+1)'} = n$. There is an alternative description of $\Phi_{k+\frac{1}{2}}(d)$: the space $V^{\otimes k}$ is isomorphic to $V^{\otimes k} \otimes e_n$, and the last space is invariant under the action of $\Phi_{k+1}(d)$ when $d \in P_{k+\frac{1}{2}}$ and $(k+1)$ and $(k+1)'$ belong to the same part of d, and therefore if $i_{k+1} = n$, then we must also have $i_{(k+1)'} = n$ to get $[\Phi_{k+1}(d)]^{i_1, i_2, \ldots, i_k, i_{(k+1)}}_{i_{1'}, i_{2'}, \ldots, i_{k'}, i_{(k+1)'}} = 1$ (see (8.42) and (8.43)). Then $\Phi_{k+\frac{1}{2}}(d)$ coincides with the restriction of $\Phi_{k+1}(d)$ to $V^{\otimes k} \otimes e_n$. This second description immediately yields the following as a consequence of Lemma 8.3.12.

Lemma 8.3.14 $\Phi_{k+\frac{1}{2}}$ *is a representation of $\mathcal{P}_{k+\frac{1}{2}}(n)$ on $V^{\otimes k}$.*

We now describe the orbits of $\mathfrak{S}_{n-1}$ on $[n]^k \times [n]^k$. Denote by $P_{k+\frac{1}{2}}(\leq n)$ the set of all partitions $d \in P_{k+\frac{1}{2}}$ with at most n blocks. To each $d \in P_{k+1}(\leq n)$ we associate a subset $\Omega_d \subseteq [n]^k \times [n]^k$ in the following way: ω_d is the set of all $(i_1, i_2, \ldots, i_k, i_{1'}, i_{2'}, \ldots, i_{k'}) \in [n]^k \times [n]^k$ such that the subsets on which the function $\{1, 2, \ldots, k, 1', 2', \ldots, k'\} \ni r \mapsto i_r$ is constant, with the constant value different from n, are the blocks of d not containing $\{k+1, (k+1)'\}$, while the subset $\{i_r : r = n\}$ coincides with $B \setminus \{k+1, (k+1)'\}$, where B is the block of d containing $\{k+1, (k+1)'\}$. Then

$$[n]^k \times [n]^k = \coprod_{d \in P_{k+\frac{1}{2}}(\leq n)} \Omega_d$$

is the decomposition of $[n]^k \times [n]^k$ into $\mathfrak{S}_{n-1}$-orbits. Indeed, the orbit of $\mathfrak{S}_{n-1}$ containing $(i_1, i_2, \ldots, i_k, i_{1'}, i_{2'}, \ldots, i_{k'}) \in [n]^k \times [n]^k$ is determined by the equivalence relation $s \overset{d}{\sim} t \Leftrightarrow i_s = i_t$, together with the distinguished block $B = \{r : i_r = n\}$. From Lemma 8.3.12.(i), (8.47) and the subsequent discussion, it follows that $\Phi_{k+\frac{1}{2}}(x_d)$, for $d \in P_{k+\frac{1}{2}}(\leq n)$, coincides with the intertwining operator associated to the $\mathfrak{S}_{n-1}$-orbit Ω_d. This yields the Schur–Weyl duality for $\mathcal{P}_{k+\frac{1}{2}}(n)$.

Theorem 8.3.15 *For $k \in \mathbb{N}$, the map $\Phi_{k+\frac{1}{2}}$ is a surjective homomorphism of $\mathcal{P}_{k+\frac{1}{2}}(n)$ onto $\mathcal{A}_{k+\frac{1}{2}}(n)$. Moreover, for $n \geq 2k+1$ it is an isomorphism*

$$\mathcal{P}_{k+\frac{1}{2}}(n) \cong \mathcal{A}_{k+\frac{1}{2}}(n),$$

while for $n < 2k+1$ we have:

$$\mathrm{Ker}\Phi_{k+\frac{1}{2}} = \langle x_d : d \in P_{k+\frac{1}{2}} \text{ and it has more that } n \text{ blocks}\rangle.$$

Remark 8.3.16 For the values $n \geq 2k$ in Theorem 8.3.13 and $n \geq 2k+1$ in Theorem 8.3.15 we also deduce that $\mathcal{P}_k(n)$ is semisimple (see Remark 7.4.6 and Theorem 7.4.7). In [58] it is proved that $\mathcal{P}_k(n)$ is semisimple if and only if $k \leq \frac{n+1}{2}$, where $n \in \{2, 3, \ldots, \}$ and $k \in \{0, \frac{1}{2}, 1, 1+\frac{1}{2}, \ldots, \}$. This result is based on previous work of Martin: [88] and [91].

In order to complete the connection between the representation theory of the partition algebras and the representation theory of the algebras $\mathcal{A}_k(n)$ (see Section 7.5.5), we need to check the compatibility between the representation Φ_k and the inclusions in the towers of algebras of the $\mathcal{A}_k$'s and the $\mathcal{P}_k$'s.

Proposition 8.3.17 *The following diagram*

$$\begin{array}{ccccc} \mathcal{P}_k(n) & \longrightarrow & \mathcal{P}_{k+\frac{1}{2}}(n) & \longrightarrow & \mathcal{P}_{k+1}(n) \\ \downarrow{\scriptstyle \Phi_k} & & \downarrow{\scriptstyle \Phi_{k+\frac{1}{2}}} & & \downarrow{\scriptstyle \Phi_{k+1}} \\ \mathcal{A}_k(n) & \longrightarrow & \mathcal{A}_{k+\frac{1}{2}}(n) & \longrightarrow & \mathcal{A}_{k+1}(n) \end{array}$$

where the horizontal maps are the inclusion in the towers (7.32) *and* (8.41)*, is commutative.*

Proof First we prove the commutativity of the first part of the diagram. Let $d \in P_k$ and let $d' = d \coprod \{(k+1), (k+1)'\}$ be its image in the inclusion $\mathcal{P}_k(n) \hookrightarrow \mathcal{P}_{k+\frac{1}{2}}(n)$ (see (8.40)). It suffices to check that $\Phi_k(d) = \Phi_{k+\frac{1}{2}}(d')$ as operators on $V^{\otimes k}$. But this fact follows from the Definition (8.43) and from the fact that now in (8.47) (applied to d') $\{k+1, (k+1)'\}$ is an isolated part. We now prove the commutativity of the second part of the diagram. We show that, for all $d \in P_{k+\frac{1}{2}}$, $\Phi_{k+1}(x_d)$ is the image of $\Phi_{k+\frac{1}{2}}(x_d)$ in the inclusion $\mathcal{A}_{k+\frac{1}{2}} \hookrightarrow \mathcal{A}_{k+1}$. This fact is clear: $\Phi_{k+\frac{1}{2}}(x_d)$ is, by definition, the restriction of $\Phi_{k+1}(d)$ to $V^{\otimes k} \otimes e_n$, and this is exactly the inclusion $\mathcal{A}_{k+\frac{1}{2}} \hookrightarrow \mathcal{A}_{k+1}$ described in an abstract form in Proposition 7.5.15 (see also Example 7.5.17) and (7.29)). □

We now want to reformulate the results in Theorem 7.5.18 (and Example 7.5.19) in the setting of partition algebras. For $k = \{0, \frac{1}{2}, 1, 1+\frac{1}{2}, \dots, \}$ let $\widehat{\mathcal{A}}_k(n)$ be the dual of $\mathcal{A}_k(n)$, as defined in Example 7.5.19. Construct the Bratteli diagram as in that example and for $\lambda \in \widehat{\mathcal{A}}_k(n)$ let Z_k^λ be the irreducible representation of $\widehat{\mathcal{A}}_k(n)$ associated to λ. From Theorem 7.5.18, Example 7.5.19, Theorem 8.3.13, Theorem 8.3.15 and Proposition 8.3.17 we deduce immediately the following theorem.

Theorem 8.3.18 (Double commutant theory for partition algebras) *Let* $k = 0, 1, 2, \dots$

(i) *Define a representation* η_k *of* $\mathcal{P}_k(n) \otimes L(\mathfrak{S}_n)$ *on* $V^{\otimes k}$ *by setting* $\eta_k(d, \pi) = \Phi_k(d)\rho_k(\pi)$*,* $d \in P_k$ $\pi \in \mathfrak{S}_n$*. Then*

$$V^{\otimes k} \cong \bigoplus_{\lambda \in \widehat{\mathcal{A}}_k(n)} (Z_k^\lambda \otimes S^\lambda)$$

is the decomposition of η_k *into irreducibles* $\mathcal{P}_k(n) \otimes L(\mathfrak{S}_n)$ *representations. Moreover,* Z_k^λ *is an irreducible representation of* $\mathcal{P}_k(n)$ *and its dimension is equal to the number of paths starting at* $\widehat{\mathcal{A}}_k(0)$ *and ending at* λ *in the Bratteli diagram of the chain* $\mathcal{A}_0(n) \subseteq \mathcal{A}_{\frac{1}{2}}(n) \subseteq \cdots$

(ii) *Define a representation* $\eta_{k+\frac{1}{2}}$ *of* $\mathcal{P}_{k+\frac{1}{2}}(n) \otimes L(\mathfrak{S}_{n-1})$ *on* $V^{\otimes k}$ *by setting* $\eta_{k+\frac{1}{2}}(d, \pi) = \Phi_{k+\frac{1}{2}}(d)\rho_k(\pi)$, *for all* $d \in P_{k+\frac{1}{2}}$, $\pi \in \mathfrak{S}_{n-1}$. *Then*

$$V^{\otimes k} \cong \bigoplus_{\mu \in \widehat{\mathcal{A}}_{k+\frac{1}{2}}(n)} (Z^{\mu}_{k+\frac{1}{2}} \otimes S^{\mu})$$

is the decomposition of $V^{\otimes k}$ *into irreducible* $(P_{k+\frac{1}{2}}(n) \otimes L(\mathfrak{S}_{n-1}))$*-representations. Moreover,* $Z^{\mu}_{k+\frac{1}{2}}$ *is an irreducible representation of* $\mathcal{P}_{k+\frac{1}{2}}(n)$ *and its dimension is equal to the number of paths starting at* $\widehat{\mathcal{A}}_k(0)$ *and ending at* μ *in the Bratteli diagram of the chain* $\mathcal{A}_0(n) \subseteq \mathcal{A}_{\frac{1}{2}}(n) \subseteq \cdots$

(iii) *If* $\lambda \in \widehat{\mathcal{A}}_k(n)$ *and* $\mu \in \widehat{\mathcal{A}}_{k+\frac{1}{2}}(n)$ *then* $\mathrm{Res}^{\mathcal{P}_{k+\frac{1}{2}}(n)}_{\mathcal{P}_k(n)} Z^{\mu}_{k+\frac{1}{2}}$ *decomposes without multiplicity and it contains* Z^{λ}_k *if and only if* S^{μ} *is contained in* $\mathrm{Res}^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}} S^{\lambda}$.

(iv) *If* $\lambda \in \widehat{\mathcal{A}}_{k+1}(n)$ *and* $\mu \in \widehat{\mathcal{A}}_{k+\frac{1}{2}}(n)$ *then* $\mathrm{Res}^{\mathcal{P}_{k+1}(n)}_{\mathcal{P}_{k+\frac{1}{2}}(n)} Z^{\lambda}_{k+1}$ *decomposes without multiplicity and it contains* $Z^{\mu}_{k+\frac{1}{2}}$ *if and only if* S^{λ} *is contained in* $\mathrm{Ind}^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}} S^{\mu}$.

In the remark below, we briefly descibe how to obtain a GZ-basis for the chain of the $\mathcal{A}_k(n)$'s and a related set of YJM elements.

Remark 8.3.19 It is possible to define a Gelfand–Tsetlin basis for the modules Z^{λ}_k in the following way. With each p in the Bratteli diagram

$$p : \mu^{(0)} \to \mu^{(\frac{1}{2})} \to \mu^{(1)} \to \mu^{(1+\frac{1}{2})} \to \cdots$$
$$\cdots \to \mu^{(k)} = \lambda \text{ (or ending in } \to \mu^{(k+\frac{1}{2})} = \lambda) \quad (8.48)$$

we associate the vector v_p obtained as in Section 2.2.1 for group algebras: at each stage, the restrictions $\mathrm{Res}^{\mathcal{A}_\ell(n)}_{\mathcal{A}_{\ell-\frac{1}{2}}(n)}$ and $\mathrm{Res}^{\mathcal{A}_{\ell+\frac{1}{2}}(n)}_{\mathcal{A}_\ell(n)}$ are multiplicity-free. Therefore, as in (2.13), we have

$$Z^{\lambda}_k = \bigoplus_p Z_p$$

where Z_p are one-dimensional subspaces (representations of $\mathcal{A}_0(n)$) and the sum is over all paths p starting at $\mu^{(0)} = (n)$ and ending at λ. Then we can choose $v_p \in Z_p$, $\|v_p\| = 1$ (defined up to a constant of modulus one). We can also define a set of YJM elements. Let T_n be the sum of all transpositions in $\mathfrak{S}_n$ (see the proof of Corollary 3.2.7). Then $\rho_k(T_n)$ belongs to the center of $\mathcal{A}_k(n)$ and $\rho_k(T_{n-1})$ belongs to the center of $\mathcal{A}_{k+\frac{1}{2}}(n)$ (see Proposition 7.3.10). Moreover, since $T_n = \sum_{j=2}^{n} X_j$ (where X_j are the YJM elements for $\mathfrak{S}_n$ (see again Corollary 3.2.7), from the spectral analysis in Section 3.3 it follows that

$$Z^{\lambda}_k \otimes S^{\lambda} \quad (\text{resp.} Z^{\mu}_{k+\frac{1}{2}} \otimes S^{\mu})$$

is an eigenspace of $\rho_k(T_n)$ (resp. $\rho_k(T_{n-1})$) and the corresponding eigenvalue is

$$\sum_{b\in\lambda} c(b) \quad (\text{resp. } \sum_{b\in\mu} c(b)),$$

where the sums are all over all boxes b of the Young frame λ (resp. μ) and $c(b)$ is the content of b. Now set $\widetilde{M}_k = \rho_k(T_n) - \rho_{k-1}(T_{n-1})$ (using $\mathcal{A}_{k-\frac{1}{2}}(n) \subseteq \mathcal{A}_k(n)$) and $\widetilde{M}_{k+\frac{1}{2}} = \rho_k(T_{n-1}) - \rho_k(T_n)$. Then the elements $\widetilde{M}_0, \widetilde{M}_{\frac{1}{2}}, \ldots, \widetilde{M}_{k-\frac{1}{2}}, \widetilde{M}_k$ commute (see Exercise 3.2.9). Moreover, from the spectral analysis of $\rho_k(T_n)$ and $\rho_k(T_{n-1})$ it follows that if p is as in (8.48) then

$$\widetilde{M}_k v_p = c(\lambda/\mu^{(k-\frac{1}{2})})v_p \quad (\text{resp. } \widetilde{M}_{k+\frac{1}{2}} v_p = -c(\mu^{(k)}/\lambda))$$

where $c(\lambda/\mu)$ is the content of the box b when λ is obtained from μ adding b. Therefore, as in Section 3.3 we can associate with v_p the vector of eigenvalues

$$(-c(\mu^{(0)}/\mu^{(\frac{1}{2})}), c(\mu^{(1)}/\mu^{(\frac{1}{2})}), -c(\mu^{(1)}/\mu^{(1+\frac{1}{2})}), \ldots,).$$

It follows that *the GZ-vectors v_p are determined by the eigenvalues of the YJM elements $\widetilde{M}$'s*. Therefore, the YJM elements constitute a basis for the GZ-algebra associated with the GZ-basis $\{v_p\}$.

Exercise 8.3.20 Check all the details in Remark 8.3.19.

In [58], the authors construct explicit elements $M_k \in \mathcal{P}_k(n)$ such that $\Phi_k(M_k) = \widetilde{M}_k$ for all the values of n (in their paper n is allowed to belong to $\mathbb{C}$) such that $\mathcal{P}_k(n)$ is semisimple (actually, they use a slightly different expression for the $\widetilde{M}_k$).

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